Internat. J. Math. & Math. Sci.
VOL. 16 NO. 2 (1993) 283-288

APPROXIMATION BY FINITE RANK OPERATORS WITH RANGES IN c_o

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(Received July 25, 1991 and in revised form August 13, 1991)

ABSTRACT. In this paper the author characterizes all those spaces X, for which $K_n(X,c_o)$ is proximinal in $L(X,c_o)$. Some examples were found that satisfy this characterization.

KEY WORDS AND PHRASES. Proximinal, best approximation, selection, extremal subspaces, n-width.
1991 AMS SUBJECT CLASSIFICATION CODES. 46.

1. INTRODUCTION.

The closed subset A of the normed linear space X, is said to be "*proximinal*" in X if for each $x \in X$, there is an element $y_x \in A$, such that:

$$d(x,A) = inf\{(\| x-y \| ; y \in A\} = \| x - y_x \|,$$

where $d(x,A)$ is the distance of x from A. The element y_x is called a "*best approximation*" of x in A. The best approximation need not be unique, and the set-valued function $P_A: X \to 2^A$ defined by

$$P_A = \{(y \in A; d(x,A) = \| x-y \| \}$$

is called the *metric projection* of X into A. If A is proximinal in X then $P_A(x) \neq \phi$ for each $x \in X$, in this case any function $f: X \to A$ satisfying that $f(x) \in P_A(x)$ for each $x \in X$, is called a "*selection*" for the metric projection P_A.

If A is a subset of X, and N is a subspace of X, then the "*deviation*" of A from N is defined to be

$$\delta(A,N) = sup\{(d(x,N); x \in A\},$$

and the n-width of A in X is defined to be

$$d_n(A,X) = inf\{(\delta(A,N); N \text{ is an n-dimensional subspace of } X\}.$$

If there is an n-dimensional subspace N_o of X, such that $d_n(A,X) = \delta(A,N_o)$ then $d_n(A,X)$ is said to be "*attained*", and the subspace N_o is said to be an "*extremal subspace*" for $d_n(A,X)$. It is well known (see Garkavi [4]), that if X^* is the dual space of the normed linear space X, then $d_n(A,X^*)$ is attained.

If X and Y are two normed linear spaces, then $L(X,Y)$ denotes the set of all bounded linear operators from X to Y, $K(X,Y)$ the set of all compact operators in $L(X,Y)$, and $K_n(X,Y)$ the subset of $K(X,Y)$ consisting of all operators of rank n.

The proximinality of $K(X,Y)$ in $L(X,Y)$ were studied by several authors, (see for examples Feder [3], Lau [8], Mach [9], Mach and Ward [10], and Saatkamp [11]). Duetsch, Mach, and

Saatkamp [1], Kamal ([5], [6], and [7]) studied the proximinality of $K_n(X,Y)$ in $L(X,Y)$ and $K(X,Y)$ in details, however, one of the problems left unsolved is the problem 5.2.2 mentioned by Duetsch. Mach and Saatkamp [1], concerning the proximinality of $K_n(X,c_o)$, in $L(X,c_o)$ where c_o is the space of all real sequences that converges to zero. The problem is divided into two parts, the first part is to characterize all those spaces X for which $K_n(K,c_o)$ is proximinal in $L(X,c_o)$, and the second part is to show whether $K_n(X,c_o)$ is proximinal in $L(X,c_o)$ or not, when$X = c$ or 1_∞. Kamal [7] showed that $K_n(c,c_o)$ is not proximinal in $L(c,c_o)$, given a partial solution for the second part of the mentioned problem. Deutsch, Mach, and Saatkamp [1] showed that if $X = c_o$ or if X^* is uniformly convex, then $K_n(X,c_o)$ is proximinal in $L(X,c_o)$, Kamal [6] showed that $K_n(1_1,c_o)$ is not proximinal in $L(\ell_1,c_o)$, also Kamal [7] showed that if Q is a compact Hausdorff space that contains an infinite convergent sequence, then $K_n(C(Q),c_o)$ is not proximinal in $L(C(Q),c_o)$. In this paper a theorem is proved to characterize all those spaces X, for which $K_n(X,c_o)$ is proximinal in $L(X,c_o)$, this characterization includes $X = c_o$, X for which X^* is uniformly convex, and X such that the metric projection P_N from X^* onto any of its n-dimensional subspaces N, has a selection which is ω^*-continuous at zero. A point worth mentioning is that although c_o is a one codimensional subspace of c, there are spaces X for which $K_n(X,c_o)$ is proximinal in $L(X,c_o)$, meanwhile $K_n(X,c)$ is not proximinal in $L(X,c)$, for example Deutsch, Mach and Saatkamp [1] showed that $K_n(c_o,c_o)$ is proximinal in $L(c_o,c_o)$, meanwhile Kamal [7] showed that $K_n(c_o,c)$ is not proximinal in $L(c_o,c)$.

The rest of introduction will cover some definitions, and known results that will be used later in Section 2.

If X is a normed linear space then $c_o(X^*,\omega^*)$ denotes the Banach space of all bounded sequences $\{x_i\}$ in X^* that converge to zero in the ω^*-topology induced on X^* by X, $c_o\ (X^*)$ is the Banach space of all sequences $\{x_i\}$ in X^* that converge to zero in the topology defined on X^* by its norm, and if $n \geq 1$ is any positive integer, then $c_o(X^*,n)$ denotes the union of all $c_o(N)$, where N is an n-dimensional subspace of X^*. The norm on $c_o(X^*,\omega^*)$ is the suprimum norm. If $\{x_i\}$ is an element in $c_o(X^*,\omega^*)$ then for any positive integer $n \geq 1$, define

$$a_n(\{x_i\}) = inf\{\,\|\{x_i\} - \{y_i\}\|\,;\{y_i\} \in c_o(X^*,n)\}$$

The following theorem can be obtained as a corollary, from the theorem of Dunford and Shwartz [2, p. 490].

THEOREM 1.1. Let X be normed linear space. The mapping $\alpha: L(X,c_o) \to c_o(X^*,\omega^*)$ defined by $\alpha(T)_i(x) = T(x)_i$ where $i = 1,2,\ldots$, and $x \in X$, is an isometric isomorphism. Furthermore $\alpha(K(X,c_o)) = c_o(X^*)$ and $\alpha(K_n(X,c_o)) = c_o(X^*,n)$.

As corollary of the Theorem 1.1, one can obtain the following:

COROLLARY 1.2. If X is a normed linear space then for any positive integer $n \geq 1$, the set $K_n(X,c_o)$ is proximinal in $L(X,c_o)$ (resp. $K_n(X,c_o)$) if and only if $c_o(X^*,n)$ is proximinal in $c_o(X^*,\omega^*)$ (resp. $c_o(X^*)$).

According to Corollary 1.2 to study the proximinality of $K_n(X,c_o)$ in $L(X,c_o)$ (resp. $K(X,c_o)$), it is enough to study the proximinality of $c_o(X^*,n)$ in $c_o(X^*,\omega^*)$ (resp. $c_o(X^*)$).

2. THE PROXIMINALITY OF $K_n(X,c_o)$ IN $L(X,c_o)$.

In this paper if $\{x_i\}$ is an element in $c_o(X^*,\omega^*)$, then $d_n(\{x_i\},X^*)$ (resp. $\delta(\{x_i\},N)$ denotes the n-width (resp. the deviation from N) of the subset $\{x_1,\ x_2,\ x_3,\ldots\}$ of X^*.

THEOREM 2.1 Let X be a normed linear space, and let $n \geq 1$ be any positive integer. If $\{x_i\}$ is a bounded sequence in X^* then

$$a_n(\{x_i\}) = max\{d_n(\{x_i\},\ X^*),\ \overline{lim}\,\|x_i\|\,\}.$$

Furthermore there is an n-dimensional subspace N_o of X^*, such that $a_n(\{x_i\}) = d(\{x_i\},\ c_o(N))$.

PROOF. First it will be shown that $a_n(\{x_i\}) \geq max\{d_n(\{x_i\}, N^*), \overline{lim} \| x_i \| \}$. By Garkavi [4], there is an n-dimensional subspace N_o of X^* such that $\delta(\{x_i\}, N) = d_n(\{x_i\}, X^*)$. For each $i = 1,2,\ldots,$ let z_i be a best approximation for x_i from N_o, and let $\varepsilon > 0$ be given, there is a positive integer $n \geq 1$ such that for each $i \geq m$, $\| x_i \| \leq \overline{lim} \| x_i \| + \varepsilon$. Define the sequence $\{y_i\}$ in $c_o(N_o)$ as follows.

$$y_i = \begin{cases} z_i & \text{if } i \leq m \\ 0 & \text{if } i > m. \end{cases}$$

Then

$$\begin{aligned} a_n(\{x_i\}) \leq \| \{x_i\} - \{y_i\} \| &= sup\{ \| x_i - y_i \| ; i = 1,2,\ldots, \} \\ &= max\{max\{ \| x_i = y_i \| ; i = 1,2,\ldots,m\},\ sup\{ \| x_i \| ; 1 = m+1, m+2, \ldots\}\} \\ &\leq max\{d_n(\{x_i\}, X^*), \overline{lim} \| x_i \| + \varepsilon\}. \end{aligned}$$

Since ε is arbitrary it follows that $a_n(\{x_i\}) \leq max\{d_n(\{x_i\}, X^*), \overline{lim} \| x_i \| \}$. Second to show that $a_n(\{x_i\} \geq max\{d_n(\{x_i\}, X^*), \overline{lim} \| x_i \| \}$, one should notice first that $a_n(\{x_i\}) \geq \overline{lim} \| x_i \|$, indeed if $\{y_i\} \in c_o(X^*, n)$ then

$$\| \{x_i\} - \{y_i\} \| = sup\{ \| x_i - y_i \| \} \geq \overline{lim} \| x_i - y_i \| = \overline{lim} \| x_i \|.$$

Let $\varepsilon > 0$ be given, there is an n-dimensional subspace N' of X^*, and a sequence $\{y_i\} \in c_o(N')$ such that $a_n(\{x_i\}) \geq \| \{x_i\} - \{y_i\} \| - \varepsilon$. Therefore

$$\| \{x_i\} - \{y_i\} \| = sup\{ \| x_i - y_i \| \} \geq sup d(x_i, N) = \delta(\{x_i\}, N') \geq d_n(\{x_i\}, X^*).$$

Hence $a_n(\{x_i\}) = d(\{x_i\}, X^*) - \varepsilon$, and since ε is arbitrary it follows that

$$a_n(\{x_i\}) \geq d_n(\{x_i\}, X^*).$$

To prove the fact that there is an n-dimensional subspace N of X^*, such that $a_n(\{x_i\}) = d_n(\{x_i\}, c_o(N))$, Let N be an extremal subspace for $d_n(\{x_i\}, X^*)$, and for each $i = 1,2,\ldots,$ let z_i be a best approximation for x_i from N. Let $\varepsilon > 0$ be given, and define the sequence $\{y_i\}$ in $c_o(N)$ as in the first part of the proof, then

$$\begin{aligned} \| \{x_i\} - \{y_i\} \| &= sup\{ \| x_i - y_i \| \} \\ &= max\{max\{ \| x_i - z_i \| ; i = 1,2,\ldots,m\}, sup\{ \| x_i \| ; i = m+1, m+2, \ldots\}\} \\ &\leq max\{\delta(\{x_i\}, N),\ \overline{lim} \| x_i \| + \varepsilon\} \\ &\leq max\{d_n(\{x_i\}, X^*), \overline{lim} \| x_i \| \} + \varepsilon \\ &= a_n(\{x_i\}) + \varepsilon. \end{aligned}$$

But ε is arbitrary so $d(\{x_i\}, c_o(N)) = a_n(\{x_i\})$.

THEOREM 2.2. Let X be a normed linear space. For any positive integer $n \geq 1$, $K_n(X, c_o)$ is proximinal in $K(X, c_o)$.

PROOF. Let $\{x_i\}$ be an element in $c_o(X^*)$, by Corollary 1.2, it is enough to find an element $\{y_i\}$ in $c_o(X^*, n)$ such that $\| \{x_i\} - \{y_i\} \| = a_n(\{x_i\})$. Since $\lim_{i \to \infty} \| x_i \| = 0$ it follows that $\overline{lim} \| x_i \| = 0$, thus by Theorem 2.1, $a_n(\{x_i\}) = d_n(\{x_i\}, X^*)$. Let N_o be an extremal subspace for $d_n(\{x_i\}, X^*)$, and for each $i = 1,2,\ldots,$ let y_i be a best approximation for x_i from N_o. Since $\lim_{i \to \infty} \| x_i \| = 0$, it follows that $\lim_{i \to \infty} \| y_i \| = 0$; that is, $\{y_i\} \in c_o(N_o)$. Thus

$$\| \{x_i\} - \{y_i\} \| = sup\{ \| x_i - y_i \| \} = \delta(\{x_i\}, N_o) = d_n(\{x_i\})X^*) = a_n(\{x_i\}).$$

LEMMA 2.3. Let X be a normed linear space, and let $\{x_i\}$ be a bounded sequence in X^*.

a) If $d_n(\{x_i\}, X^*) > \overline{lim} \| x_i \|$, then $a_n(\{x_i\})$ is attained.

b) If $d_n(\{x_i\},X^*) \leq \overline{lim} \| x_i \|$, and there is an extremal subspace N_o for $d_n(\{x_i\},X^*)$ such that $\overline{lim}\ d(x_i, N_o) < \overline{lim} \| x_i \|$, then $a_n(\{x_i\})$ is attained.

PROOF. a) Assume that N is an extremal subspace for $d_n(\{x_i\},X^*)$, and let $\alpha = d_n(\{x_i\},X^*) - \overline{lim} \| x_i \|$, then there is a positive integer $m \geq 1$ such that for each $i \geq m$, one has $\| x_i \| \leq \overline{lim} \| x_i \| + \alpha$. For each $i \leq m$, let z_i be a best approximation for x_i from N_o, and define the sequence $\{y_i\}$ in $c_o(N_o)$ as follows.

$$y = \begin{cases} z_i & \text{if } i \leq m \\ 0 & \text{if } i > m. \end{cases}$$

Then

$$\begin{aligned} \| \{x_i\} - \{y_i\} \| &= max\{max\{ \| x_i - z_i \| ; i = 1,2,...,m\},\ sup\{ \| x_i \| ; i = m+1, m+2, ...,\}\} \\ &\leq max\{\delta(\{x_i\},N_o), \overline{lim} \| x_i \| + \alpha\} \\ &= d_n(\{x_i\},X^*) = a_n(\{x_i\}). \end{aligned}$$

b) let $\alpha = \overline{lim} \| x_i \|, \beta = \overline{lim} d(x_i, N_o)$, and $\gamma = \alpha - \beta$. Then $\gamma > 0$.
Let $\{\varepsilon_i\}$ be a sequence of positive real numbers, satisfying that $\lim_{i \to \infty} \varepsilon_i = 0$, for each $i = 1,2,..., d(x_i, N_o) \leq \beta + \varepsilon_i$ and for each $i = 1,2,...,\ \| x_i \| \leq \alpha + \varepsilon_i$. For each $i = 1,2,...,$ let z_i be a best approximation for x_i from N_o, and define the sequence $\{y_i\}$ in N_o as follows,

$$y_i + \begin{cases} z_i & \text{if } \varepsilon \geq \gamma \\ \frac{\varepsilon_i}{\gamma} \cdot z_i & \text{if } \varepsilon_i < \gamma. \end{cases}$$

Since $\{z_i\}$ is a bounded sequence in N_o, and $\lim_{i \to \infty} \varepsilon_i = 0$, it follows that $\{y_i\} \in c_o(N_o)$.
Furthermore for each $i = 1,2,...,$ if $\varepsilon_i > \gamma$ then

$$\| x_i - y_i \| \leq d(x_i, N_o) \leq d_n(\{x_i\},X^*) \leq a(\{x_i\}),$$

and if $\varepsilon_i < \gamma$ then

$$\begin{aligned} \| x_i - y_i \| &\leq (1 - \frac{\varepsilon_i}{\gamma}) \| x_i \| + \frac{\varepsilon_i}{\gamma} \| x_i - z_i \| \\ &\leq (1 - \frac{\varepsilon_i}{\gamma})(\alpha + \varepsilon_i) + \frac{\varepsilon_i}{\gamma}(\alpha - \gamma + \varepsilon_i) \\ &= \alpha = a_n(\{x_i\}). \end{aligned}$$

Thus $\| \{x_i\} - \{y_i\} \| = a_n(\{x_i\})$.
Lemma 2.4 is a continuation for Lemma 2.3.

LEMMA 2.4. Let X be a normed space, and let $\{x_i\}$ be a bounded sequence in X^*. Assume that $d_n(\{x_i\},X^*) = \overline{lim} \| x_i \|$, and for each extremal subspace N for $d_n(\{x_i\},X^*)$ one has $\overline{lim}\ d(x_i, N) = \overline{lim} \| x_i \| = \alpha$. Let N be a extremal subspace for $d_n(\{x_i\},X^*)$, and for each $i = 1,2,...,$ define

$$\varepsilon_i = \begin{cases} 0 & \text{if } \| x_i \| \leq \alpha \\ \| x_i \| - \alpha & \text{if } \| x_i \| > \alpha \end{cases}, \qquad \delta_i = \alpha - d(x_i, N_o), \text{ and } \alpha_i = \begin{cases} 0 & \text{if } \varepsilon_i + \delta_i = 0 \\ \frac{\varepsilon_i}{\varepsilon_i + \delta_i} & \text{if } \varepsilon_i + \delta_i \neq 0. \end{cases}$$

If $\lim_{i \to \infty} \alpha_i = 0$ then $a_n(\{x_i\})$ is attained.

PROOF. Let z_i be a best approximation for x_i from N_o, and let $y_i = \alpha_i,\ z_i$, then the sequence $\{y_i\}$ is an element in $c_o(N_o)$. Furthermore for each $i = 1,2,...,$

$$\begin{aligned} \| x_i - y_i \| &\leq (1 - \alpha_i) \| x_i \| + \alpha_i \| x_i - z_i \| \\ &\leq (1 - \alpha_i)(\alpha + \varepsilon_i) + \alpha_i(\alpha - \delta_i) \\ &= \alpha + \varepsilon_i - \alpha_i(\varepsilon_i + \delta_i). \end{aligned}$$

If $\alpha_i = 0$ then $\varepsilon_i = 0$ so $\| x_i - y_i \| = \alpha$, and if $\alpha_i \neq 0$ then

$$\| x_i - y_i \| \leq \alpha + \varepsilon_i - \frac{\varepsilon_i}{\varepsilon_i + \delta_i}(\varepsilon_i + \delta_i) = \alpha.$$

DEFINITION 2.5. Let X be a normed linear space. The bounded sequence $\{x_i\}$ in $c_o(X^*, \omega^*)$ is said to be an "*n-border*" sequence if it satisfies the following,

1. $\lim_{i\to\infty} \| x_i \|$ exists, and for each extremal subspace N for $d_n(\{x_i\}, X^*)$, one has

$$\overline{\lim}\, d(x_i, N) = \lim_{i\to\infty} \| x_i \| = d_n(\{x_i\}, X^*).$$

2. For each extremal subspace N for $d_n(\{x_i\}, X^*)$ if ε_i, δ_i and α_i as in Lemma 2.4 then $\overline{\lim}\, \alpha_i > 0$.

THEOREM 2.6. Let X be a normed linear space, and let $n \geq 1$ be a positive integer. Then $K_n(X, c_o)$ is proximinal in $L(X, c_o)$ if and only if for each n-border sequence $\{x_i\}$ in X^*, $a_n(\{x_i\})$ is attained.

PROOF. If there is an n-border sequence $\{x_i\}$ in X^* such that $a_n(\{x_i\})$ is not attained, then since $\{x_i\} \in c_o(X^*, \omega^*)$, it follows by Corollary 1.2 that $K_n(X, c_o)$ is not proximinal in $L(X, c_o)$. To prove the other part, let $\{x_i\}$ be an element in $c_o(X^*, \omega^*)$. If $d_n(\{x_i\}, X^*) > \overline{\lim} \| x_i \|$, or if $d_n(\{x_i\}, X^*) \leq \overline{\lim} \| x_i \|$ and there is an extremal subspace N for $d_n(\{x_i\}, X^*)$, such that $\lim d(x_i, N) < \overline{\lim} \| x_i \|$ then by Lemma 2.3, $a_n(\{x_i\})$ is attained.
Assume that $d_n(\{x_i\}, X^*) = \overline{\lim} \| x_i \|$, and for each extremal subspace N for $d_n(\{x_i\}, X^*)$, one has $\overline{\lim}\, d(x_i, N) = \overline{\lim} \| x_i \|$, let ε_i, δ_i and α_i be as in Lemma 2.4. If there is an extremal subspace N for $d_n(\{x_i\}, X^*)$ such that $\lim_{i\to\infty} \alpha_i = 0$ then by Lemma 2.4, $a_n(\{x_i\})$ is attained. Therefore one may assume that for any extremal subspace N for $d_n(\{x_i\}, X^*)$ one has $\overline{\lim}\, \alpha_i > 0$. Let $\alpha = d_n(\{x_i\}, X^*)$ and let $\{x_{i_k}\}$ be the largest subsequence of $\{x_i\}$ satisfying that $\| x_{i_k} \| > \alpha$ for each i_k. Thus for each i, if x_i is not an element in $\{x_{i_k}\}$ then $\| x_i \| \leq \alpha$. The sequence $\{x_{i_k}\}$ is an n-border sequence in X^*, so there is an n-dimensional subspace N of X^*, and a sequence $\{z_{i_k}\} \in c_o(N)$ such that $\| \{x_{i_k}\} - \{z_{i_k}\} \| = a_n(\{x_{i_k}\}) = \alpha$.
Define the sequence $\{y_i\}$ in N as follows.

$$y_i = \begin{cases} z_i & \text{if } \| x_i \| > \alpha \\ 0 & \text{if } \| x_i \| \leq \alpha. \end{cases}$$

Then $\{y_i\} \in c_o(N)$, and $\| \{x_i\} - \{y_i\} \| = \alpha = a_n(\{x_i\})$.

COROLLARY 2.7. Let X be a normed linear space, and let $n \geq 1$ be a positive integer. If X^* is uniformly convex then $K_n(X, c_o)$ is proximinal in $L(X, c_o)$.

PROOF. Let $\{x_i\}$ be an n-border sequence in X^*, and let $\alpha = \lim_{i\to\infty} \| x_i \|$. Without loss of generality assume that $x_i \neq 0$ for each i. Let N be any extremal subspace for $d_n(\{x_i\}, X^*)$, and let y_i be the best approximation for x_i from N. Since $\| \frac{x_i}{\| x_i \|} \| = 1$, $\frac{\| x_i - y_i \|}{\alpha} \leq 1$, and

$$\lim_{i\to\infty} \| \frac{x_i}{\| x_i \|} + \frac{x_i - y_i}{\alpha} \| = \lim_{i\to\infty} \left(\frac{\alpha + \| x_i \|}{\alpha \| x_i \|}\right) \| x_i - \frac{y_i}{\alpha + \| x_i \|} \| \geq \lim_{i\to\infty} \left(\frac{\alpha + \| x_i \|}{a \| x_i \|}\right) \| x_i - y_i \| = 2.$$

It follows by the fact that X^* is uniformly convex that $\lim_{i\to\infty} \| \frac{x_i}{\| x_i \|} - \frac{x_i - y_i}{\alpha} \| = 0$. But then $\lim_{i\to\infty} y_i = 0$, so $\{y_i\} \in c_o(N)$ and $\| \{x_i\} - \{y_i\} \| = a_n(\{x_i\})$.
Corollary 2.7 was proved by Deutsch, Mach, and Saatkamp [1] in a different way.

COROLLARY 2.8. Let X be a normed linear space, and let $n \geq 1$ be a positive integer. If for each n-dimensional subspace N of X^*, the metric projection P_n has a selection which is ω^*-continuous at zero, then $K_n(X, c_o)$ is proximinal in $L(X, c_o)$.

PROOF. Let $\{x_i\}$ be an element in $c_o(X^*, \omega^*)$ and let N be an extremal subspace for $d_n(\{x_i\}, X^*)$. Since the metric projection P_N has a selection which is ω^*-continuous at zero, it follows that there is a sequence $\{y_i\}$ in N, satisfying that $y_i \in P_N(x_i)$ for each i, and that $\{y_i\}$

converges ω^*-to zero. But N is of finite dimension, thus $\{y_i\} \in c_o(N)$.
Furthermore

$$\| \{x_i\} - \{y_i\} \| = \delta(\{x_i\}, N) = d_n(\{x_i\}, X^*) = a_n(\{x_i\}).$$

From Corollary 2.8 one concludes that for each positive integer $n \geq 1$, if $X = c_o$ or $l_p, l < p < \infty$, then $K_n(X, c_o)$ is proximinal in $L(X, c_o)$. Proposition 2.9 clarify that. The fact that $K_n(c_o, c_o)$ is proximinal in $L(c_o, c_o)$ was proved first by Deutsch, Mach, and Saatkamp [1].

PROPOSITION 2.9. Let $n \geq 1$ be a positive integer and let $X = c_o$ or $l_p, l < p < \infty$. The metric projection P_N from X^* onto any of its n-dimensional subspace N, has a selection which is ω^*-continuous at zero.

PROOF. Let N be any n-dimensional subspace of X^*, $\{x_i\}$ be any bounded sequence in X^* that converges ω^*-to zero, and let $\{y_i\}$ by any sequence in N, satisfying that $y_i \in P_N(x_i)$ for each i. It will be shown that $\{y_i\} \in c_o(N)$. The sequence $\{y_i\}$ is a bounded sequence in a finite dimensional subspace of X^*, so it has a convergent subsequence $\{y_{i_k}\}$ that converges to y_o in N, it will be shown that $y_o = 0$. Assume not, and without loss of generality assume that $\{y_i\}$ converges to y_o, and that $X^* = l_p, l \leq p < \infty$. Let $t_i = x_i - (y_i - y_o)$, $r_i = x_i - y_i$, and let $\varepsilon > 0$ be such that $\varepsilon < \| y_o \|^p$, then as in Proposition 3 of Mach [9], there is a positive integer $m \geq 1$ such that for each $i \geq m$ one has, $| \, \| t_i - y_o \|^p - \| t_i \|^p - \| y_o \|^p \, | < \varepsilon$, thus $\| t_i - y_o \|^p \geq \| t_i \|^p + \| y_o \|^p - \varepsilon$, that is

$$\| x_i - y_i \|^p \geq \| x_i - (y_i - y_o) \|^p + \| y_o \|^p - \varepsilon > \| x_i - (y_i - y_o) \|^p.$$

So for each $i > m$ one has $\| x_i - (y_i - y_o) \| > \| x_i - y_i \|$, which contradict the fact that $\| x_i - y_i \| = d(x_i, N)$, therefore $y_o = 0$.

REFERENCES

1. DEUSTCH, F., MACH, J., & SAATKAMP, K., Approximation by finite rank operators, J. Approx. Theory 33 (1981), 199-213.
2. DUNFORD, N., & SCHWARTZ, J., Linear Operators I, Wiley Interscience, New York, 1958.
3. FEDER, A., On certain subsets of $L_1[0,1]$ and nonexistence of best approximation in some spaces of operators, J. Approx. Theory 29 (1980), 170-177.
4. GARKAVI, A., The best possible net and the best possible cross section of a set in a normed linear spaces, Amer. Math. Soc. Transl. 39 (1964), 111-132.
5. KAMAL, A., On proximinality and sets of operators I, Best approximation by finite rank operators, J. Approx. Theory 47 (1986), 132-145.
6. KAMAL, A., On proximinality and sets of operators II, nonexistence of best approximation from the sets of finite rank operators, J. Approx. Theory 47 (1986), 146-155.
7. KAMAL, A., On proximinality and sets of operators III, approximation by finite rank operators on spaces of continuous functions, J. Approx. Theory 47 (1986), 156-171.
8. LAU, K., Approximation by continuous vectors valued functions, Studia, Math. 68 (1980), 291-298.
9. MACH, J., On the proximinality of compact operators with ranges in $C(S)$, Proc. Am. Math. Soc. 72 (1978), 99-104.
10. MACH, J. & WARD, J., Approximation by compact operators on certain Banach spaces, J. Approx. Theory 23 (1978), 274-286.
11. SAATKAMP, K., Best approximation in the space of bounded operators and applications, Math. Ann. 25 (1980), 35-54.