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ANALYTICAL SOLUTION OF A CLASS OF COUPLED SECOND ORDER DIFFERENTIAL-DIFFERENCE EQUATIONS

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ABSTRACT. In this paper coupled systems of second order differential-difference equations are considered. By means of the concept of co-solution of certain algebraic equations associated to the problem, an analytical solution of initial value problems for coupled systems of second order differential-difference equations is constructed.

KEY WORDS AND PHRASES. Differential-difference equation, initial value problem, algebraic matrix equation, co-solution.

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1. INTRODUCTION.

Coupled systems of differential-difference equations are frequent in physics, engineering, economics and biology [1] . In this paper we consider second order systems of differential-difference equations of the type

$$\begin{aligned} &X''(t)+A_1X'(t)+A_2X'(t-w)+B_oX(t)=F(t), \quad t>w \\ &X(t)=G(t), \quad 0\le t\le w \end{aligned} \tag{1.1}$$

where A_1,A_2 and B_o are matrices in $\mathbb{C}^{nxn}$, $G(t)$ is a continuously differentiable $\mathbb{C}^n$ valued function in $[0,w]$, $F(t)$ is a continuous $\mathbb{C}^n$ valued function in $[w,\infty)$ and the unknown $X(t)$ takes values in $\mathbb{C}^n$.

Systems of the type (1.1) have been studied using the Laplace transform [1] , however such an approach has some computational drawbacks. First of all, it involves an increase of the problem dimension derived from the change $Z=\begin{bmatrix} X \\ X' \end{bmatrix}$ and the consideration of the transformed equivalent problem

$$\hat{A}Z'(t)+\hat{B}Z(t)+\hat{C}Z(t-w)=\hat{F}(t), \quad t>w$$
$$Z(t)=\begin{bmatrix} G(t) \\ G'(t) \end{bmatrix}, \; 0\le t\le w$$

where

$$\hat{A}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \hat{B}=\begin{bmatrix} 0 & -I \\ B_o & A_1 \end{bmatrix}, \quad \hat{C}=\begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \hat{F}(t)=\begin{bmatrix} 0 \\ F(t) \end{bmatrix},$$

see [1],p.164. The main inconvenience of the Laplace transform approach is that the expression of the solution is given in terms of the exact roots of the transcendental equation

$$\det((s\hat{A}+\hat{B})+\exp(-sw)\hat{C})=0$$

see [1],p.166 and Theorem 6.5 of [1]. Since the exact computation of these roots is not available in practice, the Laplace transform method is not interesting from the computational point of view, and it motivates the search of an alternative.

The aim of this paper is to construct the exact solution of problem (1.1) in an explicit way, avoiding the increase of the problem dimension and the determination of roots of transcendental equations. The method proposed here is based on the concept of co-solution of the associated algebraic matrix equation

$$Z^2+A_1Z+B_o=0 \tag{1.2}$$

recently given in [2]. The paper is organized as follows. In Section 2, and for the sake of clarity in the presentation, we adapt some results of [2] related to problem (1.1). We introduce an integral operator and we prove some of its properties that will be used in Section 3 in order to construct the solution of problem (1.1).

If A is a matrix in $\mathbb{C}^{nxm}$ we denote by A^T the transpose matrix of A. The set of all repeated variations of two elements taken of r elements in r elements will be denoted by $Q_{2,r}$.

2. ALGEBRAIC PRELIMINARIES AND PROPERTIES OF AN INTEGRAL OPERATOR.

We begin this section by introducing the concept of co-solution of the equation (1.2), recently given in [2], which allows us to solve initial value problems for second order differential equations without considering an extended first order system.

DEFINITION 2.1 ([2]). We say that (X,T) is a (n,q) co-solution of the algebraic equation (1.2) if $X\in\mathbb{C}^{nxq}$, $X\neq 0$, $T\in\mathbb{C}^{qxq}$ and $XT^2+A_1XT+B_oX=0$.

DEFINITION 2.2 ([2]). Let (X_i,T_i) be a (n,m_i) co-solution of equation (1.2). We say that $\{(X_i,T_i),\ 1\le i\le k\}$ is a k-complete set of co-solutions of (1.2) if m_1+m_2+ $+\dots+m_k=2n$ and the block matrix W defined by

$$W=\begin{bmatrix} X_1 & X_2 & \dots & X_k \\ X_1T_1 & X_2T_2 & \dots & X_kT_k \end{bmatrix} \tag{2.1}$$

is invertible in $\mathbb{C}^{2nx2n}$.

The next result shows the existence of k-complete sets of co-solutions of (1.2), for some appropriated value of k, and it permits the construction of such sets of co-solutions.

THEOREM 1 ([2]). Let $C=\begin{bmatrix} 0 & I \\ -B_o & -A_1 \end{bmatrix}$ and let $M=(M_{ij})$ an invertible matrix in $\mathbb{C}^{2nx2n}$ such that $M_{ij}\in\mathbb{C}^{nxm_j}$, $1\le i\le 2$, $1\le j\le k$, $m_1+m_2+\dots+m_k=2n$, such that for some block diagonal matrix $J=\operatorname{diag}(J_1,\dots,J_k)$, one satisfies $MJ=CM$. Then equation (1.2) admits a k-complete set of co-solutions given by (M_{1j},J_j), for $1\le j\le k$.

COROLLARY 1 ([2]). Let (M_{1j},J_j), $1\le j\le k$ be a k-complete set of co-solutions of equation (1.2), then the general solution of the system

$$X''(t)+A_1X'(t)+B_oX(t)=0$$

is given by

$$X(t)=\sum_{j=1}^{k} M_{1j}\exp(tJ_j)D_j\ , \tag{2.2}$$

where D_j is an arbitrary vector in $\mathbb{C}^{m_j}$, for $1\le j\le k$.

Now we are looking for the general solution of the system

$$X''(t)+A_1X'(t)+B_oX(t)=P(t), \tag{2.3}$$

where P(t) is a continuous function. Let us consider the k-complete set of co-solutions given by corollary 1, and let us denote

$$V=W^{-1}=\begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1k} \\ V_{21} & V_{22} & \cdots & V_{2k}\end{bmatrix}^T, \quad V_{ij}\in\mathbb{C}^{m_j\times n}, \ 1\le i\le 2, \quad 1\le j\le k \tag{2.4}$$

From [2] it follows that the general solution of (2.3) is given by

$$X(t)=\sum_{j=1}^{k} M_{1j}\exp(tJ_j)D_j(t), \tag{2.5}$$

$$D_j(t)=D_j+\int_w^t \exp(-uJ_j)V_{2j}P(u)du \tag{2.6}$$

where D_j is an arbitrary vector in $\mathbb{C}^{m_j}$ for $1\le j\le k$. For fixed initial conditions $X(w)=C_o$, $X'(w)=C_1$, with C_o,C_1 vectors in $\mathbb{C}^n$, the vectors D_j, $1\le j\le k$, are determined by the equations

$$C_o=\sum_{j=1}^{k} M_{1j}\exp(wJ_j)D_j, \qquad C_1=\sum_{j=1}^{k} M_{1j}J_j\exp(wJ_j)D_j, \tag{2.7}$$

and since $M_{2j}=M_{1j}J_j$, see [2], from (2.7) it follows that

$$\begin{bmatrix} D_1 \\ \vdots \\ D_k\end{bmatrix} = \mathrm{Diag}(\exp(-wJ_j);\ 1\le j\le k)\, M^{-1}\begin{bmatrix} C_o \\ C_1\end{bmatrix} \tag{2.8}$$

REMARK 1. It is interesting to recall that the Jordan canonical form of any matrix may be efficiently computed using MACSYMA [3] ,so taking into account Theorem 1, the construction of a k-complete set of co-solutions for equation (1.2) is an easy matter. On the other hand, we recall that the exponential $\exp(tJ_j)$ of a Jordan block J_j, has a well known expression in terms of J_j, see [4],p.66.

Now we introduce an integral operator that will play an important role in the following.

DEFINITION 2.3. Let $\{(M_{1j},J_j);\ 1\le j\le k\}$ be the k-complete set of co-solutions of (1.2) provided by Theorem 1. Let h and p be positive integers with $h\le p$, and $t\ge pw$, where w is a positive real number. If H is continuous $\mathbb{C}^n$ valued and defined on $[w,\infty)$ we define the operator Φ by the expression

$$\Phi[p,h,t,j,H(u_{h+d}-w(d+n))] =$$

$$\begin{cases} \int_{hw}^{t} \exp(-u_h J_j) V_{2j} H(u_h - nw)\, du_h\,, & \text{if } p=h \\ \int_{pw}^{t} \exp(-u_p J_j) V_{2j} \sum_{(i_{p-1}, i_{p-2}, \ldots, i_h) \in Q_{2,p-h}} L_{i_{p-1},p-1} L_{i_{p-2},p-2} \cdots L_{i_h,h} H(u_{h+d} - w(d+n))\, du_p\,, & \text{if } p>h \end{cases} \tag{2.9}$$

where $n=0,1$; $h \le q \le p-1$; $1 \le i_q \le 2$,

$$L_{1,q} = A_2 \sum_{j_q=1}^{k} M_{1j_q} J_{j_q} \exp((u_{q+1+d} - (d+1)w) J_{j_q}) \int_{qw}^{u_{q+1+d}-(d+1)w} \exp(-u_q J_{j_q}) V_{2j_q}\, du_q$$

$$L_{2,q} = A_2 \sum_{j_q=1}^{k} M_{1j_q} V_{2j_q} \tag{2.10}$$

and d is the number of factors of the type L_2 which appear on the left of each factor of the type L_1 or $H(u_{h+d} - w(d+n))$ before the appearance of a new factor L_1.

EXAMPLES

a) $$\Phi[3,2,t,j,H(u_{2+d} - dw)] = \int_{3w}^{t} \exp(-u_3 J_j) V_{2j} \sum_{(i_2) \in Q_{2,1}} L_{i_2,2} H(u_{2+d} - dw)\, du_3 =$$

$$\int_{3w}^{t} \exp(-u_3 J_j) V_{2j} A_2 \sum_{j_2=1}^{k} M_{1j_2} J_{j_2} \exp((u_3 - w) J_{j_2}) \int_{2w}^{u_3-w} \exp(-u_2 J_{j_2}) V_{2j_2} H(u_2)\, du_2\, du_3 +$$

$$+ \int_{3w}^{t} \exp(-u_3 J_j) V_{2j} A_2 \sum_{j_2=1}^{k} M_{1j_2} V_{2j_2} H(u_3 - w)\, du_3$$

b) $$\Phi[1,1,t,j,H(u_{1+d} - dw)] = \int_{w}^{t} \exp(-u_1 J_j) V_{2j} H(u_1)\, du_1$$

c) $$\Phi[3,1,5w,j,H(u_{2+d} - (d+1)w)] = \int_{3w}^{5w} \exp(-u_3 J_j) V_{2j} \sum_{(i_2,i_1) \in Q_{2,2}} L_{i_2,2} L_{i_1,1} H(u_{1+d} - (d+1)w)\, du_3$$

$$= \int_{3w}^{5w} \exp(-u_3 J_j) V_{2j} A_2 \sum_{j_2=1}^{k} M_{1j_2} J_{j_2} \exp((u_3 - w) J_{j_2}) \int_{2w}^{u_3-w} \exp(-u_2 J_{j_2}) V_{2j_2} A_2 \sum_{j_1=1}^{k} M_{1j_1} J_{j_1} \exp((u_2 - w) J_{j_1})$$

$$\int_{w}^{u_2-w} \exp(-u_1 J_{j_1}) V_{2j_1} H(u_1 - w)\, du_1\, du_2\, du_3 +$$

$$\int_{3w}^{5w} \exp(-u_3 J_j) V_{2j} A_2 \sum_{j_2=1}^{k} M_{1j_2} J_{j_2} \exp((u_3 - w) J_{j_2}) \int_{2w}^{u_3-w} \exp(-u_2 J_{j_2}) V_{2j_2} A_2 \sum_{j_1=1}^{k} M_{1j_1} V_{2j_1} H(u_2 - 2w)\, du_2\, du_3 +$$

$$\int_{3w}^{5w} \exp(-u_3 J_j) V_{2j} A_2 \sum_{j_2=1}^{k} M_{1j_2} V_{2j_2} A_2 \sum_{j_1=1}^{k} M_{1j_1} J_{j_1} \exp((u_3 - 2w) J_{j_1}) \int_{w}^{u_3-2w} \exp(-u_1 J_{j_1}) V_{2j_1} H(u_1 - w)\, du_1\, du_3 +$$

$$\int_{3w}^{5w} \exp(-u_3 J_j) V_{2j} A_2 \sum_{j_2=1}^{k} M_{1j_2} V_{2j_2} A_2 \sum_{j_1=1}^{k} M_{1j_1} V_{2j_1} H(u_3 - 3w)\, du_3$$

Now we prove two lemmas that will be used in the following section to construct the solution of problem (1.1).

LEMMA 1. If $t \geq (m+1)w$ and $h \leq p \leq m$, then for $n=0,1$, it follows that

$$\Phi[p+1,h,t,j,H(u_{h+d}-(d+n)w)] = \Phi[p+1,h,(m+1)w,j,H(u_{h+d}-(d+n)w)] + \int_{(m+1)w}^{t} \exp(-zJ_j)V_{2j}A_2 \Big[\sum_{j_m=1}^{k} M_{1j_m} J_{j_m} \exp((z-w)J_{j_m}) \Phi[p,h,z-w,j_m,H(u_{h+d}-(d+n)w)] + \sum_{j_m=1}^{k} M_{1j_m} \exp((z-w)J_{j_m}) \frac{d}{dt}\big(\Phi[p,h,t,j_m,H(u_{h+d}-(d+n)w)] \big)_{t=z-w} \Big] dz \tag{2.11}$$

PROOF. From definition 2.3 it follows that

$$\Phi[p+1,h,t,j,H(u_{h+d}-(d+n)w)]$$

$$= \int_{(p+1)w}^{t} \exp(-u_{p+1}J_j)V_{2j} \sum_{(i_p,i_{p-1},\ldots,i_h)\in Q_{2,p+1-h}} L_{i_p,p} L_{i_{p-1},p-1} \cdots L_{i_h,h} H(u_{h+d}-(d+n)w)\, du_{p+1}$$

$$= \int_{(p+1)w}^{(m+1)w} \exp(-u_{p+1}J_j)V_{2j} \sum_{(i_p,\ldots,i_h)\in Q_{2,p+1-h}} L_{i_p,p} \cdots L_{i_h,h} H(u_{h+d}-(d+n)w)\, du_{p+1} +$$

$$+ \int_{(m+1)w}^{t} \exp(-u_{p+1}J_j)V_{2j} \sum_{(i_p,\ldots,i_h)\in Q_{2,p+1-h}} L_{i_p,p} \cdots L_{i_h,h} H(u_{h+d}-(d+n)w)\, du_{p+1}$$

$$= \int_{(p+1)w}^{(m+1)w} \exp(-u_{p+1}J_j)V_{2j} \sum_{(i_p,\ldots,i_h)\in Q_{2,\,p+1-h}} L_{i_p,p} \cdots L_{i_h,h} H(u_{h+d}-(d+n)w)\, du_{p+1} +$$

$$+ \int_{(m+1)w}^{t} \exp(-zJ_j)V_{2j}A_2 \sum_{j_m=1}^{k} M_{1j_m} \exp((z-w)J_{j_m}) \int_{pw}^{z-w} \exp(-u_pJ_{j_m})V_{2j_m}$$

$$\Big(\sum_{(i_{p-1},\ldots,i_h)\in Q_{2,p-h}} L_{i_{p-1},p-1} \cdots L_{i_h,h} H(u_{h+d}-(d+n)w)du_p\Big)dz + \int_{(m+1)w}^{t} \exp(-zJ_j)V_{2j}A_2 \sum_{j_m=1}^{k} M_{1j_m} \exp((z-w)J_{j_m})$$

$$\frac{d}{dt}\Big(\int_{pw}^{t} \exp(-u_pJ_{j_m})V_{2j_m} \sum_{(i_{p-1},\ldots,i_h)\in Q_{2,p-h}} L_{i_{p-1},p-1} \cdots L_{i_h,h} H(u_{h+d}-(d+n)w)\, du_p \Big)_{t=z-w} dz$$

Hence we obtain the right hand side of (2.11) and the proof of the lemma is concluded.

LEMMA 2. If $t \geq (m+1)w$, $h \leq p \leq m$ and $n=0,1$, it follows that

$$\sum_{p=h}^{m+1} (-1)^{p-1} \Phi[p,h,t,j,H(u_{h+d}-(d+n)w)] \tag{2.12}$$

$$= \sum_{p=h}^{m} (-1)^{p-1} \Phi[p,h,(m+1)w,H(u_{h+d}-(d+n)w)] - \int_{(m+1)w}^{t} \exp(-zJ_j)V_{2j}$$

$$\Big\{(-1)^h H(z-nw) + A_2\Big[\sum_{j_m=1}^{k} M_{1j_m} J_{j_m} \exp((z-w)J_{j_m}) \sum_{p=h}^{m} (-1)^{p-1} \Phi[p,h,z-w,j_m,H(u_{h+d}-(d+n)w] + \sum_{j_m=1}^{k} M_{1j_m} \exp((z-w)J_{j_m}) \sum_{p=h}^{m} (-1)^{p-1} \frac{d}{dt}\big(\Phi[p,h,t,j_m,H(u_{h+d}-(d+n)w)] \big)_{t=z-w} \Big]\Big\} dz \tag{2.13}$$

PROOF. From definition 2.3 and lemma 1, it follows that

$$\sum_{p=h}^{m+1} (-1)^{p-1}\,\Phi[p,h,t,j,H(u_{h+d}-(d+n)w)]=$$

$$(-1)^{h-1}\,\Phi[h,h,t,j,H(u_{h+d}-(d+n)w)]+\sum_{p=h+1}^{m}(-1)^{p-1}\,\Phi[p,h,t,j,H(u_{h+d}-(d+n)w)]+$$

$$+(-1)^{m}\,\Phi[m+1,h,t,j,H(u_{h+d}-(d+n)w)]=$$

$$=(-1)^{h-1}\{\int_{hw}^{(m+1)w}\exp(-u_hJ_j)V_{2j}H(u_h-nw)du_h+\int_{(m+1)w}^{t}\exp(-zJ_j)V_{2j}H(z-nw)\,dz\}+$$

$$+\sum_{p'=h}^{m-1}(-1)^{p'}\Phi[p'+1,h,(m+1)w,j,H(u_{h+d}-(d+n)w)]+$$

$$+\sum_{p=h}^{m-1}(-1)^{p}\int_{(m+1)w}^{t}\exp(-zJ_j)V_{2j}A_2\,\{\sum_{j_m=1}^{k}M_{1j_m}J_{j_m}\exp((z-w)J_{j_m})\,\Phi[p,h,z-w,j_m,H(u_{h+d}-(d+n)w)]+$$

$$+\sum_{j_m=1}^{k}M_{1j_m}\exp((z-w)J_{j_m})\frac{d}{dt}(\Phi[p,h,t,j_m,H(u_{h+d}-(d+n)w)])_{t=z-w}\}\,dz+$$

$$+(-1)^{m}\int_{(m+1)w}^{t}\exp(-zJ_j)V_{2j}A_2\Big[\sum_{j_m=1}^{k}M_{1j_m}J_{j_m}\exp((z-w)J_{j_m})\,\Phi[m,h,z-w,j_m,H(u_{h+d}-(d+n)w)]+$$

$$+\sum_{j_m=1}^{k}M_{1j_m}\exp((z-w)J_{j_m})\,\frac{d}{dt}(\Phi[m,h,t,j_m,H(u_{h+d}-(d+n)w)])_{t=z-w}\Big]\,dz=$$

$$=(-1)^{h-1}\int_{(m+1)w}^{t}\exp(-zJ_j)V_{2j}H(z-nw)dz-\int_{(m+1)w}^{t}\exp(-zJ_j)V_{2j}A_2\,\{\sum_{j_m=1}^{k}M_{1j_m}J_{j_m}\exp((z-w)J_{j_m})$$

$$\sum_{p=h}^{m}(-1)^{p-1}\,\Phi[p,h,z-w,H(u_{h+d}-(d+n)w)]+\sum_{j_m=1}^{k}M_{1j_m}\exp((z-w)J_{j_m})$$

$$\sum_{p=h}^{m}(-1)^{p-1}\frac{d}{dt}(\Phi[p,h,t,j_m,H(u_{h+d}-(d+n)w)])_{t=z-w}\}\,dz$$

Hence the proof of lemma 2 is established.

3. CONSTRUCTION OF THE SOLUTION.

In this section we provide an analytical expression for the solution of problem (1.1) by means of a constructive way. Note that for $t>w$ the system (1.1) may be written in the form

$$X''(t)+A_1X'(t)+B_oX(t)=F(t)-A_2X'(t-w),\quad t>w \tag{3.1}$$

where $X(t)=G(t)$ for $0\le t\le w$. From (2.5)-(2.8), the solution of (3.1) on the interval $[w,2w]$ is given by (2.5) where $D_i(t)$ takes the form

$$D_j(t)=D_j(w)+\int_{w}^{t}\exp(-zJ_j)V_{2j}\{F(z)-A_2X'(z-w)\}\,dz$$

In general, for $t\varepsilon[mw,(m+1)w]$, the solution of (3.1) may be written in the form (2.5) where $D_j(t)$ is given by

$$D_j(t)=D_j(mw)+\int_{mw}^{t}\exp(-zJ_j)V_{2j}\{F(z)-A_2X'(z-w)\}\,dz, \quad t\varepsilon[mw,(m+1)w] \tag{3.2}$$

Taking into account that $X(u)=G(u)$ for $u\varepsilon[0,w]$, it follows that

$$D_j(t)=D_j(w)+\int_{w}^{t}\exp(-zJ_j)V_{2j}F(z)dz-\int_{w}^{t}\exp(-zJ_j)V_{2j}A_2G'(z-w)dz, \quad t\varepsilon[w,2w] \tag{3.3}$$

and from (2.5) and (3.3), we have

$$X(t)=\sum_{j=1}^{k}M_{1j}\exp(tJ_j)D_j(t)=$$

$$\sum_{j=1}^{k}M_{1j}\exp(tJ_j)\{D_j(w)+\int_{w}^{t}\exp(-zJ_j)V_{2j}F(z)dz-\int_{w}^{t}\exp(-zJ_j)V_{2j}A_2G'(z-w)dz\} \tag{3.4}$$

If $t\varepsilon[2w,3w]$, one gets

$$D_j(t)=D_j(2w)+\int_{2w}^{t}\exp(-zJ_j)V_{2j}F(z)dz-\int_{2w}^{t}\exp(-zJ_j)V_{2j}A_2X'(z-w)dz$$

and from (3.3)-(3.4), it follows that

$$\begin{aligned}
D_j(t)=&D_j(w)+\int_{w}^{2w}\exp(-zJ_j)V_{2j}F(z)dz-\int_{w}^{2w}\exp(-zJ_j)V_{2j}A_2G'(z-w)dz+\\
&+\int_{2w}^{t}\exp(-zJ_j)V_{2j}F(z)dz-\int_{2w}^{t}\exp(-zJ_j)V_{2j}A_2\{\sum_{j_1=1}^{k}M_{1j_1}J_{j_1}\exp((z-w)J_{j_1})\Big[D_{j_1}(w)+\int_{w}^{z-w}\exp(-uJ_{j_1})V_{2j_1}F(u)du\\
&-\int_{w}^{z-w}\exp(-uJ_{j_1})V_{2j_1}A_2G'(u-w)du\Big]+\sum_{j_1=1}^{k}M_{1j_1}\exp((z-w)J_{j_1})\Big[\exp(-(z-w)J_{j_1})V_{2j_1}F(z-w)+\\
&-\exp(-(z-w)J_{j_1})V_{2j_1}A_2G'(z-2w)\Big]\}dz=\\
=&D_j(w)-\int_{2w}^{t}\exp(-zJ_j)V_{2j}A_2\sum_{j_1=1}^{k}M_{1j_1}J_{j_1}\exp((z-w)J_{j_1})D_{j_1}(w)\,dz+\\
&+\int_{w}^{t}\exp(-zJ_j)V_{2j}F(z)dz-\int_{2w}^{t}\exp(-zJ_j)V_{2j}A_2\sum_{j_1=1}^{k}M_{1j_1}J_{j_1}\exp((z-w)J_{j_1})\int_{w}^{z-w}\exp(-uJ_{j_1})V_{2j_1}F(u)dudz+\\
&-\int_{2w}^{t}\exp(-zJ_j)V_{2j}A_2\sum_{j_1=1}^{k}M_{1j_1}V_{2j_1}F(z-w)dz\\
&+\int_{2w}^{t}\exp(-zJ_j)V_{2j}A_2\sum_{j_1=1}^{k}M_{1j_1}J_{j_1}\exp((z-w)J_{j_1})\int_{w}^{z-w}\exp(-uJ_{j_1})V_{2j_1}A_2G'(u-w)dudz\\
&+\int_{2w}^{t}\exp(-zJ_j)V_{2j}A_2\sum_{j_1=1}^{k}M_{1j_1}A_2G'(z-2w)dz-\int_{w}^{2w}\exp(-zJ_j)V_{2j}A_2G'(z-w)dz
\end{aligned} \tag{3.5}$$

Now by using the induction principle we are going to prove that the solution of

(1.1) for $t\varepsilon[mw,(m+1)w]$, is given by (2.5) where $D_j(t)$ is expressed in terms of the operator Φ in the form

$$D_j(t)=D_j(w)+\sum_{p=2}^{m}(-1)^{p-1}\Phi[p,2,t,j,A_2\sum_{j_1=1}^{k}M_{1j_1}J_{j_1}\exp((u_{2+d}-(d+1)w)J_{j_1})D_{j_1}(w)]+$$

$$+\sum_{p=1}^{m}(-1)^{p-1}\Phi[p,1,t,j,F(u_{d+1}-dw)]+(-1)^m\Phi[m,1,t,j,A_2G'(u_{1+d}-(d+1)w)]+$$

$$+\sum_{p=2}^{m}\{(-1)^{p-1}\Phi[p-1,1,pw,j,A_2G'(u_{1+d}-(d+1)w)]+ \tag{3.6}$$

$$+\sum_{r=p+1}^{m}(-1)^{r-1}\Phi[r,p+1,t,j,A_2\sum_{j_p=1}^{k}M_{1j_p}J_{j_p}\exp((u_{p+1}-w)J_{j_p})\Phi[p-1,1,pw,j_p,A_2G'(u_{1+d}-(d+1)w)]]\}$$

It is easy to prove that in (3.3)-(3.5), $D_j(t)$ coincides with (3.6) for $m=1,2$. Let us suppose that for $t\varepsilon[mw,(m+1)w]$, $D_j(t)$ is given by (3.6), and let $t\varepsilon[(m+1)w,(m+2)w]$. Then from (2.5),(3.2), it follows that

$$D_j(t)=D_j((m+1)w)+\int_{(m+1)w}^{t}\exp(-zJ_j)V_{2j}F(z)dz-\int_{(m+1)w}^{t}\exp(-zJ_j)V_{2j}A_2X'(z-w)dz \tag{3.7}$$

If we apply the induction hypothesis and we take into account the expression of $X(z-w)$ for $z\varepsilon[(m+1)w,t]$, and (3.7), it follows that

$$D_j(t)=D_j(w)+\sum_{p=2}^{m}(-1)^{p-1}\Phi[p,2,(m+1)w,j,A_2\sum_{j_1=1}^{k}M_{1j_1}J_{j_1}\exp((u_{2+d}-(d+1)w)J_{j_1})D_{j_1}(w)]-$$

$$\int_{(m+1)w}^{t}\exp(-zJ_j)V_{2j}\{A_2\sum_{j_m=1}^{k}M_{1j_m}J_{j_m}\exp((z-w)J_{j_m})D_{j_m}(w)+$$

$$A_2\Big[\sum_{j_m=1}^{k}M_{1j_m}J_{j_m}\exp((z-w)J_{j_m})\sum_{p=2}^{m}(-1)^{p-1}\Phi[p,2,z-w,j_m,A_2\sum_{j_1=1}^{k}M_{1j_1}J_{j_1}\exp((u_{2+d}-(d+1)w)J_{j_1})D_{j_1}(w)]$$

$$+\sum_{j_m=1}^{k}M_{1j_m}\exp((z-w)J_{j_m})\sum_{p=2}^{m}(-1)^{p-1}\frac{d}{dt}(\Phi[p,2,t,j_m,A_2\sum_{j_1=1}^{k}M_{1j_1}J_{j_1}\exp((u_{2+d}-(d+1)w)J_{j_1})D_{j_1}(w)])_{t=z-w}\Big]\}dz$$

$$+\sum_{p=1}^{m}(-1)^{p-1}\Phi[p,1,(m+1)w,j,F(u_{1+d}-dw)]-\int_{(m+1)w}^{t}\exp(-zJ_j)V_{2j}\{(-1)F(z)+$$

$$A_2\Big[\sum_{j_m=1}^{k}M_{1j_m}J_{j_m}\exp((z-w)J_{j_m})\sum_{p=1}^{m}(-1)^{p-1}\Phi[p,1,z-w,j_m,F(u_{1+d}-dw)]+$$

$$\sum_{j_m=1}^{k}M_{1j_m}\exp((z-w)J_{j_m})\sum_{p=1}^{m}(-1)^{p-1}\frac{d}{dt}(\Phi[p,1,t,j_m,F(u_{1+d}-dw)])_{t=z-w}\Big]\}dz+$$

$$(-1)^{m+1}\int_{(m+1)w}^{t}\exp(-zJ_j)V_{2j}A_2\Big[\sum_{j_m=1}^{k}M_{1j_m}J_{j_m}\exp((z-w)J_{j_m})\Phi[m,1,z-w,j_m,A_2G'(u_{1+d}-(d+1)w)]+$$

$$+ \sum_{j_m=1}^{k} M_{1j_m} \exp((z-w)J_{j_m}) \frac{d}{dt}\left(\Phi[m,1,t,j_m,A_2 G'(u_{1+d}-(d+1)w)] \right)_{t=z-w} \Big] dz +$$

$$\sum_{p=2}^{m} \{ (-1)^{p-1} \Phi[p-1,1,pw,j,A_2G'(u_{1+d}-(d+1)w)] + (-1)^{m} \Phi[m,1,(m+1)w,j,A_2G'(u_{1+d}-(d+1)w)] +$$

$$\sum_{r=p+1}^{m} (-1)^{r-1} \Phi[r,p+1,(m+1)w,j, A_2 \sum_{j_p=1}^{k} M_{1j_p} J_{j_p} \exp((u_{p+1}-w)J_{j_p}) \Phi[p-1,1,pw, j_1, A_2G'(u_{1+d}-(d+1)w)] -$$

$$\int_{(m+1)w}^{t} \exp(-zJ_j)V_{2j} [(-1)^{p-1}A_2 \sum_{j_m=1}^{k} M_{1j_m} J_{j_m} \exp((z-w)J_{j_m}) \Phi[p-1,1,pw,j_m,A_2G'(u_{1+d}-(d+1)w)] +$$

$$A_2 \Big[\sum_{j_m=1}^{k} M_{1j_m} J_{j_m} \exp((z-w)J_{j_m}) \sum_{r=p+1}^{m} (-1)^{r-1} \Phi[r,p+1,z-w,j_m,A_2 \sum_{j_p=1}^{k} M_{1j_p} J_{j_p} \exp((u_{p+1}-w)J_{j_p})$$

$$\Phi[p-1,1,pw,j_p,A_2G'(u_{1+d}-(d+1)w)] +$$

$$\sum_{j_m=1}^{k} M_{1j_m} \exp((z-w)J_{j_m}) \sum_{r=p+1}^{m} (-1)^{r-1} \frac{d}{dt}\Big($$

$$\Phi[r,p+1,t,j_m,A_2 \sum_{j_p=1}^{k} M_{1j_p} J_{j_p} \exp((u_{p+1}-w)J_{j_p}) \Phi[p-1,1,pw, j_p, A_2G'(u_{1+d}-(d+1)w)] \Big] \Big)_{t=z-w} \Big] \Big] dz \}$$

Now taking into account the lemma 2 and the proof of lemma 1, we may write

$$D_j(t)=D_j(w) + \sum_{p=2}^{m+1} (-1)^{p-1} \Phi[p,2,t,j,A_2 \sum_{j_1=1}^{k} M_{1j_1} J_{j_1} \exp((u_{2+d}-(d+1)w)J_{j_1})D_{j_1}(w)] +$$

$$\sum_{p=1}^{m+1} (-1)^{p-1} \Phi[p,1,t,j,F(u_{1+d}-dw)] +(-1)^{m+1} \Phi[m+1,1,t,j,A_2G'(u_{1+d}-(d+1)w] +$$

$$\sum_{p=2}^{m+1} \{ (-1)^{p-1} \Phi[p-1,1,pw,j, A_2G'(u_{1+d}-(d+1)w)] +$$

$$\sum_{r=p+1}^{m+1} (-1)^{r-1} \Phi[r,p+1,t,j,A_2 \sum_{j_p=1}^{k} M_{1j_p} \exp((u_{p+1}-w)J_{j_p}) \Phi[p-1,1,pw,j_p,A_2G'(u_{1+d}-(d+1)w)] \Big] \}$$

Note that this expression coincides with (3.6) replacing m by m+1. Thus the following result has been established:

THEOREM 2. Let us consider the notation of Theorem 2 and let $\{(M_{1j},J_j);\ 1\le j\le k\}$ be a k-complete set of co-solutions of equation (1.2). Let F(t) be a continuous function in $[w,\infty)$ and let G(t) be a continuously differentiable function in $[0,w]$. Then the solution of problem (1.1) is given by (2.5), where $D_j(t)$ is defined by (3.6) for $1\le j\le k$ $t\in[mw,(m+1)w]$, $m\ge 1$.

In the following we illustrate with an example in $\mathbb{C}^2$ the previous results and we show the availability of the construction of the solution of the problem (1.1).

EXAMPLE. Let us consider the coupled differential-difference system

$$X''(t)+\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}X'(t)+\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}X'(t-w)+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}X(t)=\begin{bmatrix}\exp(t)\\ 1\end{bmatrix},\quad t>w>0$$
$$X(t)=\begin{bmatrix}t\\ 1\end{bmatrix},\quad 0\le t\le w \tag{3.8}$$

The companion matrix C defined in Theorem 1 takes the form

$$C=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & -1\\ 0 & -1 & 0 & 2\end{bmatrix}$$

Straightforward computations yield that the matrices J and M of Theorem 1 as well as the k-complete set of co-solutions of the associated algebraic matrix equation

$$Z^2+\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}Z+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}=0$$

are given by

$$J=\operatorname{diag}(J_1,J_2),\qquad J_1=(0),\qquad J_2=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

$$M_{11}=\begin{bmatrix}1\\ 0\end{bmatrix},\qquad M_{12}=\begin{bmatrix}-1 & 1 & -1\\ 0 & 1 & -1\end{bmatrix},\quad M=\begin{bmatrix}1 & -1 & 1 & -1\\ 0 & 0 & 1 & -1\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

From Theorem 2, the solution of problem (3.8) is given by

$$\begin{aligned}X(t)&=M_{11}\exp(tJ_1)D_1(t)+M_{12}\exp(tJ_2)D_2(t)\\ &=M_{11}D_1(t)+M_{12}\exp(t)\begin{bmatrix}1 & t & t^2/2\\ 0 & 1 & t\\ 0 & 0 & 1\end{bmatrix}\end{aligned}$$

where $D_j(t)$ for $j=1,2$, are defined by (3.6) and $D_1(w)$, $D_2(w)$, are determined by the solution of the corresponding system (2.7):

$$\begin{bmatrix}w\\ 1\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}D_1(w)+\begin{bmatrix}-1 & 1 & -1\\ 0 & 1 & -1\end{bmatrix}\exp(w)\begin{bmatrix}1 & w & w^2/2\\ 0 & 1 & w\\ 0 & 0 & 1\end{bmatrix}D_2(w)$$

$$\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}-1 & 1 & -1\\ 0 & 1 & -1\end{bmatrix}+\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\exp(w)\begin{bmatrix}1 & w & w^2/2\\ 0 & 1 & w\\ 0 & 0 & 1\end{bmatrix}D_2(w)$$

Solving this system it follows that

$$D_1(w)=w-2\ ,\qquad D_2(w)=\exp(-w)\begin{bmatrix}-1-w^2/2\\ w\\ -1\end{bmatrix}$$

If $t\varepsilon[w,2w]$, then from (3.6) it follows that

$$D_1(t)=\exp(w)-\exp(t)+t-2$$

and

$$D_2(t)=\begin{bmatrix} w-t-(1+t^2/2)\exp(-t) \\ t\exp(-t) \\ -\exp(-t) \end{bmatrix}$$

Hence from (2.5),(3.6), the solution of (3.8) in $[w,2w]$ is given by

$$X(t)=\exp(t)\begin{bmatrix} t-w-1 \\ 0 \end{bmatrix}+\begin{bmatrix} t \\ 1 \end{bmatrix}$$

If $t\varepsilon[2w,3w]$, then from (3.6) it follows that

$$D_1(t)=(t-2w-1)\exp(t-w)-\exp(t)+2\exp(w)-2+t$$

and

$$D_2(t)=\begin{bmatrix} w-t+(t^2/2-2wt)\exp(-w)-(1+t^2/2)\exp(-t) \\ t\exp(-t) \\ -\exp(-t) \end{bmatrix}$$

From (2.5),(3.6), it follows that the solution X(t) of (3.8) in $[2w,3w]$ is given by

$$X(t)=\begin{bmatrix} t-2w-1 \\ 0 \end{bmatrix}\exp(t-w)+\begin{bmatrix} -1 \\ 0 \end{bmatrix}\exp(t)+\begin{bmatrix} 2\exp(w)-2+t \\ 0 \end{bmatrix}+$$

$$+\exp(t)\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} w-t+\exp(-w)(t^2/2-2wt)-(1+t^2/2)\exp(-t) \\ t\exp(-t) \\ -\exp(-t) \end{bmatrix}$$

$$=\begin{bmatrix} t-w-1 \\ 0 \end{bmatrix}\exp(t)+\begin{bmatrix} t+2wt-2w-1-t^2/2 \\ 0 \end{bmatrix}\exp(t-w)+\begin{bmatrix} 2\exp(w)+t \\ 1 \end{bmatrix}$$

In an analogous way using (2.5) and (3.6) we may obtain the expression of the solution of the problem in any interval $[mw,(m+1)w]$.

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