Internat. J. Math. & Math. Sci.
VOL. 16 NO. 2 (1993) 403-404

COINCIDENCE POINTS IN UNIFORM SPACES

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(Received April 13, 1992 and in revised form May 30, 1992)

ABSTRACT. In this note, we give a coincidence point theorem in sequentially complete Hausdorff uniform spaces. Our result reduces to a result of Acharya [1].

KEY WORDS AND PHRASES. Uniformity, sequentially Hausdorff uniform spaces, fixed points, and coincidence points.
1991 AMS SUBJECT CLASSIFICATION CODE. 54H25.

1. INTRODUCTION.

Let $(X,\mathfrak{U})$ be a uniform space and let $\mathbb{N}$ denote the set of all natural numbers. If p is a pseudo-metric on X, we let

$$V_{(p,r)}=\{(x,y)\in X\times X: p(x,y)<r\}(r>0).$$

For a family P of pseudo-metrics on X generating the uniformity $\mathfrak{U}$,

$$\mathcal{G}=\{\bigcap_{i=1}^{n} V_{(p_i,r_i)}: p_i\in P, r_i>0, n\in\mathbb{N}\}$$

is a base of $\mathfrak{U}$.

If $V=\bigcap_{i=1}^{n} V_{(p_i,r_i)}$ and $k>0$, define $kV=\bigcap_{i=1}^{n} V_{(p_i,kr_i)}$. If $Y\subset X$, then $\mathfrak{U}\,|\,Y=\{U^Y: U\in\mathfrak{U}\}$ is a uniformity on Y, where $U^Y=U\cap(Y\times Y)$. Moreover, $\mathcal{G}\,|\,Y=\{V^Y: V\in\mathcal{G}\}$ is a base for $\mathfrak{U}\,|\,Y$. For more details, see [1] and [2]. We need the following lemma and a theorem of Acharya ([1]) for our main theorem:

LEMMA 1.1. Let $Y\subset X, V\in\mathcal{G}$ and $\alpha,\beta>0$. Then we have the following :

(1) $\alpha(\beta V)=(\alpha\beta)V$.

(2) $\alpha V\circ\beta V\subset(\alpha+\beta)V$.

(3) $\alpha V\subset\beta V$ if $\alpha<\beta$.

(4) $(x,y)\in\alpha V_{(p,r_1)}\circ\beta V(p,r_2)$ implies $p(x,y)<\alpha r_1+\beta r_2$.

(5) $\alpha V^Y=(\alpha V)^Y$.

Recall also that for each $V \in \mathcal{G}$ there exists a pseudo-metric p (called the Minkowski pseudo-metric of V such that $V = V_{(p,1)}$.

THEOREM 1.1. Let $(X, \mathcal{U})$ be a sequentially complete Hausdorff uniform space and let $f: X \to X$ be a mapping. Let $0 < \alpha < 1$. If for every $V \in \mathcal{G}$ and $x, y \in X$,

$$(fx, fy) \in \alpha V \text{ if } (x,y) \in V, \tag{1.1}$$

then f has a unique fixed point in X.

2. MAIN RESULTS.

Now, by using Lemma 1.1 and Theorem 1.1 we give our main theorem:

THEOREM 2.1. Let $(X, \mathcal{U})$ be a uniform space. Let A be a set in $X, S, T: A \to X$ be mappings and $S(A) \subset T(A) = Y$. If $(Y, \mathcal{U} \mid Y)$ is a sequentially complete Hausdorff uniform space, $0 < \alpha < 1$, and

$$(Sx, Sy) \in \alpha V \qquad \text{if } (Tx, Ty) \in V \tag{2.1}$$

for every $V \in \mathcal{G}$ and $x, y \in A$, then there exists a point $u \in A$ such that $Tu = Su$.

PROOF. As in [3], define $fa = S(T^{-1}\{a\})$ for $a \in Y$. Then f is a well-defined self-mapping of Y. For if $b_1, b_2 \in fa$, there exist $x_1, x_2 \in T^{-1}\{a\}$ such that $b_1 = Sx_1$ and $b_2 = Sx_2$. By (2.1), for any $V \in \mathcal{G}$ we have $(b_1, b_2) = (Sx_1, Sx_2) \in \alpha V$ if $(Tx_1, Tx_2) \in V$, and since $(Tx_1, Tx_2) = (a,a) \in V$, it follows that $(b_1, b_2) \in \alpha V \subset V$. Since $S(A) = Y$, we have $(b_1, b_2) \in V^Y$ for each $V \in \mathcal{G}$. Thus, since $(Y, \mathcal{U} \mid Y)$ is Hausdorff, it follows that $b_1 = b_2$.

Now, let $a, b \in Y$, $x \in T^{-1}\{a\}$ and $y \in T^{-1}\{b\}$. By (2.1), $(a,b) = (Tx, Ty) \in V$ implies $(fa, fb) = (Sx, Sy) \in \alpha V$ for every $V \in \mathcal{G}$. Hence, by Lemma 1.1(5), for each $a, b \in Y$ and $V \in \mathcal{G}, (a,b) \in V^Y$ implies $(fa, fb) \in (\alpha V)^Y = \alpha V^Y$. Thus the mapping $f: Y \to V$ satisfies all the conditions of Theorem 2.1. Consequently, there exists a unique point $a \in Y$ with $fa = a$. Therefore, for each $u \in T^{-1}\{a\}$, we have $Su = S(T^{-1}\{a\}) = fa = a = Tu$, that is, u is a coincidence point of S and T. This completes the proof.

REMARK. If we put $A = X$ and $T = id_x$ (: the identity mapping on X), then Theorem 2.1 reduces to Theorem 1.1.

REFERENCES

1. ACHARYA, S.P. Some results on fixed points in uniform spaces, Yokohama Math. J. 22 (1974), 105-116.
2. CSASZAR, A. General Topology, Akademiai Kiado, Budapest, 1978.
3. GOEBEL, K. A coincidence theorem, Bull. Acad. Pol. Math. 16 (1968), 733-735.