Internat. J. Math. & Math. Sci.
VOL. 16 NO. 1 (1993) 87-94

COINCIDENCE POINT THEOREMS FOR MULTIVALUED MAPPINGS

SHIH-SEN CHANG

Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064
People's Republic of China

YONG-CHENG PENG

Department of Mathematics
Shaoguan University
Shaoguan, Guangdong 512005
People's Republic of China

(Received December 17, 1990 and in revised form March 2, 1991)

ABSTRACT. Some new coincidence point and fixed point theorems for multivalued mappings in complete metric space are proved. The results presented in this paper enrich and extend the corresponding results in [5-16, 20-25, 29].

KEY WORDS AND PHRASES: Multivalued mapping, coincidence point, and fixed point.
1991 AMS SUBJECT CLASSIFICATION CODES. 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES.

In recent years, the existence and uniqueness of coincidence points and fixed points for commuting mappings, weakly commuting mappings and compatible mappings have been considered by several authors (see [2, 3, 6, 8, 17-28]). The purpose of this paper is to study the existence of coincidence points and fixed point for multivalued mappings in complete metric space from different aspects. The results presented in this paper enrich and extend the corresponding results in [5-16, 20-25, 29]. ,

Throughout this paper, let $R^+ = [0, +\infty)$ and (X, d) a complete metric space. For any nonempty subsets A and B of X, we denote

$$d(x,A) = \inf\{d(x,a):\ a \in A\}\ (x \in X),$$

$$d(A,B) = \inf\{d(a,b):\ a \in A,\ b \in B\},$$

$$H(A,B) = \max\left\{\sup_{a \in A} d(a,b),\ \sup_{b \in B} d(b,A)\right\},$$

$$CC(X) = \{A:\ A \text{ is a nonempty compact subset of } X\},$$

$$CB(X) = \{A:\ A \text{ is a nonempty closed and bounded subset of } X\}.$$

and $H(\cdot,\cdot)$ is called the Hausdorff metric on $CB(X)$.

LEMMA 1 [4, Lemma 2.2]. Let (X,d) be a metric space, $A \subset X$ a nonempty compact subset and $B \subset X$ a closed subset. If $d(A,B) = 0$, then $A \cap B \neq \varnothing$.

REMARK 1. Even if A and B are both bounded closed subsets, the conclusion of Lemma 1 need not hold. This can be seen from the following

EXAMPLE 1. Let $X = R^2$ and d the Euclidean metric on R^2. Letting

$$\rho(\cdot,\cdot) = \min\{1, d(\cdot,\cdot)\},$$

it is easy to verify that $\rho(\cdot,\cdot)$ is a metric on R^2. Therefore (R^2,ρ) is also a metric space, and it is bounded. Now we consider the following subsets of (R^2,ρ):

$$A = \left\{(x,y) \in R^2 \colon\ y = \frac{1}{x},\ x \geq \frac{1}{2}\right\}$$

$$B = \{(x,y) \in R^2 \colon\ y = 0\}$$

Then A and B are both bounded and closed and $d(A,B) = 0$, but $A \cap B = \varnothing$.

LEMMA 2 [5, Theorem 1]. Let $\Phi\colon R^+ \to R^+$ be an increasing function such that

$$\Phi(t+) < t \text{ for all } t > 0 \tag{1.1}$$

and

$$\sum \Phi^n(t) \text{ is finite for all } t > 0. \tag{1.2}$$

Then there exists a strictly increasing function $\phi\colon R^+ \to R^+$ such that

$$\Phi(t) < \phi(t) \text{ for all } t > 0 \tag{1.3}$$

and

$$\sum \phi^n(t) \text{ is finite for } t > 0. \tag{1.4}$$

LEMMA 3 [5]. (i) If $\Phi\colon R^+ \to R^+$ is strictly increasing and satisfies (1.2), then Φ satisfies (1.1).

(ii) Let $\Phi\colon R^+ \to R^+$ be increasing and satisfies (1.1). If $\Sigma\Phi^n(t_1)$ is convergent for some $t_1 > 0$. Then (1.2) holds.

(iii) Let $\Phi\colon R^+ \to R^+$ be increasing and satisfies (1.1). If $t \leq \Phi(t)$ then $t = 0$.

2. MAIN RESULTS

Recently, Kaneko and Sessa [6] extended the definition of compatibility to include multivalued mappings and proved the following theorem:

THEOREM 1. Let $f\colon X \to X$ and $T\colon X \to CB(X)$ be compatible continuous mapping such that $T(X) \subset f(X)$ and

$$H(Tx,Ty) \leq \max\left\{d(fx,fy),\ d(fx,Tx),\ d(fy,Ty),\ \frac{1}{2}(d(fx,Ty) + d(fy,Tx))\right\}$$

for all $x,\ y$ in X, where $0 \leq h < 1$. Then there exists a point $x_* \in X$ such that $fx_* \in Tx_*$.

As an improvement and generalization of Theorem 1, we have the following

THEOREM 2. Let $F\colon X \to CC(X)$, $S,\ T\colon X \to CB(X)$ be three multivalued mappings such that $S(X) \cup T(X) \subset F(X)$, $F(X)$ is closed and

$$H(Sx,Ty) \leq \Phi\left(\max\left\{d(Fx,Fy),\ d(Fx,Sx),\ d(Fy,Ty),\ \frac{1}{2}(d(Fx,Ty) + d(Fy,Sx))\right\}\right)$$

for all x, y in X, where Φ: $R^+ \to R^+$ is an increasing function satisfying conditions (1.1) and (1.2). Then there exists a point $z \in X$ such that

$$Fz \cap Sz \cap Tz \neq \varnothing.$$

PROOF. By Lemma 2, there exists a strictly increasing function ϕ: $R^+ \to R^+$ satisfying conditions (1.3) and (1.4). For any x, y in X let us denote

$$A(x,y) = \max\left\{d(Fx,Fy),\ d(Fx,Sx),\ d(Fy,Ty),\ \frac{1}{2}(d(Fx,Ty)+d(Fy,Sx))\right\}.$$

Then (2.1) can be reduced as follows:

$$H(Sx,Ty) \le \Phi(A(x,y)).$$

For any $x_0 \in X$, since $S(X) \subset F(X)$, there exists an $x_1 \in X$ such that $Tx_1 \cap Sx_0 \neq \varnothing$. Let $y_1 \in Fx_1 \cap Sx_0$, then we have

$$\begin{aligned} d(y_1, Tx_1) &\le H(Sx_0, Tx_1) \ (\text{since } y_1 \in Sx_0) \\ &\le \Phi(A(x_0,x_1)) \end{aligned}$$

(a) If $A(x_0,x_1) = 0$, then $d(Fx_0,Sx_0) = 0$. By Lemma 1, $Fx_0 \cap Sx_0 \neq \varnothing$. Taking $z \in Fx_0 \cap Sx_0$, then we have

$$\begin{aligned} H(z,Tx_0) &\le H(Sx_0,Tx_0) \le \Phi(A(x_0,x_0)) \\ &\le \Phi\left(\max\left\{0,\ 0,\ d(z,Tx_0),\ \frac{1}{2}d(z,Tx_0)\right\}\right) \\ &\le \Phi(d(z,Tx_0)). \end{aligned}$$

By Lemma 3 (iii) $d(z,Tx_0) = 0$. Since Tx_0 is closed, $z \in Tx_0$. Therefore in this case the conclusion of Theorem 2 is proved.

(b) If $A(x_0,x_1) > 0$, then, by (1.3) we have

$$d(y_1,Tx_1) \le \Phi(A(x_0,x_1)) < \phi(A(x_0,x_1))$$

Consequently, we can find an $y_2 \in Tx_1$ such that

$$d(y_1,y_2) \le \phi(A(x_0,x_1)). \tag{2.2}$$

Since $T(X) \subset F(X)$, for $y_2 \in Tx_1 \subset F(X)$, there exists a point $x_2 \in X$ such that $y_2 \in Fx_2$. This implies that we can find an $y_2 \in Fx_2 \cap Tx_1$ such that (2.2) holds.

On the other hand, by the assumption we have

$$d(Sx_2, y_2) \le H(Sx_2, Tx_1) \le \Phi(A(x_2,x_1)).$$

If $A(x_2,x_1) = 0$, by the same way as stated in the proof of (a) we can prove that the conclusion of Theorem 2 is true. If $A(x_2,x_2) > 0$, repeating the same way mentioned above, we can find an $x_3 \in X$ and $y_3 \in Fx_3 \cap Sx_2$ such that

$$d(y_3,y_2) \le \phi(A(x_2,x_1)).$$

Inductively, we can define two sequence $\{x_n\}, \{y_n\} \subset X$ such that

$$\begin{cases} y_{2n+1} \in Fx_{2n+1} \cap Sx_{2n} \\ y_{2n+2} \in Fx_{2n+2} \cap Tx_{2n+1} \end{cases} \quad n = 0,1,2,\ldots \tag{2.3}$$

and

$$\begin{cases} d(y_{2n+1}, y_{2n+2}) \le \phi(A(x_{2n}, x_{2n+1})) \\ d(y_{2n+3}, y_{2n+2}) \le \phi(A(x_{2n+2}, x_{2n+1})) \end{cases} \qquad n = 0, 1, 2, \ldots \tag{2.4}$$

Now we prove that $\{y_n\}$ is a Cauchy sequence in X. In fact, for any positive integer n we have

$$\begin{aligned} A(x_{2n}, x_{2n+1}) &= \max\{d(Fx_{2n}, Fx_{2n+1}), d(Fx_{2n}, Sx_{2n}), d(Fx_{2n+1}, Tx_{2n+1}), \\ &\qquad \tfrac{1}{2}(d(Fx_{2n}, Tx_{2n+1}) + d(Fx_{2n+1}, Sx_{2n}))\} \\ &\le \max\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \\ &\qquad \tfrac{1}{2}(d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1}))\} \\ &\le \max\{d(y_{2n}, y_{2n+1}, d(y_{2n+1}, y_{2n+2})\}. \end{aligned}$$

By the same way we can prove that

$$A(x_{2n+2}, x_{2n+1}) \le \max\{d(y_{2n+1}, y_{2n+2}), d(y_{2n+2}, y_{2n+3})\}.$$

Consequently, in general, we have

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\le \phi(A)(x_n, x_{n+1})) \\ &\le \phi(\max\{d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2})\}), \qquad n = 1, 2, \ldots \end{aligned} \tag{2.5}$$

If $d(y_{n+1}, y_{n+2}) > d(y_n, y_{n+1}) \ge 0$, then, by (2.5) and Lemma 3 we have

$$d(y_{n+1}, y_{n+2}) \le \phi(d(y_{n+1}, y_{n+2})) < d(y_{n+1}, y_{n+2})$$

a contradiction. Therefore we have $d(y_{n+1}, y_{n+2}) \le d(y_n, y_{n+1})$. Hence we have

$$\begin{aligned} d(y_n, y_{n+1}) &\le \phi(d(y_{n-1}, y_n)) \le \ldots \\ &\le \phi^{n-1} d(y_1, y_2)), \qquad n = 1, 2, \ldots \end{aligned} \tag{2.6}$$

If $d(y_1, y_2) = 0$, i.e. $y_1 = y_2$, denoting $z = y_1 = y_2$, then $z = y_1 \in Fx_1 \cap Sx_0$, $z = y_2 \in Fx_2 \cap Tx_1$. Hence $z \in Fx_1 \cap Tx_1$. Similarly using the proof in (a) we can prove $z \in Sx_1$. Hence the conclusion of Theorem 2 is proved.

If $d(y_1, y_2) > 0$, in view of condition (1.4), we know that $\Sigma\phi^{n-1}d(y_1, y_2))$ is convergent. It follows from (2.6) that $\Sigma d(y_n, y_{n+1})$ is convergent too. This implies that $\{y_n\}$ is a Cauchy sequence in X. Let it converge to some point y_* in X. Since $y_n \in Fx_n \subset F(X)$ and $F(X)$ is closed, this shows that $y_* \in F(X)$. Hence there exists $z \in X$ such that $y_* \in Fz$. By (2.1) and (2.3) we have

$$\begin{aligned} d(y_*, Sz) &\le d(y_*, y_{2n+2}) + d(y_{2n+2}, Sz) \\ &\le d(y_*, y_{2n+2}) + H(Tx_{2n+1}, Sz) \qquad (\text{since } y_{2n+2} \in Tx_{2n+1}) \\ &\le d(y_*, y_{2n+2}) + \Phi(A(z, x_{2n+1})) \\ &\le d(y_*, y_{2n+2}) + \Phi(\max\{d(Fz, Fx_{2n+1}), d(Fz, Sz), \\ &\qquad d(Fx_{2n+1}, Tx_{2n+1}), \tfrac{1}{2}(d(Fz, Tx_{2n+1}) + d(Fx_{2n+1}, Sz))\}) \\ &\le d(y_*, y_{2n+2}) + \Phi(\max\{d(d(y_*, y_{2n+1}), d(y_*, Sz), \\ &\qquad d(y_{2n+1}, y_{2n+2}), \tfrac{1}{2}(d(y_*, y_{2n+2}) + d(y_{2n+1}, Sz))\}). \end{aligned}$$

Letting $n \to \infty$, we have

$$d(y_*,Sz) \le \Phi\max\left\{0, d(y_*,Sz),\ 0,\ \frac{1}{2}d(y_*,Sz)\right\} \le \Phi d(y_*,Sz)).$$

By Lemma 3 (iii) we have $d(y_*,Sz) = 0$. Since Sz is closed, so that $y_* \in Sz$.

Similarly, we can prove that $y_* \in Tz$. Therefore we have $y_* \in Fz \cap Sz \cap Tz$.

This completes the proof.

REMARK 2. (i) Theorem 1 is a special case of Theorem 2 with F being a single-valued mapping, $S=T$ and $\Phi(t) = h \cdot t$, where $0 \le h < 1$ and $t \in R^+$.

(ii) Even if the mapping F in Theorem 2 is assumed to satisfy the condition "$F(X)$ is closed", Theorem 2 still weakens the continuity and compatibility conditions on T in Theorem 1. This can be seen from the following Example:

EXAMPLE 2. Let $X = R^+$ and f and g be two functions from R^+ into R^+ defined by

$$f(x) = \begin{cases} x, & \text{if } x < 1, \\ 1, & \text{if } x \ge 1, \end{cases} \qquad g(x) = x(x+1)^{-1}.$$

It is easy to see that $f(X)$ is closed, f and g are continuous, but they are not compatible (see [8, Example 2.5]).

(iii) Theorem 2 extends and improves also the corresponding results of [7, 8, 20-25].

As a consequence of Theorem 2 we have the following result:

COROLLARY 1. Let T_i: $X \to CB(X)(i = 1,2,\dots)$ and

$$H(T_ix,T_jy) \le \Phi(\max\left\{d(x,y),\ d(x,T_ix),\ d(y,T_jy),\ \frac{1}{2}(d(x,T_jy) + d(y,T_ix))\right\}), \qquad i \ne j \tag{2.7}$$

for all x, y in X, where Φ: $R^+ \to R^+$ is an increasing function satisfying conditions (1.1) and (1.2). Then the fixed point set $\{x\colon x \in T_ix\}$, $i = 1,2,\dots$ are nonempty and equal to each other. Moreover, if at least one of $\{T_i\}$ is continuous, then they are all closed.

PROOF. For the sake of convenience we prove the conclusions of Corollary only for the case of i=1 and j=2.

By Theorem 2, there exists an $z \in X$ such that $z \in T_1z \cap T_2z$.

Now we prove that the fixed point sets of T_1 and T_2 are equal to each other. In fact, if u is a fixed point of T_1, i.e. $u \in T_1u$, then we have

$$\begin{aligned} d(u,T_2u) &\le H(T_1u,T_2u) \\ &\le \Phi(\max\left\{d(u,u),\ d(u,T_1u),\ d(u,T_2u),\ \frac{1}{2}(d(u,T_2u) + (d(u,T_1u))\}\right) \\ &\le \Phi\left(\max\left\{0,\ 0,\ d(u,T_2u),\ \frac{1}{2}d(u,T_2u)\right\}\right) \\ &\le \Phi(d(u,T_2u)). \end{aligned}$$

By Lemma 3 (iii), we have $d(u,T_2u) = 0$. Since T_2u is closed, $u \in T_2u$.

By the same way we can prove that if w is a fixed point of T_2 then w is also a fixed point of T_1. Hence we have $\{x \in X\colon x \in T_1x\} = \{x \in X\colon x \in T_2x\}$.

Next, we prove that if T_1 (or T_2) is continuous, then the set of fixed points $\{x \in X: x \in T_1x\}$ is a closed set. In fact, let $\{x_n\} \subset \{x \in X: x \in T_1x\}$ and $x_n \to x$ as $n \to \infty$. Since $x_n \in T_1x_n$ and $T_1x_n \to T_1x$ as $n \to \infty$, we have

$$\begin{aligned} d(x,T_1x) &\le d(x,x_n) + d(x_n,T_1x) \\ &\le d(x,x_n) + H(T_1x_n,T_1x) \to 0 \text{ as } n \to \infty, \end{aligned}$$

i.e. $d(x,T_1x) = 0$. Therefore $x \in T_1x$.

This completes the proof.

REMARK 3. If all the mapping T_i, $i = 1,2,\ldots$ in Corollary 1 are single-valued, then T_i, $i = 1,2,\ldots$ have a unique common fixed point in X.

In fact, let u, $v \in X$ be two common fixed points of T_i, $i = 1,2,\ldots$, then we have

$$\begin{aligned} d(u,v) &= d(T_iu,T_jv) \\ &\le \Phi(\max\{d(u,v),\ d(u,T_iu),\ d(v,T_jv), \\ &\qquad \frac{1}{2}(d(u,T_jv) + d(v,T_iu))\}) \\ &= \Phi(\max\{d(u,v),\ 0,\ 0,\ d(u,v)\}) \\ &\le \Phi(d(u,v)), \text{ for all } i,\ j,\ i \ne j. \end{aligned}$$

Hence we have $d(u,v) = 0$, i.e. $u = v$.

REMARK 4. The results of [5, Theorem 9], [9, 10, 11, 12, 13, Theorem 1], [14, Theorem] and [15, Theorem 1, 3, 4] are all the special cases of Corollary.

DEFINITION. A function $\Psi(t_1,t_2,t_3,t_4,t_5)$: $R^{+5} \to R^+$ is called to satisfy the condition (Ψ), if it is nondecreasing in each variable and there exists an increasing function $\Phi(t)$: $R^+ \to R^+$ satisfying the conditions (1.1) and (1.2) such that

$$\Psi(t,t,t,at,bt) \le \Phi(t), \qquad \forall t \ge 0,\ a+b = 3,\ a,\ b = 1,2.$$

THEOREM 3. Let F: $X \to CC(X)$, S, T: $X \to CB(X)$ be three multivalued mappings such that $S(X) \cup T(X) \subset F(X)$, $F(X)$ is closed and satisfies the following conditions:

$$H(Sx,Ty) \le \Psi(d(Fx,Fy),\ d(Fx,Sx),\ d(Fy,Ty),\ d(Fx,Ty),\ d(Fy,Sx)) \tag{2.8}$$

for all x, y in X, where $\Psi(t_1,t_2,t_3,t_4,t_5)$: $R^{+5} \to R^+$ satisfies condition (Ψ). Then there exists a point $z \in X$ such that $Fz \cap Sz \cap Tz \ne \emptyset$.

PROOF. Let

$$t^* = \max\left\{d(Fx,Fy),\ d(Fx,Sx),\ d(Fy,Ty),\ \frac{1}{2}(d(Fx,Ty) + d(Fy,Sx))\right\}.$$

Without loss of generality we can assume that $d(Fx,Ty) \ge d(Fy,Sx)$ (otherwise, it can be proved similarly). Then we have

$$t^* \ge \max\{d(Fx,Fy),\ d(Fx,Sx),\ d(Fy,Ty)\},$$

$$t^* \ge \frac{1}{2}(d(Fx,Ty) + d(Fy,Sx)) \ge d(Fy,Sx)$$

$$2t^* \ge d(Fx,Ty) + d(Fy,Sx) \ge d(Fx,Ty)$$

Using condition (Ψ) and (2.8) we have

$$H(Sx,Ty) \le \Psi(t^*,t^*,t^*,2t^*,t^*) \le \Phi(t^*)$$

$$= \Phi\left(\text{Max}\left\{d(Fx,Fy),\ d(Fx,Sx),\ d(Fy,Ty),\ \frac{1}{2}(d(Fx,Ty)+d(Fy,Sx))\right\}\right).$$

Therefore, F, S, T satisfies all conditions of Theorem 2. The conclusion of Theorem 3 follows from Theorem 2 immediately.

From Theorem 3 we can obtain the following

COROLLARY 2. Let T_i: $X \to CB(X)$, $i = 1,2,\dots$ satisfy the following condition

$$H(T_i x, T_j y) \le \Psi(d(x,y),\ d(x,T_ix),\ d(y,T_jy),\ d(x,T_jy),\ d(y,T_ix)),\ \forall i,\ j,\ i \ne j$$

for all x, y in X, where $\Psi(t_1,t_2,t_3,t_4,t_5)$: $R^{+5} \to R^+$ satisfies condition (Ψ). Then the fixed point sets $\{x \in X\colon\ x \in T_ix\}$, $i = 1,2,\dots$ are nonempty and equal to each other. Moreover, if one of T_i, $i = 1,2,\dots$ is continuous, then they are closed.

REMARK 5. Corollary 2 generalizes the corresponding result of [29].

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