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EXISTENCE AND UNIQUENESS FOR THE NONSTATIONARY PROBLEM OF THE ELECTRICAL HEATING OF A CONDUCTOR DUE TO THE JOULE-THOMSON EFFECT

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ABSTRACT. Existence of a weak solution is established for the initial-boundary value problem for the system $\frac{\partial}{\partial t}u - div(\Theta(u)\nabla u) + \sigma(u)\alpha(u)\nabla u \nabla v = \sigma(u)|\nabla v|^2, div(\sigma(u)\nabla v) = 0$. The question of uniqueness is also considered in some special cases.

KEY WORDS AND PHRASES. Joule-Thomson effect, quadratic gradient growth in the nonlinearity, existence, uniqueness.
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1. INTRODUCTION.

Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and T a positive number. In this paper we shall be concerned with the following problem:

$$u_t - div(\Theta(u)\nabla u) + \sigma(u)\alpha(u)\nabla u \nabla v = \sigma(u)|\nabla v|^2 \text{ in } Q_T = \Omega\times(0,T), \tag{1.1a}$$

$$div(\sigma(u)\nabla v) = 0 \text{ in } Q_T, \tag{1.1b}$$

$$u = 0 \text{ on } S_T = \partial\Omega\times(0,T), \tag{1.1c}$$

$$v = B(x,t) \text{ on } S_T, \tag{1.1d}$$

$$u(x,0) = U_0(x) \text{ in } \Omega\times\{0\}. \tag{1.1e}$$

Here, $\Theta(u), \sigma(u)$, and $\alpha(u)$ are known functions of their argument and B, U_0 are given data.

Problem (1.1) may be proposed as a model for the electrical heating of a conductor resulted from Thomson's effect and Joule's heating; see [1]. In this situation, u is the temperature of the conductor and v the effective potential. Equation (1.1b) represents the conservation of charge, while (1.1a) says that there are two types of heat source involved in the heat conduction; the convective term in (1.1a) corresponds to Thomson's effect and the quadratic term in (1.1a) reflects Joule's heating.

If $N = 2, \alpha = 0$, and $\sigma \in C^1(\mathbf{R})$ is such that

$$0 < m \le \sigma(s) \le M, s \in \mathbf{R}$$

for some $M \ge m$ the existence of a weak solution is established for (1.1) in [2]. A result due to Shi, Shillor, and Xu [3] asserts that the assumption that $N=2$ and $\sigma \in C^1(\mathbf{R})$ in [2] can be eliminated. The associated stationary problem of (1.1) was first considered in [1] where σ and Θ are assumed to obey the Wiedemann-Franz law, i.e.,

$$\frac{\sigma(u)}{\Theta(u)} = \frac{c}{u} \text{ for some } c > 0,$$

and α is assumed to be linear. Under these assumptions the stationary problem can be reformulated as

$$\left.\begin{array}{l} div(A(u,v) \nabla u) = 0 \\ div(A(u,v) \nabla v) = 0 \end{array}\right\} \text{ in } \Omega,$$

$$u = u_0, v = v_0 \text{ on } \partial\Omega.$$

Thus a uniform bound for the temperature can be obtained, thereby establishing an existence assertion. See [1] for details.

Our main objective is to prove an existence theorem for (1.1) under rather general assumptions on the data. Indeed, if the temperature is known to be bounded, our assumptions are much weaker than those in [1]. Of course, our approach is also different and is based upon an approximation scheme. We also consider the question of uniqueness, but we are only able to show that the uniqueness holds when $N=2$ and $\Theta(s)=s$.

The mathematical interest of our problem is due to the presence of quadratic gradient growth in the nonlinearity. In general, nonlinearities of this nature render the classical regularity and compactness results useless; see [4] for a detailed description in this regard. Our method makes full use of the explicit nonlinear structure of our problem, which enables us to extract enough extra information to obtain an existence assertion. We refer the reader to [4] for more related works in this direction.

Finally, let us make some comments on notation. The letter c will be used to denote the generic constant. When distinction among different constants is needed, we add a subscript $i \in \{0,1,2,...\}$ to c. Other notation conventions follow those employed in [5] and [6]. For example,

$$\| f \|_{p,\Omega} = \| f \|_p = \left(\int_\Omega | f |^p dx \right)^{1/p}$$

for $f \in L^p(\Omega)$.

2. EXISTENCE

In this section we first establish an existence assertion for the associated stationary problem. Then a weak solution to (1.1) is obtained via the implicit discretization in time.

Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. Consider the system

$$\left.\begin{array}{l} -div(I(u,v) \nabla u) + K(u+v) = J(u+v) \mid \nabla v \mid^2 + H(x) \\ -div(J(u+v) \nabla v) = 0 \end{array}\right\} \text{in } \Omega \tag{2.1a}$$

coupled with boundary conditions

$$u = u_0 \text{ on } \partial\Omega, v = v_0 \text{ on } \partial\Omega. \tag{2.1b}$$

With respect to the data involved, we assume the following.

(H1) $I(s,\tau), J(s), K(s)$ are all continuous;

(H2) There exist two positive numbers $m \le M$ such that

$$m \le I(s,\tau) \le M, m \le J(s) \le M$$

for all $s, \tau \in \mathbf{R}$;

(H3) K is nondecreasing and satisfies

$$|K(s)| \le c|s| \text{ for some } c > 0;$$

(H4) $u_0 \in W^{1,2}(\Omega), v_0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, and $H \in L^2(\Omega)$.

A weak solution to (2.1) is defined as a pair (u,v) such that

$$u, v \in W^{1,2}(\Omega), \tag{2.2}$$

$$\int_\Omega I(u,v) \nabla u \nabla \xi dx + \int_\Omega K(u+v)\xi dx = \int_\Omega (J(u+v)|\nabla v|^2 + H(x))\xi dx \tag{2.3}$$

for all $\xi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega J(u+v) \nabla v \nabla \eta dx = 0 \text{ for all } \eta \in W_0^{1,2}(\Omega), \tag{2.4}$$

$$u = u_0, v = v_0 \text{ on } \partial\Omega. \tag{2.5}$$

THEOREM 2.1. Let (H1) - (H4) be satisfied. Then there exists a weak solution to (2.1).

PROOF. For each k define

$$P_k(x) = \begin{cases} k & \text{if } |x|^2 \ge k, \\ |x|^2 & \text{if } |x|^2 < k, \end{cases}$$

$$K_k(s) = \begin{cases} k & \text{if } K(s) > k, \\ K(s) & \text{if } |K(s)| \le k, \\ -k & \text{if } K(s) < -k. \end{cases}$$

Denote by V the product space $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ and V^* its topological dual. Set $E = \{(u_1,v_1) \in V: u_1|_{\partial\Omega} = u_0 \text{ and } v_1|_{\partial\Omega} = v_0\}$. Clearly, E is a closed, convex subset of V. For each k define an operator $A_k: E \to V^*$ by

$$(A_k(w_1), w_2) = \int_\Omega I(u_1,v_1) \nabla u_1 \nabla u_2 dx + \int_\Omega \{K_k(u_1+v_1) - J(u_1+v_1)P_k(\nabla v_1)$$
$$- H(x)\}u_2 dx$$
$$+ \int_\Omega J(u_1+v_1) \nabla v_1 \nabla v_2 dx,$$

$$w_1 = (u_1,v_1) \in E, w_2 = (u_2,v_2) \in V,$$

where $(\cdot,\cdot)$ denotes the duality pairing between V^* and V. By the definition of P_k, A_k is well-defined. It is not difficult to verify that for each k, A_k satisfies the following conditions:

(i) A_k is bounded.

(ii) A_k is pseudomonotone.

(iii) $(A_k(w), w-w_0)/\|w\|_V \to \infty$ as $\|w\|_V \to \infty$ for $w \in E$, where $w_0 = (u_0, v_0)$.

Now we are in a position to invoke an existence result in [7, p. 169] to conclude that for each k there exists at least one vector-valued function $w_k = (u_k, v_k) \in E$ such that

$$(A_k(w_k), w - w_k) \geq 0 \text{ for all } w \in E.$$

This is easily seen to be equivalent to the following statements:

$$u_k\,|_{\partial\Omega} = u_0, v_k\,|_{\partial\Omega} = v_0, \tag{2.6}$$

$$\int_\Omega I(u_k, v_k) \nabla u_k \nabla \xi dx + \int_\Omega K_k(u_k + v_k)\xi dx = \int_\Omega (J(u_k + v_k)P_k(\nabla v_k) + H(x))\xi dx, \tag{2.7}$$

$$\int_\Omega J(u_k + v_k) \nabla v_k \nabla \xi dx = 0, \tag{2.8}$$

for all $\xi \in W_0^{1,2}(\Omega)$. Equation (2.8) allows us to use the weak maximum principle to get

$$\sup_{x \in \Omega} |v_k(x)| \leq c(k = 1, 2, \dots). \tag{2.9}$$

Set $\xi = v_k - v_0$ in (2.8) to deduce

$$\|\nabla v_k\|_2 \leq c(k = 1, 2, \dots). \tag{2.10}$$

Let $\xi = u_k - u_0$ in (2.7) to derive

$$\begin{aligned}&\int_\Omega I(u_k, v_k) |\nabla u_k|^2 dx + \int_\Omega K_k(u_k + v_k)(u_k - u_0) dx \\ &= \int_\Omega I(u_k, v_k) \nabla u_k \nabla u_0 dx + \int_\Omega J(u_k + v_k) P_k(\nabla v_k)\,(u_k - u_0) dx + \int_\Omega H(x)(u_k - u_0) dx.\end{aligned} \tag{2.11}$$

We estimate, with the aid of (H3) and (2.9), that

$$\begin{aligned}&\int_\Omega K_k(u_k + v_k)(u_k - u_0) dx \\ &= \int_\Omega K_k(u_k + v_k)(u_k + v_k) dx - \int_\Omega K_k(u_k + v_k)(v_k + u_0) dx \\ &\geq -\|K_k(u_k + v_k)\|_2 \|v_k + u_0\|_2 \\ &\geq -c_1 \|u_k\|_2 - c_2.\end{aligned} \tag{2.12}$$

For each positive integer j define

$$L_j(s) = \begin{cases} j & \text{if } s \geq j, \\ s & \text{if } |s| < j, \\ -j & \text{if } s \geq -j. \end{cases}$$

We calculate, using (2.8), that

$$\int_\Omega J(u_k+v_k)P_k(\nabla v_k)L_j(u_k-u_0)dx$$

$$= \int_\Omega J(u_k+v_k)P_k(\nabla v_k)([L_j(u_k-u_0)]^+ - [L_j(u_k-u_0)]^-)dx$$

$$\le \int_\Omega J(u_k+v_k)P_k(\nabla v_k)[L_j(u_k-u_0)]^+ dx$$

$$\le \int_\Omega J(u_k+v_k)|\nabla v_k|^2[L_j(u_k-u_0)]^+ dx$$

$$= \int_\Omega J(u_k+v_k)\nabla v_k\{\nabla(v_k[L_j(u_k-u_0)]^+) - v_k\nabla[L_j(u_k-u_0)]^+\}dx$$

$$= \int_\Omega J(u_k+v_k)\nabla v_k v_k \nabla[L_j(u_k-u_0)]^+ dx$$

$$\le c\|\nabla v_k\|_2\|\nabla(u_k-u_0)\|_2 \le c_1\|\nabla u_k\|_2 + c_2.$$

Send j to infinity to get

$$\int_\Omega J(u_k+v_k)P_k(\nabla v_k)(u_k-u_0)dx \le c_1\|\nabla u_k\|_2 + c_2.$$

Use this and (2.12) in (2.11) to obtain

$$m\int_\Omega |\nabla v_k|^2 dx \le c_1\|\nabla u_k\|_2 + c_2\|u_k\|_2 + c_3 (k=1,2,\ldots). \tag{2.13}$$

According to Poincaré's inequality,

$$\|u_k-u_0\|_2 \le c\|\nabla(u_k-u_0)\|_2 \le c(\|\nabla u_k\|_2 + \|\nabla u_0\|_2).$$

Consequently,

$$\|u_k\|_2 \le \|u_k-u_0\|_2 + \|u_0\|_2 \le c_1\|\nabla u_k\|_2 + c_2. \tag{2.14}$$

Combining (2.13) and (2.14) yields

$$\|u_k\|_2 + \|\nabla u_k\|_2 \le c (k=1,2,\ldots). \tag{2.15}$$

In view of (2.9), (2.10), and (2.15), we may assume that there exists a subsequence of $\{k\}$, still denoted by $\{k\}$, such that

$$v_k \to v \text{ weakly in } W^{1,2}(\Omega) \text{ and strongly in } L^2(\Omega), \tag{2.16}$$

$$u_k \to u \text{ weakly in } W^{1,2}(\Omega) \text{ and strongly in } L^2(\Omega). \tag{2.17}$$

Then it immediately follows from (H1), (H2), and (H3) that

$$I(u_k,v_k)\to I(u,v), K_k(u_k+v_k)\to K(u+v) \tag{2.18}$$

and

$$J(u_k+v_k)\to J(u+v) \text{ strongly in } L^2(\Omega). \tag{2.19}$$

Set $\xi = v_k - v$ in (2.8) to deduce

$$\limsup_{k\to\infty} m\int_\Omega |\nabla(v_k - v)|^2 dx \le \limsup_{k\to\infty}\int_\Omega J(u_k+v_k)|\nabla(v_k-v)|^2 dx$$

$$= \lim_{k\to\infty} -\int_\Omega J(u_k+v_k)\nabla v\nabla(v_k - v)dx = 0.$$

This implies that

$$P_k(\nabla v_k) \to |\nabla v|^2 \text{ strongly in } L^1(\Omega). \tag{2.20}$$

Then the theorem follows from taking $k\to\infty$ in (2.7) and (2.8).

Let Ω, $H(x)$, u_0, v_0 be given as before. Consider the following problem:

$$\left.\begin{array}{l} u - div(\Theta(u)\nabla u) + \sigma(u)\beta'(u)\nabla u\nabla v = \sigma(u)|\nabla v|^2 + H(x) \\ div(\sigma(u)\nabla v) = 0 \end{array}\right\} \text{ in } \Omega, \tag{2.21a}$$

$$u|_{\partial\Omega} = u_0, v|_{\partial\Omega} = v_0. \tag{2.21b}$$

We impose the following conditions on Θ, σ, β:

(H5) Θ, σ, β are continuous and satisfy

$$m \le \Theta(s) \le M, m \le \sigma(s) \le M, m \le \beta(s) \le M \text{ for some } M \ge m > 0$$

for all $s \in \mathbf{R}$.

(H6) β' is continuous and bounded.

A weak solution to (2.21) can be defined in the same manner as that to (2.1).

THEOREM 2.2. Under the above assumptions there is a weak solution to (2.21).

PROOF. Let

$$F(s) = \int_0^s \frac{\Theta(\tau)}{\sigma(\tau)\beta(\tau)}d\tau.$$

Then by (H5) there exists two positive constants c_1, c_2 such that

$$0 < c_1 \le F'(s) \le c_2 \text{ for all } s \in \mathbf{R}. \tag{2.22}$$

Denote by K the inverse of F. From (2.22), we have

$$c_4 \le K'(s) \le c_3 \text{ for all } s \in \mathbf{R}$$

for some $c_3 \ge c_4 > 0$. Thus K satisfies (H3). Now set

$$I(s,\tau) = \sigma(K(s+\tau)\beta(K(s+\tau)), \tag{2.23}$$

$$J(s) = \sigma(K(s)). \tag{2.24}$$

Clearly, I, J, K satisfy (H1)-(H3). By Theorem 2.1, there is a weak solution to the following problem:

$$\left.\begin{array}{ll} -div(I(a,b)\nabla a) + K(a+b) = J(a+b)|\nabla b|^2 + H(x) & (2.25) \\ -div(J(a+b)\nabla b) = 0 & (2.26) \end{array}\right\} \text{ in } \Omega$$

$$a|_{\partial\Omega} = F(u_0) - v_0, b|_{\partial\Omega} = v_0.$$

Let $u = K(a+b), v = b$. We wish to show that (u,v) thus defined is a weak solution to (2.21). Clearly, $u, v \in W^{1,2}(\Omega)$, and (2.21b) is satisfied. Note that

$$a = F(u) - b = F(u) - v.$$

We derive from (2.23), (2.24), and (2.25) that

$$\begin{aligned} &-div\Big(\sigma(u)\beta(u)\nabla(F(u)-v)\Big)+u \\ &\quad = -div(\sigma(u)\beta(u)F'(u)\nabla u - \sigma(u)\beta(u)\nabla v) + u \\ &\quad = -div(\Theta(u)\nabla u) + div(\sigma(u)\beta(u)\nabla v) + u = \sigma(u)|\nabla v|^2 + H(x) \text{ in } \Omega. \end{aligned} \tag{2.27}$$

We conclude from (2.24) and (2.26) that

$$-div(\sigma(u)\nabla v) = 0 \text{ in } \Omega. \tag{2.28}$$

We calculate from (2.28) that for any $\xi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$,

$$\begin{aligned} \int_\Omega \sigma(u)\beta(u)\nabla v \nabla \xi dx &= \int_\Omega \sigma(u)\nabla v\Big(\nabla(\beta(u)\xi) - \xi\nabla\beta(u)\Big)dx \\ &= -\int_\Omega \sigma(u)\beta'(u)\nabla u\nabla v \xi dx. \end{aligned}$$

Thus

$$div(\sigma(u)\beta(u)\nabla v) = \sigma(u)\beta'(u)\nabla u \nabla v \tag{2.29}$$

in the sense of distributions. Use this in (2.7) to obtain the theorem.

REMARK. In fact, we only need to assume that β is bounded. Then we can always select a number c large enough so that

$$0 < m \le c + \beta \le M.$$

Also, if we know that u is bounded a *priori*, then there is no need to assume that Θ, σ, β are bounded above. In this sense, our hypotheses are much weaker than those in [1]. However, in the generality considered here it does not seem likely that u can be bounded.

Now we are ready to prove an existence assertion for the following problem:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} - div(\Theta(u)\nabla u) + \sigma(u)\beta'(u)\nabla u\nabla v = \sigma(u)|\nabla v|^2 \\ &div(\sigma(u)\nabla v) = 0 \end{aligned}\right\} \in Q_T \equiv \Omega x(0,T), \tag{2.30a}$$

$$u = 0, v = B \text{ on } S_T = \partial\Omega \times (0,T), \tag{2.30b}$$

$$u = U_0 \text{ on } \Omega \times \{0\}. \tag{2.30c}$$

THEOREM 2.3. Let $\Omega, \Theta, \sigma, \beta$ be given as before. Assume that $B \in L^2(0,T;W^{1,2}(\Omega)) \cap L^\infty(Q_T)$, and $U_0 \in W^{1,2}(\Omega)$. Then there exists a weak solution to (2.30), i.e., there is a pair (u,v) such that

$$u, v \in L^2(0,T;W^{1,2}(\Omega)), \tag{2.31}$$

$$u, v - B \in L^2(0,T;W_0^{1,2}(\Omega)), \tag{2.32}$$

$$-\int_{Q_T} u\xi_t dxdt + \int_{Q_T} \Theta(u) \nabla u \nabla \xi dxdt + \int_{Q_T} \sigma(u)\beta'(u) \nabla u \nabla v\xi dxdt \tag{2.33}$$

$$= \int_{Q_T} \sigma(u) \mid \nabla v \mid^2 \xi dxdt + \int_{\Omega} U_0(x)\xi(x,0)dx$$

for all $\xi \in H^1(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(Q_T)$ such that $\xi(x,T) \equiv 0$,

$$\int_{Q_T} \sigma(u) \nabla v \nabla \eta dxdt = 0 \text{ for all } \eta \in L^2(0,T;W_0^{1,2}(\Omega)). \tag{2.34}$$

PROOF. We shall follow the approach presented in [3] using a discretization technique. Let $n \in \{1,2,...\}$. Set $\delta = T/n$. For $k = 1,2,...,n$, denote by $B_n^{(k)}$ the integral $\frac{1}{\delta}\int_{(k-1)\delta}^{k\delta} B(x,\tau)d\tau$. Subsequently, we may generate a set of n pairs $(u_n^{(1)}, v_n^{(1)}),...,(u_n^{(n)}, v_n^{(n)})$ via the following iteration formula:

$$\frac{u_n^{(k)} - u_n^{(k-1)}}{\delta} - div\Big(\Theta(u_n^{(k)} \nabla u_n^{(k)}\Big) + \sigma(u_n^{(k)})\beta'(u_n^{(k)}) \nabla u_n^{(k)} \nabla v_n^{(k)}$$

$$= \sigma(u_n^{(k)}) \mid \nabla v_n^{(k)} \mid^2 \text{ in } \Omega, \tag{2.35}$$

$$div\Big(\sigma(u_n^{(k)}) \nabla v_n^{(k)}\Big) = 0 \text{ in } \Omega, \tag{2.36}$$

$$u_n^{(k)} = 0 \text{ on } \partial\Omega,$$

$$v_n^{(k)} = B_n^{(k)} \text{ on } \partial\Omega,$$

$$k = 1,2,...n,$$

where

$$u_n^{(0)} = U_0.$$

Define two function u_n, v_n by

$$u_n(x,t) = \begin{cases} U_0 & \text{if } t \le 0, \\ u_n^{(k)} & \text{if } (k-1)\delta < t \le k\delta (k = 1,...,n), \end{cases}$$

$$v_n(x,t) = \begin{cases} v_n^{(1)} \text{ if } t \le \delta, \\ v_n^{(k)} \text{ if } (k-1)\delta < t \le k\delta (k = 2,...,n). \end{cases}$$

We infer from (2.36) that

$$\sup_{(x,t) \in Q_T} \mid v_n(x,t) \mid \le c, \tag{2.37}$$

$$\| \nabla v_n \|_{2,Q_T} \le c (n = 1,2,...). \tag{2.38}$$

Let $L_j(s)$ be given as before. Note that

$$\int_{\Omega} \sigma(u_n^{(k)})\beta'(u_n^{(k)}) \nabla u_n^{(k)} \nabla v_n^{(k)} L_j(u_n^{(k)}) dx$$

$$= -\int_\Omega \sigma(u_n^{(k)}) \nabla v_n^{(k)} \beta(u_n^{(k)}) \nabla L_j(u_n^{(k)}) dx$$

$$\leq c \| \nabla v_n^{(k)} \|_2 \| \nabla u_n^{(k)} \|_2, \tag{2.39}$$

and that

$$\int_\Omega \sigma(u_n^{(k)}) | \nabla v_n^{(k)} |^2 L_j(u_n^{(k)}) dx$$

$$= -\int_\Omega \sigma(u_n^{(k)}) v_n^{(k)} \nabla v_n^{(k)} \nabla L_j(u_n^{(k)}) dx$$

$$\leq c \| \nabla v_n^{(k)} \|_2 \| \nabla u_n^{(k)} \|_2. \tag{2.40}$$

Multiply (2.35) by $L_j(u_n^{(k)})$ and use (2.39) and (2.40) in the resulting equation to obtain

$$\frac{1}{\delta}\int_\Omega (u_n^{(k)} - u_n^{(k-1)}) L_j(u_n^{(k)}) dx + \int_\Omega \Theta(u_n^{(k)}) \nabla u_n^{(k)} \nabla L_j(u_n^{(k)}) dx$$

$$\leq c \| \nabla v_n^{(k)} \|_2 \| \nabla u_n^{(k)} \|_2. \tag{2.41}$$

Observe that

$$(u_n^{(k)} - u_n^{(k-1)}) u_n^{(k)} \geq \tfrac{1}{2}(u_n^{(k)})^2 - \tfrac{1}{2}(u_n^{(k-1)})^2. \tag{2.42}$$

Send j to infinity in (2.41) and use (2.42) in the resulting equation to obtain

$$\frac{1}{\delta}\int_\Omega \left\{\tfrac{1}{2}(u_n^{(k)})^2 - \tfrac{1}{2}(u_n^{(k-1)})^2\right\} dx + \frac{m}{2}\int_\Omega | \nabla u_n^{(k)} |^2 dx \leq c \| \nabla v_n^{(k)} \|_2^2.$$

Pick up an ℓ from $\{1, \ldots, n\}$ and sum for $k = 1, \ldots, \ell$ to deduce

$$\frac{1}{2}\int_\Omega u_n^2(x, \ell\delta) dx + \frac{m}{2}\int_0^{\ell\delta}\int_\Omega | \nabla u_n |^2 dx dt$$

$$\leq c \int_0^{\ell\delta}\int_\Omega | \nabla v_n |^2 dx dt$$

$$+ \frac{1}{2}\int_\Omega U_0^2(x) dx \leq c_1. \tag{2.43}$$

Consequently,

$$\sup_{0 \leq t \leq T} \int_\Omega u_n^2(x,t) dx + \int_0^T \int_\Omega | \nabla u_n |^2 dx dt \leq c. \tag{2.44}$$

In view of (2.39) and (2.40), we may rewrite (2.35) to read

$$\frac{u_n^{(k)}-u_n^{(k-1)}}{\delta}-div\Big(\Theta(u_n^{(k)})\nabla u_n^{(k)}\Big)+div\Big(\sigma(u_n^{(k)})\beta(u_n^{(k)})\nabla v_n^{(k)}\Big) \tag{2.45}$$

$$=div\Big(\sigma(u_n^{(k)})v_n^{(k)}\nabla u_n^{(k)}\Big) \text{ in } \Omega.$$

For each n define $\overline{u_n}(x,t)$ by

$$\overline{u_n}(x,t)=\frac{t-(k-1)\delta}{\delta}u_n^{(k)}+\frac{k\delta-t}{\delta}u_n^{(k-1)} \text{ if } (k-1)\delta<t\le k\delta, k=1,2,...,n.$$

We deduce from (2.45) that

$$\frac{\partial}{\partial t}\overline{u_n}-div(\Theta(u_n)\nabla u_n)+div(\sigma(u_n)\beta(u_n)\nabla v_n)$$

$$=div(\sigma(u_n)v_n\nabla v_n) \tag{2.46}$$

$$\text{in } L^2(0,T;W^{-1,2}(\Omega)).$$

In view of (2.44), (2.37) and (2.38), we obtain that $\left\{\frac{\partial}{\partial t}\overline{u_n}\right\}$ is bounded in $L^2(0,T;W^{-1,2},(\Omega))$. This allows us to invoke Lions-Aubin's theorem to conclude that

$$\{\overline{u_n}\} \text{ is precompact in } L^2(Q_T). \tag{2.47}$$

Use $u_n^{(k)}-u_n^{(k-1)}$ as a test function in (2.45) to get

$$\frac{1}{\delta}\int_\Omega(u_n^{(k)}-u_n^{(k-1)})^2dx$$

$$=\int_\Omega\Big(-\Theta(u_n^{(k)})\nabla u_n^{(k)}+\sigma(u_n^{(k)})\beta(u_n^{(k)})\nabla v_n^{(k)}$$

$$-\sigma(u_n^{(k)})v_n^{(k)}\nabla v_n^{(k)}\Big)\nabla[u_n^{(k)}-u_n^{(k-1)}]dx$$

$$\le c(\|\nabla u_n^{(k)}\|_2+\|\nabla v_n^{(k)}\|_2)(\|\nabla u_n^{(k)}\|_2+\|\nabla u_n^{(k-1)}\|_2). \tag{2.48}$$

Note that

$$\|\nabla u_n^{(k)}\|_2=\frac{1}{\delta^{1/2}}\left(\int_{(k-1)\delta}^{k\delta}\int_\Omega|\nabla u_n|^2dxdt\right)^{1/2}\le\frac{c}{\delta^{1/2}}. \tag{2.49}$$

Similarly,

$$\|\nabla v_n^{(k)}\|_2\le\frac{c}{\delta^{1/2}}. \tag{2.50}$$

Use (2.49) and (2.50) in (2.48) and then sum for $k=1,...,n$ to derive

$$\int_0^T\int_\Omega(u_n(x,t)-u_n(x,t-\delta))^2dx\le c\delta^{1/2}.$$

On the other hand,

$$\int_{Q_T}(u_n-\overline{u_n})^2dxdt=\frac{1}{3}\int_0^T\int_\Omega(u_n(x,t)-u_n(x,t-\delta))^2dxdt$$

$$\leq c\delta^{1/2} \to 0 \text{ as } n \to \infty.$$

Thus $\{u_n\}$ is also precompact in $L^2(Q_T)$.

There exists a subsequence of $\{n\}$, still denoted by $\{n\}$, such that

$$u_n \to u \text{ strongly in } L^2(Q_T) \text{ and weakly in } L^2(0,T;W^{1,2}(\Omega)), \text{ and}$$
$$v_n \to v \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)).$$

To pass to the limit in (2.46), we still need to show that

$$v_n \to v \text{ strongly in } L^2(Q_T) \tag{2.51}$$

We infer from (2.36) that

$$\int_{Q_T} \sigma(u_n) \nabla v_n \nabla \xi dxdt = 0 \text{ for all } \xi \in L^2(0,T;W_0^{1,2}(\Omega)). \tag{2.52}$$

Let $B_n(x,t) = B_n^{(k)}(x)$ if $(k-1)\delta < t \leq k\delta, k = 1,2,\ldots,n$. It is easy to see that

$$B_n \to B \text{ strongly in } L^2(0,T;W^{1,2}(\Omega)). \tag{2.53}$$

Set $\xi = v_n - v - B_n + B$ in (2.52) to deduce

$$\int_{Q_T} \sigma(u_n) \mid \nabla (v_n - v) \mid^2 dxdt = \int_{Q_T} \sigma(u_n) \nabla v_n \nabla (B_n - B) dxdt$$
$$- \int_{Q_T} \sigma(u_n) \nabla v \nabla (v_n - v) dxdt$$
$$\to 0 \text{ as } n \to \infty.$$

Consequently, we have

$$v_n \to v \text{ strongly in } L^2(0,T;W^{1,2}(\Omega)).$$

Thus (2.51) follows. Now we can take $n \to \infty$ in (2.46) to get

$$\frac{\partial}{\partial t}u - div(\Theta(u)) + div(\sigma(u)\beta(u) \nabla v) = div(\sigma(u)v \nabla v)$$
$$\in L^2(0,T;W^{-1,2}(\Omega)). \tag{2.54}$$

Send n to infinity in (2.52) to get (2.34). Then it is easy to verify from (2.34) that

$$div(\sigma(u)\beta(u) \nabla v) = \sigma(u)\beta'(u) \nabla u \nabla v, \tag{2.55}$$
$$div(\sigma(u)v \nabla v) = \sigma(u) \mid \nabla v \mid^2 \tag{2.56}$$

in the sense of distributions. Use (2.55) and (2.56) in (2.54) to obtain (2.33). The proof is complete.

3. UNIQUENESS.

In this section, a uniqueness assertion is established for (1.1) in some special cases.

THEOREM 3.1. Let the assumptions of Theorem 2.3 hold. Assume that $\Theta(s) = s$ and that σ is Lipschitz continuous. Then there exists at most one solution to (1.1) in the space $L^2(0,T;W_0^{1,2}(\Omega)) \times L^\infty(0,T;W^{1,\infty}(\Omega))$.

PROOF. Suppose that there exist two solutions (u_1,v_1) and (u_2,v_2) to (1.1). First note that (2.54) is equivalent to (2.33) when (2.34) holds true. Set

$$\bar{u} = u_1 - u_2,\ \bar{v} = v_1 - v_2.$$

We derive from (2.54) that

$$\frac{\partial}{\partial t}\bar{u} - \Delta\bar{u} = div(\sigma(u_1)v_1 \nabla v_1 - \sigma(u_2)v_2 \nabla v_2) - div(\sigma(u_1)\beta(u_1) \nabla v_1 - \sigma(u_2)\beta(u_2) \nabla v_2)$$
$$\text{in } L^2(0,T;W^{-1,2}(\Omega)).$$

Thus,

$$\frac{1}{2}\|\bar{u}(x,t)\|_{2,\Omega}^2 + \int_0^t\int_\Omega |\nabla\bar{u}|^2 dx d\tau = \int_0^t\int_\Omega (\sigma(u_1)v_1 \nabla v_1 - \sigma(u_2)v_2 \nabla v_2)\nabla\bar{u}\, dx d\tau$$
$$+ \int_0^t\int_\Omega (\sigma(u_1)\beta(u_1) \nabla v_1 - \sigma(u_2)\beta(u_2) \nabla v_2)\nabla\bar{u}\, dx d\tau$$
$$\equiv I_1 + I_2. \tag{3.1}$$

Recall from our assumptions that $\nabla v_1, \nabla v_2 \in [L^\infty(Q_T)]^N$. I_1 and I_2 can be estimated as follows:

$$|I_1| \le \left|\int_0^t\int_\Omega (\sigma(u_1) - \sigma(u_2))v_1 \nabla v_1 \nabla\bar{u}\, dx d\tau\right| + \left|\int_0^t\int_\Omega \sigma(u_2)(v_1 - v_2)\nabla v_1 \nabla\bar{u}\, dx d\tau\right|$$
$$+ \left|\int_0^t\int_\Omega \sigma(u_2)v_2(\nabla v_1 - \nabla v_2)\nabla\bar{u}\, dx d\tau\right|$$
$$\le c\left\{\left(\int_0^t\int_\Omega (\sigma(u_1) - \sigma(u_2))^2 dx d\tau\right)^{1/2} + \left(\int_0^t\int_\Omega (v_1 - v_2)^2 dc d\tau\right)^{1/2}\right.$$
$$\left. + \left(\int_0^t\int_\Omega |\nabla v_1 - \nabla v_2|^2 dx dt\right)^{1/2}\right\}\left(\int_0^t\int_\Omega |\nabla\bar{u}|^2 dx d\tau\right)^{1/2}$$
$$\le c_1\left(\int_0^t\int_\Omega \bar{u}^2 dx d\tau + \int_0^t\int_\Omega \bar{v}^2 dx d\tau\right.$$
$$\left. + \int_0^t\int_\Omega |\nabla\bar{v}|^2 dx dt\right) + \frac{1}{4}\int_0^t\int_\Omega |\nabla\bar{u}|^2 dx d\tau.$$

Here we used the fact that σ is Lipschitz continuous. Similarly,

$$|I_2| \le \frac{1}{4}\int_0^t\int_\Omega |\nabla\bar{u}|^2 dx + c\left(\int_0^t\int_\Omega \bar{u}^2 dx d\tau\right.$$

$$+\int_0^t\int_\Omega |\nabla\bar{v}|^2 dxd\tau\Bigg).$$

Clearly,

$$\int_0^t\int_\Omega \sigma(u_i)\nabla v_i\nabla\xi dx = 0 \text{ for all } \xi\in L^2(0,t;W_0^{1,2}(\Omega))$$

for all $0 < t \le T$ and for $i = 1,2$.
Thus we obtain

$$\int_0^t\int_\Omega \sigma(u_2)|\nabla\bar{v}|^2 dxd\tau = -\int_0^t\int_\Omega (\sigma(u_1)-\sigma(u_2))\nabla v_1\nabla\bar{v}\,dxd\tau.$$

Consequently,

$$\int_0^t\int_\Omega |\nabla\bar{v}|^2 dxd\tau \le c\int_0^t\int_\Omega \bar{u}^2 dxd\tau. \tag{3.2}$$

By Poincaré's inequality,

$$\int_0^t\int_\Omega \bar{v}^2 dxd\tau \le c\int_0^t\int_\Omega |\nabla\bar{v}|^2 dxd\tau \tag{3.3}$$

$$\le c_1\int_0^t\int_\Omega \bar{u}^2 dxd\tau.$$

This immediately implies

$$\bar{u}\equiv 0.$$

By (3.3),

$$\bar{v}\equiv 0.$$

Thus $u_1\equiv u_2, v_1\equiv v_2$. This completes the proof.

The above theorem is not very satisfactory because it requires that ∇v be bounded, which cannot be guaranteed by the existence theorem. Thus it is interesting to investigate when ∇v becomes bounded. We summarize our results in the following theorem.

THEOREM 3.2. Let the hypothesis of Theorem 2.3 be satisfied. Assume

(i) $U_0\in C^{0,\lambda}(\bar{\Omega})$ for some $0<\lambda<1$;

(ii) $\frac{\partial}{\partial x_i}B\in L^\infty(0,T;C^{0,\lambda}(\bar{\Omega}))$ for each i;

(iii) $N=2$;

(iv) σ is Lipschitz continuous.

Then there is a $\lambda_1\in(0,1)$ such that $\frac{\partial}{\partial x_i}v\in L^\infty(0,T;C^{0,\lambda_1}(\bar{\Omega}))$ for $i=1,2$.

PROOF. Set

$$\psi = v - B.$$

Then for a.e. $t\in[0,T]$, we have

$$\int_\Omega \sigma(u(x,t))\nabla\psi(x,t)\nabla\xi(x)dx = -\int_\Omega \sigma(u(x,t))\nabla B(x,t)\nabla\xi(x)dx \tag{3.4}$$

for all $\xi\in W_0^{1,2}(\Omega)$. That is to say, we view (2.34) as a family of elliptic equations. Then for a.e. t in $(0,T)$ we appeal to a result due to Meyers [8, p. 36] to conclude that there is a positive number c depending only on m, M in (H5) and on Ω such that

$$\| \nabla \psi(x,t) \|_{p,\Omega} \leq c \| \nabla B(x,t) \|_{p,\Omega} \leq c_1$$

for some $p > 2$. Thus $v \in L^{\infty}(0,T;W^{1,p}(\Omega))$. Since u satisfies (2.54) and $p > N = 2$, we may invoke the classical regularity theory for linear parabolic equations [5, pp. 181-204] to get

$$u \in H^{\lambda, \lambda/2}(\overline{Q_T}) \text{ for some } \lambda > 0.$$

It immediately follows that $u \in L^{\infty}(0,T;C^{0,\lambda}(\overline{\Omega}))$. Recall that σ is Lipschitz continuous. Hence $\sigma(u(x,t)) \in C^{0,\lambda}(\overline{\Omega})$ for all $t \in [0,T]$. We are in a position to apply a result in [6, p. 210] to (3.4), thereby establishing

$$| \psi(\cdot,t) |_{1,\lambda} \leq c_1 + c_2 | \sigma(u(\cdot,t) \nabla B(\cdot,t) |_{0,\lambda}$$

$$\leq c \text{ for some } 1 > \lambda > 0 \text{ and for all } t \in [0,T].$$

The proof is complete.

Combining Theorems 3.1 and 3.2 yields the following:

THEOREM 3.3. Let the assumptions of Theorem 3.2 hold. Assume that $\Theta(s) = s$. Then there exists a unique solution to (1.1).

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REFERENCES

1. CIMATTI, G., Existence and uniqueness for the equations of the Joule-Thomson effect, Applicable Anal. 41 (1991), 131-144.
2. CIMATTI, G., Existence of weak solutions for the nonstationary problem of the Joule heating of a conductor, Ann. Mat. Pura Appl., (To appear).
3. SHI, P.; SHILLOR, M. & XU, X., Existence of a solution to the Stefan problem with Joule's heating, J. Differential Equations (To appear).
4. EVANS, L.C., Weak Convergence Methods for Nonlinear Partial Differential Equations, AMS, Providence, Rhode Island, 1990.
5. LADYZENSKAJA, O.A.; SOLONNIKOV, V.A. & URALCEVA, N.N., Linear and quasilinear equations of parabolic type, Tran. of Math. Monographs 23, AMS, Providence, Rhode Island, 1968.
6. GILBARG, D. & TRUDINGER, N.S., Elliptic Partial Differential Equations of Second Order, 2nd Ed., Springer-Verlag, Berlin, 1983.
7. ODEN, J.T., Qualitative Methods in Nonlinear Mechanics, Prentice Hall, Inc. Englewood Cliffs, New Jersey, 1986.
8. BENSOUSSAN, A.; LIONS, J.L. & PAPANICOLAOU, G., Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.