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REAL HYPERSURFACES OF INDEFINITE KAEHLER MANIFOLDS

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ABSTRACT. We show the existence of (ϵ)-almost contact metric structures and give examples of (ϵ)-Sasakian manifolds. Then we get a classification theorem for real hypersurfaces of indefinite complex space-forms with parallel structure vector field. We prove that (ϵ)-Sasakian real hypersurfaces of a semi-Euclidean space are either open sets of the pseudosphere S_{2s}^{2n+1} (1) or of the pseudohyperbolic space $H_{2s-1}^{2n+1}(1)$. Finally, we get the causal character of (ϵ)-cosymplectic real hypersurfaces of indefinite complex space-forms.

KEYWORDS. (ϵ)-Sasakian manifolds, real hypersurfaces, indefinite complex space-forms, (ϵ)-cosymplectic real hypersurfaces, globally hyperbolic space.

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0. INTRODUCTION. Indefinite Kahler manifolds have been introduced by Barros-Romero [1]. Because of the signature of the metric we expect some essential changes in the study of submanifolds in such spaces. Some new results on this matter are obtained in the present paper.

Our purpose is first to investigate the induced structures on real hypersurfaces of an indefinite Kahler manifold and then to study some particular classes of such structures. Thus in the first section we introduce (ϵ)-Sasakian manifolds which enclose the class of usual Sasakian manifolds. It has to be noted that in the definition of an (ϵ)-Sasakian manifold it is essential that the causal character of the characteristic vector field of the structure is preserved. We close this section with examples of (ϵ)-Sasakian structures on R^{2n+1} . As far as we know till now, Takahashi [9] and Duggal [5] have been concerned with Sasakian manifolds with indefinite metric.

In section 2 we define an (ϵ)-almost contact metric structure on a real hypersurface of an indefinite Kahler manifold and obtain its principal properties. The next two sections are concerned with two classes of such structures on real hypersurfaces: (ϵ)-Sasakian and (ϵ)-cosymplectic structures. In section 3 we show that both the pseudosphere $S_{2s}^{2n+1}(1)$ and the pseudohyperbolic space $H_{2s-1}^{2n+1}(1)$ are examples of space-like Sasakian manifolds and time-like Sasakian manifolds respectively.

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1. (ϵ)-SASAKIAN MANIFOLDS. Let M be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact structure (f, ξ, η). This means that f is a tensor field of type (1,1), ξ is a vector field and η is a 1-form on M satisfying

$$f^2 = -I + \eta \otimes \xi \,;\; \eta(\xi) = 1. \tag{1.1}$$

It follows that

$$\eta \circ f = 0 \,;\; f(\xi) = 0 \,;\;\; \text{rank} \quad f = 2n. \tag{1.2}$$

We then say that M is an almost contact manifold (see Blair [4]).

The manifold M is supposed to be paracompact and differentiable of class C^∞. Denote $F(M)$ the algebra of real differentiable functions on M and by $\Gamma(TM)$ the $F(M)$-module of differentiable vector fields on M. The same notation is used for the set of sections of a vector bundle over M or over any other manifold.

Throughout the paper, by a semi-Riemannian metric on M we understand a non-degenerate symmetric tensor field g of type (0,2), (cf. O'Neill [8]). We now suppose on M there exists a semi-Riemannian metric g (see Duggal [5]) that satisfies

$$g(fX, fY) = g(X,Y) - \epsilon\eta(X)\eta(Y), \forall X, Y \epsilon \Gamma(TM) \tag{1.3}$$

where $\epsilon = \pm 1$. It follows that

$$\eta(X) = \epsilon g(X, \xi), \;\; \forall X \epsilon \Gamma(TM) \tag{1.4}$$

and

$$\epsilon = g(\xi, \xi). \tag{1.5}$$

Hence ξ is never a light-like vector field on M. This implies that the contact distribution $D = \{X \in \Gamma(TM), \eta(X) = 0\}$ is always non-degenerate on M. Moreover, the index of g is an odd number $\nu = 2r+1$ in case ξ is time-like and an even number $\nu = 2r$ otherwise. This follows as a consequence of the fact that on M we may consider an orthonormal field frame $\{E_1, \dots, E_n, fE_1, \dots, fE_n, \xi\}$ with $E_i \in \Gamma(D)$ and such that $g(E_i, E_i) = g(fE_i, fE_i)$.

We are now concerned with the existence of semi-Riemannian metrics satisfying (1.3). In the particular case $\epsilon = 1$ and $\nu = 0$ there exists a Riemannian metric g satisfying (1.3) and M is the usual almost contact metric manifold (cf. Blair [4]). For the general case, following Blair [4], and subject to the above mentioned restrictions of the index of g, we have the following result.

THEOREM 1. Let (f, ξ, η) be an almost contact structure and h_0 be a semi-Riemannian metric on M such that ξ is not a light-like vector field. Then there exists on M a symmetric tensor field g of type (0,2) satisfying (1.3)

PROOF. We first define two semi-Riemannian metrics $h_1 = -\frac{\epsilon}{\alpha} h_0$, where $\alpha = h_0(\xi, \xi)$ and

$$h(X,Y) = h_1(f^2X, f^2Y) + \epsilon\eta(X)\eta(Y), \forall X, Y \in \Gamma(TM).$$

In order to prove that h is a semi-Riemannian metric we first note that

$$\eta(X) = \epsilon h(X, \xi) \, and \; h(\xi, \xi) = \epsilon.$$

Then denote by $\{\xi\}$ the distribution spanned by ξ on M and by D_1 the complementary orthogonal distribution to $\{\xi\}$ with respect to h_1. Then for any $X \in \Gamma(D_1)$, we have

$$h(X,X) = h_1(-X+\eta(X)\xi, -X+\eta(X)\xi) + \epsilon\eta(X)^2 = h_1(X,X),$$

since $h_1(X,\xi) = 0$ and $h_1(\xi,\xi) = -\epsilon$. Thus h is a semi-Riemannian metric on M of the same index as h_1 on D_1 . Finally, we define the symmetric tensor field

$$g(X,Y) = \frac{1}{2}\{h(X,Y) + h(fX,fY) + \epsilon\eta(X)\eta(Y)\}$$

and we have

$$\begin{aligned} g(fX,fY) &= \frac{1}{2}\{h(fX,fY) + h(-X+\eta(X)\xi, -Y+\eta(Y)\xi)\} \\ &= g(X,Y) - \epsilon\eta(X)\eta(Y), \end{aligned}$$

as desired.

Therefore, in general, the above theorem does not provide us a semi-Riemannian metric on M satisfying (1.3). However, we may prove the existence of Lorentz metrics satisfying (1.3).

COROLLARY 1. Let (f,ξ,η) be an almost contact structure on M . Then there exists a Lorentz metric g on M satisfying (1.3) with $\epsilon = -1$.

PROOF. Since M is paracompact there exists a Riemannian metric h_0 on M . We define h_1, h and g as in Theorem 1 with $\epsilon = -1$. Then it is easy to see that both h and g are Lorentz metrics on M . Besides, g satisfies (1.3) with $\epsilon = -1$.

We call (f,ξ,η,g) satisfying (1.1) and (1.3) an (ϵ)-almost contact metric structure and M an (ϵ)-almost contact metric manifold. Thus we have the following new classes of manifolds.

1 $\epsilon = 1$ and $\nu = 2r$. M is called a space-like almost contact metric manifold.

2 $\epsilon = -1$ and $\nu = 2r+1$. M is called a time-like almost contact metric manifold.

An important subclass of the second class is the Lorentz almost contact manifold ($\epsilon = -1$, $\nu = 1$),recently studied by the second author (see Duggal [5]). As ξ is globally defined, following the terminology of Duggal [5] and the definition of space-time (see Beem-Ehrlich [2]) a time orientable Lorentz almost contact manifold will be called a contact space-time. Here for the sake of completeness, we state the following result (proved in Duggal [5]) on contact space-times.

THEOREM 2. (Duggal[5]). For an (ϵ)-almost contact metric manifold M, the following are equivalent:

(1) M is contact space-time .

(2) The characteristic vector field ξ is time-like and the 2n-dimensional contact distribution $(D,f,g/D)$ is space-like.

Next, we consider the fundamental 2-form Φ of the (ϵ)-almost contact metric structure defined by

$$\Phi(X,Y) = g(X,fY), \forall X,Y \in \Gamma(TM) \tag{1.6}$$

Then we say that (f,ξ,η,g) is an (ϵ)-contact metric structure if we have

$$\Phi(X,Y) = d\eta(X,Y), \qquad \forall X,Y \in \Gamma(TM). \tag{1.7}$$

In this case M is an (ϵ)-contact metric manifold. Besides we recall that the almost contact structure (f,ξ,η) is normal if

$$[f,f] + 2d\eta \otimes \xi = 0, \tag{1.8}$$

where $[f,f]$ is the Nijenhuis tensor field associated to f. An (ϵ)-contact metric structure which is normal is called an (ϵ)-Sasakian structure. A manifold endowed with an (ϵ)-Sasakian structure is called an (ϵ)-Sasakian manifold. As in the case of Riemannian Sasakian manifolds we have.

THEOREM 3. An (ϵ)-almost contact metric structure (f,ξ,η,g) is (ϵ)-Sasakian if and only if

$$(\nabla_X f)Y = g(X,Y)\xi - \epsilon\eta(Y)X, \ \forall X,Y \in \Gamma(TM) \tag{1.9}$$

where ∇ is the Levi-Civita connection with respect to g .

If we replace Y by ξ in (1.9) we get

$$\nabla_X\xi = -\epsilon fX, \forall X \in \Gamma(TM). \tag{1.10}$$

Thus, we have:

COROLLARY 2. The characteristic vector field ξ on an (ϵ)-Sasakian manifold is a Killing vector field.

Sasakian manifolds with indefinite metrics have been first considered by Takahashi [9]. Their importance for physics has been pointed out by one of the present authors (see Duggal [5]).

According to the causal character of ξ we have two new classes of (ϵ)-Sasakian manifolds. Thus in case ξ is space-like ($\epsilon = 1$ and $\nu = 2r$), (resp. time-like, $\epsilon = -1$ and $\nu = 2r+1$) we say that M is a space-like Sasakian manifold (resp. time-like Sasakian manifold). In case $\epsilon = 1$ and $\nu = 0$ we get the well-known concept of Riemannian Sasakian manifold. Certainly for physics it is important to consider Lorentz metrics. In this case $\epsilon = -1$, $\nu = 1$ and we call M a Lorentz-Sasakian manifold or a Sasakian-spacetime (cf. Duggal [5]).

As Takahashi [9] pointed out, from a space-like Sasakian structure (f,ξ,η,g,ϵ) we always get a time-like Sasakian structure $(f',\xi',\eta',g',\epsilon')$, where $f' = f$, $\xi' = -\xi, \eta' = -\eta, g' = -g, \epsilon' = -\epsilon$ and vice versa. However, taking into account that the causal character of ξ determines one or another structure we shall consider the general case of (ϵ)-Sasakian structures.

We close the section with some examples of (ϵ)-Sasakian structures on R^{2n+1}. Other examples we shall give in section 3.

First we make the following notations:

$O_{p,k}$ = the $p \times k$ null matrix ; I_k = the $k \times k$ unit matrix. For any non-negative integer $s \leq n$ we put

$$\epsilon^a = \begin{cases} -1 \text{ for } a\epsilon\{1,\dots,s\} \\ \ \ 1 \text{ for } a \in \{s+1,\dots,n\}' \end{cases}, \quad \text{in case } s \neq 0 \,,$$

and $\epsilon^a = 1$ in case $s = 0$.

Then we consider (x^i, y^i, z), $i = 1,\dots,n$ as cartesian coordinates on R^{2n+1} and define with respect to the natural field of frames $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial z}\right\}$ a tensor field f of type (1,1) by its matrix.

$$[f] = \begin{bmatrix} 0_{n,n} & I_n & 0_{n,1} \\ -I_n & 0_{n,n} & 0_{n,1} \\ 0_{1,n} & \epsilon^a y^a & 0 \end{bmatrix} \tag{1.11}$$

The differential 1-form η is defined by

$$\eta = \frac{\epsilon}{2}\left\{dz + \sum_{i=1}^{s} y^i dx^i - \sum_{i^*=r+1}^{n} y^{i*} dx^{i*}\right\}, \tag{1.12}$$

if $s \neq 0$, and

$$\eta = \frac{\epsilon}{2}\left\{dz - \sum_{i=1}^{n} y^i dx^i\right\}, \text{ if } s = 0. \tag{1.13}$$

The vector field ξ is defined for each s by

$$\xi = 2\epsilon\frac{\partial}{\partial z}. \tag{1.14}$$

It is easy to check (1.1) and thus (f,ξ,η) is an almost contact structure on R^{2n+1} for each $s \in \{0,1,\ldots,n\}$. Finally, we define the semi-Riemannian metric g by the matrix

$$[g] = \frac{\epsilon}{4}\begin{bmatrix} -\delta_{ij} + y^i y^j & -y^i y^{j*} & 0_{s,s} & 0_{s,n-s} & y^i \\ -y^i y^{j*} & \delta_{i^*j^*} + y^{i*}y^{j*} & 0_{n-s,s} & 0_{n-s,n-s} & -y^{i*} \\ 0_{s,s} & 0_{s,n-s} & -I_s & 0_{s,n-s} & 0_{s,1} \\ 0_{n-s,s} & 0_{n-s,n-s} & 0_{n-s,s} & I_{n-s} & 0_{n-s,1} \\ y^i & -y^{i^*} & 0_{1,s} & 0_{1,n-s} & 1 \end{bmatrix} \tag{1.15}$$

for $s \neq 0$, and

$$[g] = \frac{\epsilon}{4}\begin{bmatrix} \delta_{ij} + y^i y^j & 0_{n,n} & y^i \\ 0_{n,n} & I_n & 0_{n,1} \\ y^i & 0_{1,n} & 1 \end{bmatrix}, \tag{1.16}$$

with respect to the natural field of frames. In order to help the reader to see the right form of [g] we write it down for $n = 4$ and $s = 1$:

$$[g] = \frac{\epsilon}{4}\begin{bmatrix} -1+(y^1)^2 & -y^1y^2 & -y^1y^3 & -y^1y^4 & 0 & 0 & 0 & 0 & y^1 \\ -y^1y^2 & 1+(y^2)^2 & y^2y^3 & y^2y^4 & 0 & 0 & 0 & 0 & -y^2 \\ -y^1y^3 & y^2y^3 & 1+(y^3)^2 & y^3y^4 & 0 & 0 & 0 & 0 & -y^3 \\ -y^1y^4 & y^2y^4 & y^3y^4 & 1+(y^4)^2 & 0 & 0 & 0 & 0 & -y^4 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ y^1 & -y^2 & -y^3 & -y^4 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

An orthonormal field of frames with respect to the semi-Riemannian metric (1.15) is

$$\left\{\begin{array}{c} E_i = 2\dfrac{\partial}{\partial y^i}, E_{i*} = 2\dfrac{\partial}{\partial y^{i*}};\ fE_i = 2\left(\dfrac{\partial}{\partial x^i} - y^i\dfrac{\partial}{\partial z}\right), \\ fE_{i^*} = 2\left(\dfrac{\partial}{\partial x^i} + y^i\dfrac{\partial}{\partial z}\right), \xi \end{array}\right\}$$

It is easy to check that (f,ξ,η,g) given by (1.11)-(1.16) is an (ϵ)-Sasakian structure on R^{2n+1} for any $s \in \{0,1,\ldots,n\}$. In case $s = 0$ and $\epsilon = 1$ we obtain the classical Sasakian structure on R^{2n+1} (see Blair [4]). In other cases we get either a space-like Sasakian structure on R^{2n+1}_{2s} $(\epsilon = 1,\ s \neq 0)$ or a time-like Sasakian structure on $R^{2n+1}_{2(n-s)+1}(\epsilon = -1, s \neq 0)$.

The Lorentz-Sasakian structure is obtained from the latter for $s = n$.

PHYSICAL EXAMPLE. First we need the following information (for details see [2,8]. Let M be a spacetime manifold, with a Lorentz metric g of signature $(-,+,\ldots,+)$. A spacetime M is called globally hyperbolic if M is a product manifold of the form $(M = R\times S, g = -dt^2 + G)$ with (S,G) a compact Riemannian manifold. Recently the second author, Duggal [5], has proved the following physical result, also valid for Sasakian structures.

THEOREM 4 (Duggal [5]). An odd dimensional globally hyperbolic spacetime can carry a Lorentz-Sasakian structure.

Well known examples are Minkowski-spacetime, Lorentz spheres and Robertson-Walker spacetime [2,8].

In another direction, physically, Corollary 2 of Theorem 3 is important for the special case of Sasakian spacetimes since ξ is a Killing vector field. The existence of Killing vector fields in spacetimes has often been used as the most effective symmetry. In fact, many exact solutions of Einstein field equations have been found by assuming one or more Killing vector fields (Kramer-Stephani-Herlt [6]).

2. REAL HYPERSURFACES OF INDEFINITE KAHLER MANIFOLDS.

Let $\tilde{M}$ be a real $2(n+1)$-dimensional manifold. Suppose $\tilde{M}$ is endowed with an almost complex structure $\tilde{J}$ and a semi-Riemannian metric $\tilde{g}$ satisfying

$$\tilde{g}(\tilde{J}X,\tilde{J}Y)=\tilde{g}(X,Y),\forall X,Y\in\Gamma(T\tilde{M}). \tag{2.1}$$

It follows that the index of $\tilde{g}$ is an even number $\nu=2(r+1)$. Then we say that $\tilde{M}$ is an indefinite almost Hermitian manifold. Moreover, if on $\tilde{M}$ we have

$$(\tilde{\nabla}_X\tilde{J})Y=0, \text{ for any } X,Y\in\Gamma(T\tilde{M}), \tag{2.2}$$

where $\tilde{\nabla}$ is the Levi-Civita connection with respect to $\tilde{g}$, we say that $\tilde{M}$ is an indefinite Kahlerian manifold (see Barros-Romero [1]).

Now suppose M is an orientable non-degenerate real hypersurface of $\tilde{M}$. Let N be the normal unit vector field of M . Thus by (2.1) and taking account of the orientability of M we see that $\xi=-\tilde{J}N$ is a vector field tangent to M. Then the equations of Gauss and Weingarten are given by

$$\tilde{\nabla}_XY=\nabla_XY+h(X,Y)N,\forall X,Y\in\Gamma(TM), \tag{2.3}$$

and

$$\tilde{\nabla}_XN=-AX,\forall X\in\Gamma(TM), \tag{2.4}$$

respectively, where ∇ is the Levi-Civita connection with respect to the semi-Riemannian metric g induced by $\tilde{g}$ on M , A is the shape operator of M and h is a symmetric tensor field of type (0,2) on M . Suppose now $\tilde{g}(N,N)=\epsilon$ and by (2.1) we have $g(\xi,\xi)=\epsilon$. Then from (2.3) and (2.4) we get

$$h(X,Y)=\epsilon g(AX,Y),\forall X,Y\in\Gamma(TM).$$

Hence (2.3) becomes

$$\tilde{\nabla}_XY=\nabla_XY+\epsilon g(AX,Y)N,\forall X,Y\in\Gamma(TM). \tag{2.5}$$

We now denote by $\{\xi\}$the distribution spanned by ξ on M and by D the complementary orthogonal distribution to $\{\xi\}$ in TM. Certainly D is invariant by $\tilde{J}$ and the distribution $\{\xi\}$ is carried by $\tilde{J}$ into the normal bundle. Thus any real hypersurface of an indefinite Kahler manifold is an example of a CR-submanifold (see Bejancu [3]). The projection morphism of TM to D is then denoted by P . Hence any vector field X on M is written as follows

$$X=PX+\eta(X)\xi \tag{2.6}$$

where η is a 1-form on M defined by

$$\eta(X)=\epsilon g(X,\xi). \tag{2.7}$$

Thus we have

$$\eta(\xi)=1. \tag{2.8}$$

Further, we define a tensor field f on M by

$$fX = \tilde{J}PX, \forall X \in \Gamma(TM). \tag{2.9}$$

Then taking account that D is invariant by $\tilde{J}$ we get

$$f^2X = -X + \eta(X)\xi. \tag{2.10}$$

Moreover, by using (2.1), (2.7) and (2.9) we get

$$g(fX, fY) = g(X,Y) - \epsilon\eta(X)\eta(Y), \forall X, Y \in \Gamma(TM). \tag{2.11}$$

Hence, we obtain

PROPOSITION 1. An orientable non-degenerate real hypersurface of an indefinite almost Hermitian manifold of index $\nu = 2r$ inherits an (ϵ)-almost contact metric structure (f, ξ, η, g). Moreover, we have

PROPOSITION 2. The (ϵ)-almost contact metric structure on M immersed in an indefinite Kahlerian manifold $\tilde{M}$ satisfies

$$(\nabla_X f)Y = \eta(Y)AX - \epsilon g(AX, Y)\xi, \tag{2.12}$$

$$(\nabla_X \eta)Y = \epsilon g(fAX, Y), \tag{2.13}$$

and

$$\nabla_X \xi = fAX, \tag{2.14}$$

for any $X, Y \in \Gamma(TM)$.

PROOF. By direct calculations in (2.2) using (2.4) and (2.5) we obtain (2.12) and (2.13). Then we replace Y in (2.12) by ξ and obtain (2.14).

From Proposition 2 we easily obtain

COROLLARY 3. Let M be as in Proposition 2. Then the following assertions are equivalent

(i) f is parallel on M

(ii) η is parallel on M

(iii) ξ is parallel on M

(iv) The shape operator satisfies

$$AX = \eta(AX)\xi, \forall X \in \Gamma(TM). \tag{2.15}$$

We now recall from general theory of hypersurfaces in semi-Riemannian manifolds that the Gauss and Codazzi equations are given by

$$\begin{aligned} g(\tilde{R}(X,Y)Z, W) = g(R(X,Y)Z, W) + g(AX, Z)g(AY, W) \\ - g(AY, Z)g(AX, W), \end{aligned} \tag{2.16}$$

and

$$g(\tilde{R}(X,Y)Z, N) = g(\nabla_X A)Y - (\nabla_Y A)X, Z), \tag{2.17}$$

respectively, for any $X, Y, Z, W \in \Gamma(TM)$, where $\tilde{R}$ and R are the curvature tensor fields of $\tilde{M}$ and M respectively. On the other hand, we recall (see Barros-Romero [1]) that the curvature tensor field of an indefinite complex-space form $\tilde{M}(c)$ is given by

$$\begin{aligned} \tilde{R}(X,Y)Z = \frac{c}{4}\Big\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y + \tilde{g}(\tilde{J}Y,Z)\tilde{J}X - \tilde{g}(\tilde{J}X,Z)\tilde{J}Y + \Big\} \\ + 2\tilde{g}(X,\tilde{J}Y)\tilde{J}Z \; \} \end{aligned} \tag{2.18}$$

for any $X, Y, Z \in \Gamma(T\tilde{M})$.

Then we have

THEOREM 5. Let M be a connected real hypersurface with $dimM \geq 3$ of an indefinite complex space-form $\tilde{M}(c)$ satisfying one of the assertions of Corollary 3. Then $c = 0$ and M is a semi-Euclidean space.

PROOF. We replace Z by PZ in (2.17) and by using (2.15) and assertion (iii) of Corollary 3, we obtain $g(\tilde{R}(X,Y)PZ,N) = 0$. Then from (2.18), taking account of (2.6) we get

$$\frac{c}{4}\left\{\tilde{g}(PZ,\tilde{J}Y)\tilde{g}(X,\xi) - \tilde{g}(PZ,\tilde{J}X)\tilde{g}(Y,\xi)\right\} = 0, \forall X,Y,Z \in \Gamma(TM)$$

which implies

$$\frac{c\epsilon}{4}\tilde{g}(\tilde{J}PZ,Y) = 0 \tag{2.19}$$

Suppose now $c \neq 0$ and from (2.19) we get $PZ = 0$ for any $Z \in \Gamma(TM)$, which contradicts the hypothesis $dimM \geq 3$. Thus $c = 0$ and by using (2.15) in (2.16) we obtain $R = 0$ which completes the proof.

3. (ϵ)-SASAKIAN REAL HYPERSURFACES OF AN INDEFINITE KAHLER MANIFOLD.

First we obtain the following theorem of characterization for (ϵ)-Sasakian real hypersurfaces of indefinite Kahler manifolds.

THEOREM 6. Let M be an orientable real hypersurface of an indefinite Kahler manifold $\tilde{M}$. Then the following assertions with respect to the (ϵ)-almost contact metric structure inherited by M are equivalent:

(i) M is an (ϵ)-Sasakian manifold,

(ii) The (ϵ)-characteristic vector field satisfies (1.10).

(iii) The shape operator satisfies

$$AX = -\epsilon X + (\epsilon + \eta(A\xi))\eta(X)\xi,\ \forall X \in \Gamma(TM). \tag{3.1}$$

PROOF. $(i) \Rightarrow (ii)$ was shown in section 1.

$(ii) \Rightarrow (iii)$. By using (1.10) and (2.14) we get

$$PAX = -\epsilon PX,\ \forall X \in \Gamma(TM).$$

Hence by (2.6) we have

$$AX = -\epsilon PX + \eta(AX)\xi,\ \forall X \in \Gamma(TM). \tag{3.2}$$

From (3.2) follows

$$A\xi = \eta(A\xi)\xi. \tag{3.3}$$

Finally, taking account that A is a symmetric operator with respect to g and by using (1.4), (2.6) and (3.3) in (3.2) we obtain (3.1).

$(iii) \Rightarrow (i)$. Replace AX from (3.1) in (2.12) and obtain (1.9).

In order to state the next result we recall (see O'Neill [8]), the definitions of hyperspheres and pseudohyperbolic spaces in semi-Euclidean spaces. Consider the semi-Euclidean space $R^{2(n+1)}_{2s}$ with the indefinite Kahlerian structure (cf. Barros-Romero [1]). The pseudosphere of radius $r > 0$ in $R^{2(n+1)}_{2s}$ is the hyperquadric

$$S^{2(n+1)}_{2s}(r) = \left\{x \in R^{2(n+1)}_{2s};\ g(x,x) = r^2\right\},$$

of dimension $(2n+1)$ and index $2s$. In a similar way, the pseudohyperbolic space of radius $r > 0$ in $R_{2s}^{2(n+1)}$ is the hyperquadric

$$H_{2s-1}^{2n+1}(r) = \left\{ x \in R_{2s}^{2(n+1)};\, g(x,x) = -r^2 \right\},$$

of dimension $(2n+1)$ and index $(2s-1)$.

We now state

THEOREM 7. Let M be an (ϵ)-Sasakian connected real hypersurface of $R_{2s}^{2(n+1)}$. Then M is an open set either of $S_{2s}^{2n+1}(1)$ or of $H_{2s-1}^{2n+1}(1)$.

PROOF. Since $R_{2s}^{2(n+1)}$ is a flat indefinite complex space form, from (2.17) we obtain

$$(\nabla_X A)Y - (\nabla_Y A)X = 0,\ \forall X, Y \in \Gamma(TM) \tag{3.4}$$

Next, from (3.1), we get

$$AX = -\epsilon X,\ \forall X \in \Gamma(D), \text{ and } (3.3). \tag{3.5}$$

Then we take $X \in \Gamma(D)$ and $Y = \xi$ in (3.4) and by using (3.5), (3.3) and taking into account that $\nabla_X \xi$ and $\nabla_\xi X$ belong to the contact distributions D we obtain

$$\eta(A\xi) = -\epsilon \tag{3.6}$$

Hence by using (3.6) in (3.1) we get

$$AX = -\epsilon X,\ \forall X \in \Gamma(TM). \tag{3.7}$$

Therefore M is a totally umbilical hypersurface (but not totally geodesic) with normal curvature $k = -\epsilon$. Hence by Lemma 35 and Proposition 36 from O'Neill [8], p.116, we obtain that M has constant curvature ϵ and it is an open set of $S_{2s}^{2n+1}(1)$ when $\epsilon = 1$ and an open set of $H_{2s-1}^{2n+1}(1)$ when $\epsilon = -1$.

Suppose now M is a totally umbilical real hypersurface of $\tilde{M}$, that is, $A = \rho I$, where ρ is a differentiable function and I is the identity on $\Gamma(TM)$. Then we first state

THEOREM 8. A real hypersurface of $R_{2s}^{2(n+1)}$ is (ϵ)-Sasakian if and only if it is totally umbilical and $\rho = -\epsilon$.

PROOF. The first part of the assertion follows from the proof of Theorem 7. Suppose now M is totally umbilical with $\rho = -\epsilon$. Then $A\xi = -\epsilon\xi$ and thus $\eta(A\xi) = -\epsilon$. Hence (3.1) is satisfied and this completes the proof.

REMARK 1. Tashiro [10] has constructed the Sasakian structure on a sphere of a Euclidean space and Takahashi [9], by a different approach than ours, obtained the (ϵ)-Sasakian structure on $S_{2s}^{2n+1}(1)$ and $H_{2s-1}^{2n+1}(1)$.

Now suppose M is a totally umbilical real hypersurface of an indefinite complex space form $\tilde{M}(c)$. Then we get

$$g\left((\nabla_X A)Y - (\nabla_Y A)X, \xi\right) = 0,\ \forall X, Y \in \Gamma(D). \tag{3.8}$$

On the other hand, from (2.18) we get

$$g(\tilde{R}(X,Y)\xi, N) = \frac{c\epsilon}{2}\, g(X, fY), \quad \forall X, Y \in \Gamma(D). \tag{3.9}$$

Hence from (3.8) and (3.9), taking account of (2.17) we obtain $c = 0$, which enable us to state

PROPOSITION 3. There exist no totally umbilical real hypersurfaces in an indefinite complex space form of non-null holomorphic sectional curvature.

Tashiro-Tachibana [11] first obtained such a result for positive definite complex space forms.

4. COSYMPLECTIC REAL HYPERSURFACES OF INDEFINITE KAHLER MANIFOLDS.

Suppose as in the previous section M is an orientable real hypersurface of an indefinite $2(n+1)$-dimensional Kahler manifold $\tilde{M}$. Then we say that the (ϵ)-almost contact metric structure (f,ξ,η,g) induced on M defines an (ϵ)-cosymplectic structure if both the 1-form η and the fundamental 2-form Φ given by (1.6) are closed. M is then called an (ϵ)-cosymplectic hypersurface. Therefore on M we have

$$d\eta(X,Y)=0, \qquad \text{and} \tag{4.1}$$

$$d\Phi(X,Y,Z)=0, \text{ for any } X,Y,Z\in\Gamma(TM). \tag{4.2}$$

If we express (4.2) by means of the Levi-Civita connection we obtain

$$d\Phi(X,Y,Z)=\frac{1}{3}\left\{g(X,(\nabla_Z f)Y)+g(Y,(\nabla_X f)Z)+g(Z,(\nabla_Y f)X)\right\}. \tag{4.3}$$

Then using (2.12) and (2.7) in (4.3) by direct calculations it follows that (4.2) is always satisfied on M . Hence M is an $(\epsilon)-$cosymplectic manifold if and only if (4.1) is satisfied. Furthermore

PROPOSITION 4. M is an (ϵ)-cosymplectic hypersurface if and only if the shape operator satisfies

$$A\circ f+f\circ A=0. \tag{4.4}$$

The proof follows from (4.1) taking into account that

$$d\eta(X,Y)=\frac{1}{2}\left\{(\nabla_X\eta)Y-(\nabla_Y\eta)X\right\}, \forall X,Y\in\Gamma(TM)$$

and by using (2.11) and (2.13). From this proposition we infer

COROLLARY 4. Let M be an (ϵ)-cosymplectic real hypersurface of an indefinite Kahler amnifold $\tilde{M}$. Then we have

(i) ξ is a principal curvature vector field,
(ii) the trajectories of ξ are geodesics.

PROOF. Apply (4.4) to ξ and obtain $PA\xi=0$. Hence by (2.6) we get

$$A\xi=\alpha\xi, \quad \alpha=\eta(A\xi) \tag{4.5}$$

which means that ξ is a principal curvature vector. The second assertion follows from (4.4) by using (2.14).

REMARK 2. (4.5) follows from (3.1). Hence the first assertion of Corollary 4 also holds for (ϵ)-Sasakian real hypersurfaces.

With respect to the existence of (ϵ)-cosymplectic real hypersurfaces immersed in complex space forms such that their shape operators have real eigenvalues, we obtain

THEOREM 9. Let M be an (ϵ)-cosymplectic real hypersurface of an indefinite complex space form $\tilde{M}_{2s}^{2(n+1)}(c)$ such that the shape operator A has only real eigenvalues. Then

(1) If $c=0$, then M is a semi-Euclidean space.
(2) If $c\neq 0$, then we have
(a) $c=4$, and M should be time-like
(b) $c=-4$,and M should be space-like

Moreover, in the last two cases, M has at most three principal curvatures.

PROOF. By direct calculations taking account of (1.5), (2.13), (2.14) and (4.5), we get

$$g(\xi,(\nabla_X A)Y)+g(AfAX,Y)=\epsilon X(\alpha)\eta(Y)+2g(fAX,Y), \tag{4.6}$$

for any $X,Y\in\Gamma(TM)$. On the other hand, from Codazzi equation (2.17) taking account of (2.18) we obtain

$$g((\nabla_X A)Y-(\nabla_Y A)X,\xi)=\frac{c\epsilon}{2}g(X,fY),\forall X,Y\in\Gamma(TM), \tag{4.7}$$

Then taking account of (4.4) we see that (4.6) and (4.7) imply

$$g(X,\frac{c\epsilon}{2}fY-2AfAY)=\epsilon\{X(\alpha)\eta(Y)-Y(\alpha)\eta(X)\}. \tag{4.8}$$

Take now $X=\xi$ in (4.8) and obtain

$$Y(\alpha)=\xi(\alpha)\eta(Y),\quad \forall Y\in\Gamma(TM), \tag{4.9}$$

which together with (4.4) and (4.8) imply

$$\frac{c\epsilon}{4}Y+A^2Y=0,\quad \forall Y\in\Gamma(D). \tag{4.10}$$

As we have seen in Corollary 4, ξ is a principal curvature vector field of M . Suppose now $Z\in\Gamma(TM)$ is another principal curvature vector field of M and $\lambda\in\mathbf{R}$ is the corresponding principal curvature. Then by using (2.6) and (4.5) we get

$$APZ-\lambda PZ+\eta(Z)(\alpha-\lambda)\xi=0. \tag{4.11}$$

But taking account that A is a symmetric operator with respect to g and using again (4.5) we obtain

$$g(APZ,\xi)=g(PZ,A\xi)=\alpha g(PZ,\xi)=0,$$

which together with (4.11) implies

$$APZ=\lambda PZ. \tag{4.12}$$

We now replace Y from (4.10) by PZ and obtain

$$\frac{c\epsilon}{4}+\lambda^2=0. \tag{4.13}$$

In case $c=0$ we then have $\lambda=0$ and thus $AY=0$ for each $Y\in\Gamma(D)$ since the eigen distribution of A with respect to this eigenvalue is just D . Further, by using (2.6), (2.7) and (4.5) we obtain

$$g(AX,Z)=\epsilon\alpha\eta(X)\eta(Z),\quad \forall X,Z\in\Gamma(TM). \tag{4.14}$$

Then taking account of (4.14) in (2.16) we infer $R(X,Y)Z=0$ for any $X,Y,Z\in\Gamma(TM)$. Hence we have the assertion 1 of the theorem. The assertion 2 follows from (4.13) taking into account that the eigenvalues of A are supposed to be real.

COROLLARY 5. Let M be either a space-like cosymplectic real hypersurface of an indefinite complex space-form of positive holomorphic sectional curvature or a time-like cosymplectic real hypersurface of an indefinite complex space-form of negative holomorphic sectional curvature. Then the shape operator of M has at least two eigenvalues which are not real.

REMARK 3. In the case of cosymplectic real hypersurfaces of positive definite space forms, important results have been obtained by Okumura [7].

Next by (4.1) we see that the distribution D is involutive on an (ϵ)-cosymplectic real hypersurface M . Moreover, in case 2 of Theorem 9 by using (4.4) we derive that A has eigenvalues $(+1)$ and (-1) with the same multiplicity n . Denote by D^+ and D^- the eigen distributions with respect to the above eigenvalues. Further, take $X,Y\in\Gamma(D^+)$, $Z\in\Gamma(D)$ and from (2.17) we get

$$g([X,Y]-A([X,Y]),Z)=0.$$

On the other hand, by using (4.5) and taking into account that D is involutive, we obtain

$$g([X,Y]-A([X,Y]),\xi)=0.$$

Hence $A([A,X])=[X,Y]$, which says that D^+ is involutive. In a similar way, it follows that D^- is involutive too.

Suppose now that M^+ is a leaf of D^+ and denote h^+ and $\tilde{h}^+$ the second fundamental forms of immersions of M^+ in M and $\tilde{M}(c)$ respectively. Then for any $X,Y\in\Gamma(TM^+)$ we have

$$\tilde{\nabla}_X Y=\nabla^+_X Y+h^+(X,Y)+\epsilon g(X,Y)N,$$

and

$$\tilde{\nabla}_X Y=\nabla^+_X Y+\tilde{h}^+(X,Y),$$

where ∇^+ is the Levi-Civita connection on M^+. Thus we have

PROPOSITION 5. Let M^+ be a leaf of D^+ which is totally geodesic immersed in M . Then M^+ is totally umbilical immersed in $\tilde{M}(c)$.

Certainly such a result holds for leaves of D^-too.

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