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RESEARCH NOTES

A REMARK ON RHOADES FIXED POINT THEOREM FOR NON-SELF MAPPINGS

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ABSTRACT. Let X be a Banach space, K a non-empty closed subset of X and $T: K \to X$ a mapping satisfying the contractive definition (1.1) below and the condition $T(\partial K) \subseteq K$. Then T has a unique fixed point in K. This result improves Theorem of Rhoades [1] and generalizes the corresponding theorem of Assad [2].

KEY WORDS AND PHRASES. Fixed point, Cauchy sequence, complete metric space.
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1. INTRODUCTION.

Let X be a Banach space and K a closed subset of X. In many applications the domain of a considered function is K, but the codomen is not entirely included in K. So it is of interest to amplify a class of such mappings which have a fixed point. Rhoades [1] introduced a class of non-self mappings T of K into X which satisfy the following contractive definition

$$d(Tx, Ty) \leq h \cdot max\{d(x,y)/2, d(x,Tx), d(y,Ty), [d(x,Ty) + d(y,Tx)]/q\}, \tag{1.1}$$

where h and q are reals satisfying $0 < h < 1, q \geq 1 + 2h$. Rhoades proved that if $T(\partial K) \subseteq K$, then T has a unique fixed point. As pointed out by Rhoades [1, p. 459], the method of proof used in his Theorem 1 does not extend to more general contractive definitions.

The purpose of this note is to extend the result of Rhoades [1] to a class of non-self mappings of K which satisfy the following contractive definition:
There exists a constant $h, 0 < h < 1$, such that for each $x, y \in K$,

$$d(Tx, Ty) \leq h \cdot max\{d(x,y)/a, d(x,Tx), d(y,Ty), [d(x,Ty) + d(y,Tx)]/(a+h)\}, \tag{1.2}$$

where a is a real number satisfying $a \geq 1 + (2h^2)/(1+h)$.

Note that if T satisfies the condition (1.1) then T satisfies condition (1.2) with $a = 1 + h$.

Using a new method of proof we proved the result which is an improvement of the Theorem of Rhoades [1] and generalization of the Theorem of Assad [2].

2. MAIN RESULT.

In this paper we shall use the fact that, if $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$, the boundary of K, such that $d(x,z) + d(z,y) = d(x,y)$.

THEOREM 2.1. Let X be a Banach space, K a non-empty closed subset of X and $T: K \to X$ a mapping satisfying (1.2) on K and such that $T(\partial K) \subseteq K$. Then T has a unique fixed point in K at which T is continuous.

PROOF. Let $x_0 \in K$ be arbitrary point. Define two sequences $\{x_n\}$ and $\{x'_n\}$ satisfying the following conditions:

(i) $x'_{n+1} = Tx_n$,

(ii) $x_n = x'_n$ if $x'_n \in K$,

(iii) $x_n \in \partial K$ and $d(x_{n-1}, x_n) + d(x_n, x'_n) = d(x'_{n-1}, x'_n)$ if $x'_n \notin K$.

Let $P = \{x_k \in \{x_n\}: x_k = x'_k\}$ and $Q = \{x_k \in \{x_n\}: x_k \neq x'_k\}$. Note that $\{x_n\} \subseteq K$ and if $x_n \in Q$, then x_{n-1} and x_{n+1} belong to P, since by $T(\partial K) \subseteq K$ we cannot have two consecutive points of $\{x_n\}$ in Q.

From (1.2) it is easy to obtain $d(x'_n, x'_{n+1}) \leq h \cdot d(x_{n-1}, x_n)$.

We shall estimate $d(x_n, x_{n+1})$. Actually, we have three cases to consider:

Case I: x_n, $x_{n+1} \in P$,

Case II: $x_n \in P$, $x_{n+1} \in Q$,

Case III: $x_n \in Q$, $x_{n+1} \in P$.

It is easily seen that Cases I and II imply

$$d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n). \tag{2.1}$$

Case III. $x_n \in Q$, $x_{n+1} \in P$. We shall show that

$$d(x_n, x_{n+1}) \leq h \cdot d(x_{n-2}, x_{n-1}). \tag{2.2}$$

If $d(x_n, x_{n+1}) \leq d(x_{n-1}, x'_n)$, then (2.2) holds by Case II, since $x_n \in Q$ implies $x_{n-1} \in P$. Assume now that

$$d(x_n, x_{n+1}) > d(x_{n-1}, x'_n). \tag{2.3}$$

Since $x_n \in Q$ and is a convex linear combination of x_{n-1} and x'_n it follows

$$d(x_n, x_{n+1}) \leq max\{d(x'_n, x_{n+1}),\ d(x_{n-1}, x_{n+1})\}. \tag{2.4}$$

Using (1.1) and (2.3) we obtain

$$d(x'_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq h \cdot max\{d(x_{n-1}, x_n)/a, d(x_n, x_{n+1}), 2d(x_n, x_{n+1})/(a+h)\}$$

and hence, as $a \geq 1$,

$$d(x'_n, x_{n+1}) \leq 2h \cdot d(x_n, x_{n+1})/(1+h). \tag{2.5}$$

Then from (2.4) and (2.5) we get

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_{n+1}). \tag{2.6}$$

Using again (1.1), (2.3) and the triangle inequality we have

$$\begin{aligned} d(x_{n-1}, x_{n+1}) = d(Tx_{n-2}, Tx_n) \leq h\ max\{&[d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n)]/a, \\ &d(x_{n-2}, x_{n-1}),\ d(x_n, x_{n+1}), [d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)]/(a+h)\}. \end{aligned} \tag{2.7}$$

Suppose that (2.7) implies

$$d(x_{n-1}, x_{n+1}) \leq h \cdot [d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)]/(a+h).$$

Then we have

$$d(x_{n-1},x_{n+1}) \leq h \cdot [d(x_{n-2},x_{n-1}) + d(x_{n-1},x_n)]/a.$$

Since by (2.6) $d(x_{n-1},x_{n+1}) \leq h \cdot d(x_{n-2},x_{n-1})$ immediately implies (2.2), and $d(x_{n-1},x_{n+1}) \leq h \cdot d(x_n,x_{n+1})$ is in contradiction with (2.6), we may suppose that (2.7) implies

$$d(x_{n-1},x_{n+1}) \leq h \cdot [d(x_{n-2},x_{n-1}) + d(x_{n-1},x_n)]/a. \tag{2.8}$$

Assume now that $d(x_{n-1},x_{n+1}) > [2h/(1+h)] \cdot d(x_{n-1},x'_n)$. Then by (2.1) we get

$$d(x_n,x'_n) \leq [1 - 2h/(1+h)] \cdot d(x_{n-1},x'_n) \leq [(1-h)/(1+h)] \cdot h \cdot d(x_{n-2},x_{n-1})$$

and so by the triangle inequality and (2.5) we have

$$\begin{aligned} d(x_n,x_{n+1}) &\leq d(x_n,x'_n) + d(x'_n,x_{n+1}) \\ &\leq [h(1-h)/(1+h)] \cdot d(x_{n-2},x_{n-1}) + [2h/(1+h)] \cdot d(x_n,x_{n+1}). \end{aligned}$$

Hence it follows that $d(x_n,x_{n+1}) \leq h \cdot d(x_{n-2},x_{n-1})$, i.e., the relation (2.2).

Assume now that $d(x_{n-1},x_n) \leq [2h/(1+h)] \cdot d(x_{n-1},x'_n)$. Then from (2.1) and (2.8) we get

$$d(x_{n-1},x_{n+1}) \leq h \cdot [1 + 2h^2/(1+h)] \cdot d(x_{n-2},x_{n-1})/a \leq h \cdot d(x_{n-2},x_{n-1}),$$

since by hypothesis $a \geq 1 + 2h^2/(1+h)]$. So we proved (2.2).

By (2.1) and (2.2) we conclude that in all cases

$$d(x_n,x_{n+1}) \leq h \cdot max\{d(x_{n-2},x_{n-1}),\ d(x_{n-1},x_n)\}.$$

Now it is easily shown by induction that, for $n \geq 2$,

$$d(x_n,x_{n+1}) \leq h^{(n/2)} max\{d(x_0,x_1), d(x_1,x_2)\},$$

where $(n/2)$ is the greatest integer not exceeding $n/2$. Hence for $m > n > N$,

$$d(x_n,x_m) \leq \textstyle\sum_{i=N}^{\infty} d(x_i,x_{i+1}) \leq [2h^{(N/2)}/(1-h)] \cdot max\{d(x_0,x_1), d(x_1,x_2)\},$$

so that $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\} \subseteq K$ and K is closed, $\{x_n\}$ converges to some point $p \in X$.

Let $\{x_{n(k)+1}\} \subseteq P$ be an infinite subsequence of $\{x_n\}$. From (1.2)

$$\begin{aligned} d(p,Tp) \leq d(p,Tx_{n(k)}) + d(Tx_{n(k)},Tp) \leq (p,Tx_{n(k)}) + h \cdot max\{d(x_{n(k)},p)/a, \\ d(x_{n(k)},x_{n(k+1)}), d(p,Tp), [d(x_{n(k)},Tp) + d(p,x_{n(k)+1})]/(a+h)\}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ yields $d(p,Tp) \leq h \cdot d(p,Tp)$. Hence $Tp = p$. Condition (1.2) implies uniqueness.

Mappings which satisfy (1.2) may be discontinuous but at a fixed point they are continuous. For if $y_n \rightarrow p = Tp$, then from (1.2) we have

$$\begin{aligned} d(Ty_n,p) &= d(Ty_n,Tp) \\ &\leq h \cdot max\{d(y_n,p)/a, [d(y_n,p) + d(p,Ty_n), [d(y_n,p) + d(p,Ty_n)]/(a+h). \end{aligned}$$

Hence $lim\ sup\ d(Ty_n,p) \leq h \cdot lim\ sup\ d(p,Ty_n)$. Hence $Ty_n \rightarrow p$. This completes the proof.

The following result readily follows from Theorem 2.1.

COROLLARY. Let X be a Banach space, K a non-empty closed subset of X and $T:K \to X$ a mapping satisfying

$$d(T^k x, T^k y) \leq h \cdot max\{d(x,y)/a, d(x,T^k x), d(y,T^k y), [d(x,T^k y) + d(y,T^k x)]/(a+h)]\} \tag{2.9}$$

for all $x,y \in K$, where k is a positive integer and a,h constants such that $0 < h < 1$ and $a > 1 + 2h^2/(1+h)$. If $T(\partial K) \subset K$, then T has a unique fixed point in K.

Theorem 2.1 can easily be extended to multi-valued mappings. Let (X,d) be a metric space in $BN(X)$ the set of all bounded subsets of X. For $A,B \in BN(X)$ set $\delta(A,B) = sup\{d(a,b): a \in A, b \in B\}$. Now we can state our result.

THEOREM 2.2. Let X be a Banach space, K a non-empty closed subset of X and $F: K \to BN(X)$ a multi-valued mapping satisfying

$$\delta(Fx,Fy) \leq h \cdot max\{d(x,y)/a, \delta(x,Fx), \delta(y,Fy), [D(x,Fy) + D(y,Fx)]/(a+h)\}$$

for all $x,y \in K$, where a and h are reals satisfying $0 < h < 1$, $a \geq 1 + 2h^2/(1+h)$. If $Fx \subseteq K$ for all $x \in \partial K$, then F has a unique stationary point in K (i.e., there is some $\xi \in K$ such that $F\xi = \{\xi\}$.

The proof of Theorem 2.2 is omitted, since it follows the same arguments as those of Theorem 3 of [2] and Theorem 2.1 above.

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