Internat. J. Math. & Math. Sci.
VOL. 16 NO. 1 (1993) 111-116

THE OPEN-OPEN TOPOLOGY FOR FUNCTION SPACES

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(Received February 20, 1991 and in revised form January 22, 1992)

ABSTRACT. Let (X,T) and (Y,T^*) be topological spaces and let $F \subset Y^X$. For each $U \in T, V \in T^*$, let $(U,V) = \{f \in F : f(U) \subset V\}$. Define the set $S_{oo} = \{(U,V) : U \in T \text{ and } V \in T^*\}$. Then S_{oo} is a subbasis for a topology, T_{oo} on F, which is called the open-open topology. We compare T_{oo} with other topologies and discuss its properties. We also show that T_{oo}, on $H(X)$, the collection of all self-homeomorphisms on X, is equivalent to the topology induced on $H(X)$ by the Pervin quasi-uniformity on X.

KEY WORDS AND PHRASES. Compact-open topology, admissible topology, Galois space, Pervin quasi-uniformity, self-homeomorphism, quasi-uniform convergence.
1980 AMS SUBJECT CLASSIFICATION CODES. Primary 54C35, Secondary 54H13.

1. INTRODUCTION.

The type of set-set topology which will be discussed here is one which can be defined as follows: Let (X,T) and (Y,T^*) be topological spaces. Let $\mathbf{U}$ and $\mathbf{V}$ be collections of subsets of X and Y, respectively. Let $F \subset Y^X$, the collection of all functions from X into Y. Define, for $U \in \mathbf{U}$ and $V \in \mathbf{V}$, $(U,V) = \{f \in F : f(U) \subset V\}$. Let $S(\mathbf{U},\mathbf{V}) = \{(U,V) : U \in \mathbf{U} \text{ and } V \in \mathbf{V}\}$. If $S(\mathbf{U},\mathbf{V})$ is a subbasis for a topology $T(\mathbf{U},\mathbf{V})$ on F then $T(\mathbf{U},\mathbf{V})$ is called a set-set topology.

One of the original set-set topologies is the compact-open topology, T_{co}, which was introduced in 1945 by R. Fox [1]. For this topology, as one may surmise from the name, $\mathbf{U}$ is the collection of all compact subsets of X and $\mathbf{V} = T^*$, the collection of all open subsets of Y. Fox and Arens [2] developed and examined the properties of this now well-known topology. In particular, it was shown that if $F \subset C(X,Y)$, the collection of all continuous functions on X into Y, then T_{co} on F is equivalent to the topology of uniform convergence on compacta; and, if in addition, X is compact, then T_{co} is equivalent to the topology of uniform convergence on F. Arens also defined the concept of admissible topology for function spaces and was instrumental in the study of groups of self-homeomorphisms and topological groups.

Other set-set topologies that have been of interest are: the point-open topology, T_p, also known as the topology of pointwise convergence, in which $\mathbf{U}$ is the collection of all singletons in X and $\mathbf{V} = T^*$; the closed-open topology, where $\mathbf{U}$ is the collection of all closed subsets of X and the set $\mathbf{V} = T^*$; and the bounded-open topology (Lambrinos [3]), where $\mathbf{U}$ is the collection of all bounded subsets of X and again, $\mathbf{V} = T^*$.

In section 2 of this paper, we shall introduce and discuss the open-open topology, T_{oo}, for function spaces. It will be shown which of the desirable properties T_{oo} possesses. In section 3, the group of all self-homeomorphisms, $H(X)$, endowed with T_{oo}, is discussed.

As will be proven in section 5, T_{oo}, on $H(X)$, is actually equivalent to the Pervin topology of quasi-uniform convergence (Fletcher [4]). One of the advantages of the open-open topology is the set-set notation which provides us with simple notation and, hence, our proofs are more concise than those using the cumbersome notation of the quasi-uniformity. Pervin spaces will be discussed in section 4.

We assume a basic knowledge of quasi-uniform spaces. An introduction to quasi-uniform spaces may be found in Fletcher and Lindgren's [5] or in Murdeshwar and Naimpally's book [6].

Throughout this paper we shall assume (X,T) and (Y,T^*) are topological spaces.

2. THE OPEN-OPEN TOPOLOGY.

If we let $\mathbf{U} = T$ and $\mathbf{V} = T^*$, then $S_{oo} = S(\mathbf{U},\mathbf{V})$ is the subbasis for the topology, T_{oo}, on any $F \subset Y^X$, which is called the open-open topology.

We first examine some of the properties of function spaces the open-open topology possesses.

THEOREM 1. Let $F \subset C(X,Y)$. If (Y,T^*) is T_i for i = 0, 1, 2, then (F,T_{oo}) is T_i for i = 0, 1, 2.

PROOF. We shall show the case i = 2; the other cases are done similarly. Let i = 2. Let $f,g \in F$ such that $f \neq g$. Then there is some $x \in X$ such that $f(x) \neq g(x)$. If Y is T_2 there exists disjoint open sets O and U in Y such that $f(x) \in U$ and $g(x) \in O$. Both f and g are continuous, so there are open sets V and W in X with $x \in V \cap W$, $f(V) \subset U$ and $g(W) \subset O$. $f \in (V,U), g \in (W,O)$, and $(V,U) \cap (W,O) = \phi$. Thus, (F,T_{oo}) is T_2.

A topology, T', on $F \subset Y^X$ is called an admissible (Arens [2]) topology for F provided the evaluation map, E: $(F,T') \times (X,T) \to (Y,T^*)$, defined by $E(f,x) = f(x)$, is continuous.

THEOREM 2. If $F \subset C(X,Y)$ then T_{oo} is admissible for F.

PROOF. Let $F \subset C(X,Y)$. Let $O \in T^*$ and let $(f,p) \in E^{-1}(O)$. Then $f(p) \in O$. Since f is continuous, there exists some $U \in T$ such that $p \in U$ and $f(U) \subset O$. So, $(f,p) \in (U,O) \times U$. If $(g,b) \in (U,O) \times U$, then $g(U) \subset O$ and $b \in U$, so $g(b) \in O$. Hence, $(U,O) \times U \subset E^{-1}(O)$. Therefore, T_{oo} is admissible for F.

Arens also has shown that if T' is admissible for $F \subset C(X,Y)$, then T' is finer than T_{co}. From this fact and Theorem 2, it follows that $T_{co} \subset T_{oo}$.

3. THE OPEN-OPEN TOPOLOGY ON $H(X)$.

We now consider T_{oo} on $H(X)$, the collection of all self-homeomorphisms on X. Note that $H(X)$ with the binary operation $\circ$, composition of functions, and identity element e, is a group.

Some of the set-set topologies previously mentioned are equivalent under certain hypotheses. For example, the closed-open topology is equal to the compact-open topology whenever X is compact T_2, the point-open topology is equivalent to the compact-open topology if all compact subsets of X are finite sets. It is always advantageous to know when topologies are or are not equivalent. In particular, it is well known that if X is T_1 then $T_p \subset T_{co}$ and as we have shown $T_{co} \subset T_{oo}$. When are T_{co} and T_{oo} distinct? One hypothesis under which these two topologies are not equivalent is: "Let X be T_2 and Galois."

A topological space is Galois provided that for each closed set, $C \subset X$ and each point $p \in X \setminus C$, there is an $h \in H(X)$ such that $h(x) = x$ for all $x \in C$ and $h(p) \neq p$. Among the spaces which are T_2 Galois are the topological vector spaces and, as Fletcher [7] has shown, locally euclidean T_2 spaces or homogeneous 0-dimensional spaces which have no isolated points.

THEOREM 3. If X is a T_2 Galois space then $T_{oo} \neq T_{co}$ on $H(X)$.

PROOF. Let X be a T_2 Galois space. Let $x \in X$; then $X \setminus \{x\}$ is open in X so that $(X \setminus \{x\}, X \setminus \{x\})$ is open in T_{oo}. But note that $(X \setminus \{x\}, X \setminus \{x\}) = (\{x\}, \{x\})$ in $(H(X), T_{oo})$.

Let e be the identity map on X then $e \in (\{x\}, \{x\})$. *Claim*: $(\{x\}, \{x\}) \notin T_{co}$ and hence $T_{co} \neq T_{oo}$. Let $\bigcap_{i=1}^{n}(C_i, U_i)$ be a basic open set in T_{co} which contains e. So, $C_i \subset U_i$ for all $i = 1, 2, 3, ..., n$.

Set $U_0 = X$ and $C_0 = \phi$. Let $P = \{U_i\}_{i=0}^{n}$ and $Q = \{C_i\}_{i=0}^{n}$. Define $P_x = \cap\{U \in P | x \in U\}$ and $Q_x = \cup\{C \in Q | x \notin C\}$. Let $L = P_x \setminus Q_x$. Note that $x \in L$ and L is open in X.

Case 1: $L = X$: Then for each $i = 1, 2, 3 ..., n$, $U_i = X$, so that $\bigcap_{i=1}^{n}(C_i, U_i) = H(X)$ and $\bigcap_{i=1}^{n}(C_i, U_i) \not\subset (\{x\}, \{x\})$.

Case 2: $L \neq X$: Thus, since X is Galois, there exists some $h \in H(X)$ such that $h(y) = y$ for all $y \in X \setminus L$ and $h(x) \neq x$. So $h \notin (\{x\}, \{x\})$ and $h \in (L, L)$. Let $q \in C_j$ for some $j \in \{1, 2, ..., n\}$. If $q \notin L$ then $h(q) = q \in C_j \subset U_j$. If $q \in L$ then $x \in C_j$ from which it follows that $L \subset U_j$. Thus, $h(L) = L \subset U_j$. In either case, $h \in \bigcap_{i=1}^{n}(C_i, U_i)$ and again $\bigcap_{i=1}^{n}(C_i, U_i) \not\subset (\{x\}, \{x\})$.

Therefore, $(\{x\}, \{x\}) \notin T_{co}$ and $T_{co} \neq T_{OO}$.

Effros' Theorem (Effros [8]) is a widely known and useful tool in the study of homogeneous spaces and continua theory. Of its several forms, the most popular is: If X is a compact homogeneous metric space then for each $x \in X$, the evaluation map, $E_x : (H(X), T_{co}) \to X$, defined by $E_x(h) = h(x)$, is an open map. It follows that, if the conclusion holds when E_x is considered on the space $(H(X), T*)$, and if $T \subset T*$ on $H(X)$, then the conclusion also holds on $(H(X), T)$. Ancel [9] has asked the following question: If the hypothesis of the Effros' Theorem is changed to "X is a compact, homogeneous, Hausdorff space," is the evaluation map on $(H(X), T_{co})$ still open *? To this end, since $T_{co} \subset T_{oo}$, we could consider whether a form of Effros' Theorem would be true for T_{oo} on $H(X)$. Unfortunately, we discover the following.

THEOREM 4. Let (X,T) be a T_1 topological space. Then, for each $x \in X$, the evaluation map, $E_x : (H(X), T_{oo}) \to X$ defined by $E_x(h) = h(x)$, is open only if T is the discrete topology.

PROOF. Let $x \in X$. Then the set $O = ((X \setminus \{x\}), (X \setminus \{x\}))$ is open in $(H(X), T_{oo})$. But $((X \setminus \{x\}), (X \setminus \{x\})) = (\{x\}, \{x\})$. So. $E_x(O) = \{x\}$. Thus E_x is open for each $x \in X$ only if X is discrete.

Let $(G, \circ)$ be a group such that (G,T) is a topological space, then (G,T) is a topological group provided the following two maps are continuous. (1) $m : G \times G \to G$ defined by $m(g_1, g_2) = g_1 \circ g_2$ and $\Phi : G \to G$ defined by $\Phi(g) = g^{-1}$. If only the first map is continuous, then we call (G,T) a quasi-topological group (Murdeshwar and Naimpally [6]).

It is not difficult to show that if (X,T) is a topological space and G is a subgroup of $H(X)$ then (G, T_{oo}) is a quasi-topological group. However, (G, T_{oo}) is not always a topological group as the following example (Fletcher [4]) shows: Let $X = R$ and let the topology on X be described as follows: $T = \{(a,b) \subset R : a < 0 < b\} \cup \{\phi, X\}$. Let $f, g : X \to X$ be defined by $f(x) = -x$ and $g(x) = -\frac{x}{2}$. Clearly, f and g are homeomorphisms on X. Note that $f(x) = f^{-1}(x)$ and $g^{-1}(x) = -2x$. Let $U = V = (-1,1)$. Then $f \in (U, V) \in T_{oo}$. Now define $\Phi : H(X) \to H(X)$ by $\Phi(h) = h^{-1}$. So, $f \in \Phi^{-1}((U,V))$. Claim: Φ is not continuous: Let $O = \bigcap_{i=1}^{n}((a_i, b_i), (c_i, d_i))$ be a basic open set in $(H(X), T_{oo})$ which contains f. Then $a_i < 0 < b_i$ and $c_i < 0 < d_i$ for each i. If $x \in (a_i, b_i)$ then $f(x) = -x \in (c_i, d_i)$. So, $g(x) = -\frac{x}{2} \in (c_i, d_i)$ and, hence, $g \in O$. Thus, every basic open set containing f also contains g. But $g^{-1}(U) \not\subset V$ and so $g \notin \Phi^{-1}((U,V))$. Therefore, any basic open set containing f is not contained in $\Phi^{-1}((U,V))$. Thus, Φ is not continuous and (G, T_{oo}) is not a topological group.

4. PERVIN SPACES.

A topological space, (X,T), is called a Pervin space (Fletcher [4]) provided that for each finite collection, $\mathcal{A}$, of open sets in X, there exists some $h \in H(X)$ such that $h \neq e$ and $h(U) \subset U$ for all $U \in \mathcal{A}$.

Topologies are rarely interesting if they are the trivial or discrete topology. To this end, we have:

THEOREM 5. $(H(X), T_{oo})$ is not discrete if and only if (X,T) is a Pervin space.

PROOF. First, assume that (X,T) is a Pervin space. Let W be a basic open set in T_{oo}

*This was recently answered in the negative by Bellamy and Porter [10].

which contains e; i.e. $W = \bigcap_{i=1}^{n}(O_i, U_i)$ where $O_i \subset U_i$ for each $i = 1,2,3,...,n$ and O_i and U_i are open in X. $\{O_i : i = 1,2,3,...,n\}$ is a finite collection of open sets in X, and X is a Pervin space, hence, there exists some $h \in H(X)$ such that $h \neq e$ and $h(O_i) \subset O_i \subset U_i$. So, $h \in W$ and $h \neq e$. Since $(H(X), T_{oo})$ is a quasi-topological group, $(H(X), T_{oo})$ is not a discrete space.

Now assume that $(H(X), T_{oo})$ is not discrete. Let V be a finite collection of open sets in X. Let $O = \bigcap_{U \in V}(U, U)$. Then, O is a basic open set in $(H(X), T_{oo})$ which is not a discrete space. Hence, there exists $h \in O$ with $h \neq e$. So, (X, T) is Pervin.

Fletcher [4] proved that the Pervin topology of quasi-uniform convergence on $H(X)$ is not discrete if and only if (X, T) is Pervin. In order to prove this, Fletcher had to first introduce numerous definitions along with some mind boggling notation. The above proof, along with the few needed definitions involving T_{oo}, is an example of the simplification that the definition of T_{oo} offers over the quasi-uniform definition and notation.

5. THE PERVIN TOPOLOGY OF QUASI-UNIFORM CONVERGENCE.

Recall that if Q is a quasi-uniformity on X, then the topology, T_Q, on X, which has as its neighborhood base at x, $B_x = \{U[x] : U \in Q\}$, is called the topology induced by Q. The ordered triple (X, Q, T_Q) is called a quasi-uniform space. A topological space, (X, T) is quasi-uniformizable provided there exists a quasi-uniformity, Q, such that $T_Q = T$. In 1962, Pervin [11] proved that every topological space is quasi-uniformizable by giving the following construction.

Let (X, T) be a topological space. For each $O \in T$, define the set $S_O = (O \times O) \cup ((X \backslash O) \times X)$. Let $S = \{S_O : O \in T\}$. Then S is a subbasis for a quasi-uniformity, P, for X, called the Pervin quasi-uniformity and, as is easily shown, $T_P = T$.

If (X, Q) is a quasi-uniform space then Q induces a topology on $H(X)$ called the topology of quasi-uniform convergence w.r.tQ, as follows: For each set $U \in Q$, let us define $W(U) = \{(f, g) \in H(X) \times H(X) : (f(x), g(x)) \in U \text{ for all } x \in X\}$. Then, $B(Q) = \{W(U) : U \in Q\}$ is a basis for Q^*, the quasi-uniformity of quasi-uniform convergence w.r.t. Q (Naimpally [12]). Let T_{Q^*} denote the topology on $H(X)$ induced by Q^*. T_{Q^*} is called the topology of quasi-uniform convergence w.r.t. Q^*. If P is the Pervin quasi-uniformity on X T_{P^*} is the Pervin topology of quasi-uniform convergence.

At this time one could, once again, prove that T_{P^*} is not discrete if and only if (X, T) is a Pervin space, this time using the quasi-uniform structure [4]. We leave this to the reader.

We now show that the open-open topology is equivalent to the Pervin topology of quasi-uniform convergence.

THEOREM 6. Let (X, T) be a topological space and let G be a subgroup of $H(X)$. Then, $T_{oo} = T_{P^*}$ on G.

PROOF. Let (O, U) be a subbasic open set in T_{oo} and let $f \in (O, U)$. Then $f(O) \subset U$. So $f \in W(S_U)[f]$. Hence, if $g \in W(S_U)[f]$, $((f(x), g(x)) \in S_U$ for all $x \in X$. If $x \in O$, $f(x) \in U$, which implies that $g(x) \in U$. Thus, $g \in (O,U)$ and $W(S_U) \subset (O,U)$. Therefore, $T_{oo} \subset T_{P^*}$.

Now let $V \in T_{P^*}$ and let $f \in V$. Then there exists $U \in P$ such that $f \in W(U)[f] \subset V$. Since $U \in P$, there is some finite collection, $\{U_i : i = 1,2,...,n\} \subset T$ such that $\bigcap_{i=1}^{n} S_{U_i} \subset U$. Define $A = \bigcap_{i=1}^{n}(f^{-1}(U_i), U_i)$. A is an open set in T_{oo}. and $f \in A$. Assume $g \in A$ and let $x \in X$. If $f(x) \in U_j$ for some $j \in \{1,2,...,n\}$, then $x \in f^{-1}(U_j)$. So, $g(x) \in U_j$, hence, $(f(x), g(x)) \in U_j \times U_j \subset S_{U_j}$. If $f(x) \notin U_j$ for some $j \in \{1,2,...,n\}$ then $(f(x), g(x)) \in (X \setminus U_j, X) \subset S_{U_j}$ Hence, $(f(x), g(x)) \in \bigcap_{i=1}^{n} S_{U_i} \subset U$, and it follows that $g \in W(U)[f] \subset V$ and $A \subset V$. So, $T_{oo} = T_{P^*}$.

ACKNOWLEDGEMENT. The author would like to thank the Committee for Faculty and Curriculum Development at Saint Mary's College of California for their financial support.

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