ON TOTALLY UMBILICAL CR-SUBMANIFOLDS OF A KAEHLER MANIFOLD

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ABSTRACT. Let M be a compact 3-dimensional totally umbilical CR-submanifold of a Kaehler manifold of positive holomorphic sectional curvature. We prove that if the length of the mean curvature vector of M does not vanish, then M is either diffeomorphic to S^3 or RP^3 or a lens space $L^3{}_{p,q}$.

KEY WORDS AND PHRASES. CR-submanifolds. Totally umbilical submanifolds. Kaehler manifold.
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1. INTRODUCTION.

Totally umbilical CR-submanifolds of a Kaehler manifold have been considered by Bejancu [2], Blair, and Chen [3]. Recently Deshmukh and Husain [5] have also studied these submanifolds. In fact, they have proved a classification theorem when the dimension of the submanifold M is ≥ 5. In this paper we consider 3-dimensional totally umbilical CR-submanifolds of a Kaehler manifold. For this case we have obtained the following theorem:

THEOREM 1.1. Let M be a compact 3-dimensional totally umbilical CR-submanifold of a Kaehler manifold $\bar{M}$, of positive holomorphic sectional curvature. If the length of the mean curvature vector of M does not vanish then M is diffeomorphic either to S^3, RP^3 or the lens space $L^3{}_{p,q}$.

2. PRELIMINARIES.

Let $\bar{M}$ be an m-dimensional Kaehler manifold with almost complex structure J. A $(2p+q)$-dimensional submanifold M of $\bar{M}$ is called a CR-submanifold if there exists a pair of orthogonal complementary distributions D and $\overset{\perp}{D}$ such that $JD = D$ and $J\overset{\perp}{D} \subset \nu$, where ν is the normal bundle of M and $dim\ \overset{\perp}{D} = q$[1]. Thus the normal bundle ν splits as $\nu = J\overset{\perp}{D} \oplus \mu$, where μ is invariant subbundle of ν under J. A CR-submanifold is said to be proper if neither $D = \{0\}$ nor $\overset{\perp}{D} = \{0\}$.

We denote by $\bar{\nabla}, \nabla, \overset{\perp}{\nabla}$ the Reimannian connection on $\bar{M}, M$ and the normal bundle respectively. They are related by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.1}$$

$$\bar{\nabla}_X N = -A_N X + \overset{\perp}{\nabla}_X N, \qquad N \in \nu \tag{2.2}$$

where $h(X,Y)$ and $A_N X$ are the second fundamental forms which are related by

$$g(h(X,Y),N) = g(A_N X, Y) \tag{2.3}$$

Now a CR-submanifold is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H$$

where $H = \frac{1}{n}(trace\ h)$ is the main curvature vector. If M is totally umbilical CR-submanifold, then equations (2.1) and (2.2) become

$$\bar{\nabla}_X Y = \nabla_X Y + g(X,Y)H \tag{2.4}$$

$$\bar{\nabla}_X N = -g(H,N)X + \overset{\perp}{\nabla}_X N \tag{2.5}$$

For $X,Y,Z,W \in X(M)$, the equation of Gauss is given by

$$R(X,Y;Z,W) = \bar{R}(X,Y;Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \tag{2.6}$$

3. 3-DIMENSIONAL CR-SUBMANIFOLD OF A KAEHLER MANIFOLD.

(A) Let M be a compact totally umbilical 3-dimensional CR-submanifold of a Kaehler manifold $\bar{M}$. If $dim\ D = 0$, then M will be totally real. Therefore, we assume that $dim\ D \neq 0$. Since M is 3-dimensional it follows that $dim\ D = 2$. We can then choose a frame field $\{X, JX, Z\}$ on M, where $X \in D$ and $Z \in \overset{\perp}{D}$. We first have the following:

LEMMA 1. Let $\{X, JX, Z\}$ be a frame field on M, $X \in D$, $Z \in \overset{\perp}{D}$. Then $\nabla_Z Z = 0$. and $H \in J\overset{\perp}{D}$.

PROOF. Using (2.4) and (2.5) in the equation $\bar{\nabla}_Z JZ = J\bar{\nabla}_Z Z$, we obtain

$$-g(H,JZ)JZ + J\overset{\perp}{\nabla}_Z JZ = -\nabla_Z Z - h(Z,Z) \tag{3.1}$$

Taking inner produce in (3.1) with $W \in D$ we have

$$g(\nabla_Z Z, W) = 0 \qquad\qquad W \in D \tag{3.2}$$

From (3.2) we have $\nabla_Z Z \in \overset{\perp}{D}$. Since $g(Z,Z) = 1$, we also have $\nabla_Z Z \in D$. Therefore $\nabla_Z Z = 0$.
Now for $X, Y \neq 0$ in D we use (2.4) and the equation $J\bar{\nabla}_X Y = \bar{\nabla}_X JY$ to get

$$J\nabla_X Y + g(X,Y)JH = \nabla_X JY + g(X,JY)H \tag{3.3}$$

Taking inner produce in (3.1) with $N \in \mu$ we have

$$g(X,Y)g(JH,N) = g(X,JY)g(H,N) \tag{3.4}$$

In particular if we let $Y = JX$ in (3.4) we get

$$\|X\|\, g(H,N) = 0, \qquad N \in \mu. \qquad\qquad \text{Therefore } H \in J\overset{\perp}{D}. \tag{3.5}$$

Consider the frame field $\{X, JX, Z\}$ on M. Since M is totally umbilical the equation $h(Y,W) = g(Y,W)H$ for $Y, W \in X(M)$ implies that

$$h(X,JX) = h(X,Z) = h(JX,Z) = 0$$

$$h(X,X) = h(JX,JX) = h(Z,Z) = H \equiv \alpha JZ \tag{3.6}$$

for some smooth function α on M, since $H \in JD^{\perp}$.

Using (2.3) with $N = JZ$ we get

$$AX = \alpha X, \qquad AJX = \alpha JX, \qquad AZ = \alpha Z \tag{3.7}$$

So the frame field $\{X, JX, Z\}$ diagonalizes A. Now using the equation $(\bar{\nabla}_X J)(X) = 0$ and $(\bar{\nabla}_{JX} J)(X) = 0$ with the help of (3.6) we get

$$g(\nabla_X X, Z) = 0, \qquad g(\nabla_{JX} JX, Z) = 0 \tag{3.8}$$

Also using the equation $\nabla_Z Z = 0$ from Lemma 1 we have

$$g(\nabla_Z X, Z) = 0, \qquad g(\nabla_Z JX, Z) = 0 \tag{3.9}$$

Then using the equation $(\bar{\nabla}_X J)(Z) = 0$ and (3.7) we obtain

$$g(\nabla_X Z, X) = 0, \qquad g(\nabla_X Z, JX) = \alpha \tag{3.10}$$

and using the equation $(\bar{\nabla}_{JX} J)(Z) = 0$ we have

$$g(\nabla_{JX} Z, X) = -\alpha, \qquad g(\nabla_{JX} Z, JX) = 0 \tag{3.11}$$

Using equations (3.8), (3.9), (3.10), and (3.11) one can write the following equations for the frame field $\{X, JX, Z\}$:

$$\begin{array}{lll} \nabla_X Z = \alpha JX, & \nabla_{JX} Z = -\alpha X, & \nabla_Z Z = 0 \\ \nabla_X X = aJX, & \nabla_{JX} X = -bJX + \alpha Z, & \nabla_Z X = cJX \\ \nabla_X JX = -aX - \alpha Z, & \nabla_{JX} JX = bX, & \nabla_Z JX = -cX \end{array} \tag{3.12}$$

for some smooth functions a, b and c.

Now we are ready to prove the following:

LEMMA 2. For the frame field $\{X, JX, Z\}$ we have

(i) $R(X, Z; Z, X) = \|H\|^2$

(ii) $R(X, JX; JX, X) = \bar{R}(X, JX: JX, X) + \|H\|^2$

(iii) $R(Z, JX, JX, Z) = \|H\|^2$

PROOF. Using equations (3.12) in the equation

$R(X, Z: Z, X) = g(\nabla_X \nabla_Z Z - \nabla_Z \nabla_X Z, -\nabla_{[X,Z]} Z, X)$, we obtain (i) and (iii). (ii) follows from the Gauss equation (2.6) and the equation $h(X, Y) = g(X, Y)H$.

PROOF OF THE THEOREM. Since $\bar{R}(X, JX: JX, X) > 0$ and $\|H\| \neq 0$ it follows from (i), (ii), and (iii) of Lemma 2 that all plane sections of M have strictly positive sectional curvature. Therefore, the Ricci-curvature of M is strictly positive. Hence by Hamilton's theorem (cf. [4]) it follows that M is diffeomorphic to either S^3, RP^3 or the lens space $L^3{}_{p,q}$.

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