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PROJECTION SOLUTIONS OF FROBENIUS-PERRON OPERATOR EQUATIONS

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ABSTRACT. We construct in this paper the first order and second order piecewise polynomial finite approximation schemes for the computation of invariant measures of a class of nonsingular measurable transformations on the unit interval of the real axis. These schemes are based on the Galerkin's projection method for L^1-spaces and are proved to be convergent for the class of Frobenius-Perron operators.

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1. INTRODUCTION.

Many problems in physics and engineering are concerned with the mathematical problem of existence and computation of invariant measures of measurable transformations on measure spaces [2]. For a class of nonsingular measurable transformations from $[0,1]$ into itself. Lasota and Yorke [3] established the existence of the invariant measures. Specifically, if $S:[0,1]\to[0,1]$ is a piecewise C^2 stretching mapping, then the "time average" $\frac{1}{n}\sum_{k=0}^{n-1} P^k f$ converges strongly in $L^1(0,1)$ to some f^* of bounded variation with $Pf^* = f^*$ for any density function $f \in L^1(0,1)$. Here P is the Frobenius-Perron operator associated with S. In [5], Li and Yorke gave a sufficient condition for the uniqueness of this invariant density and thus the ergodicity of the mapping.

A straightforward numerical way to calculate the invariant measures can be obtained from the classical Birkhoff's Individual Ergodic Theorem which uses the Koopman operator instead of the Frobenius-Perron operator. By Birkhoff's theorem, if μ is an ergodic invariant probability measure under S, then for any measurable set $A \subset [0,1]$, the time average

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \chi_A\,(S^k(x)),$$

which measures the "average time" spent in A under iteration of S, exists and is $\mu(A)$ for μ-almost all x. Here χ_A is the characteristic function of $A(=1$ on A and $=0$ off $A)$. Hence, one might choose almost any x in $[0,1]$ and calculate the average time of the iterations $S^k(x)$ recurring in A to obtain $\mu(A)$. However, computer round-off error can completely dominate the calculation and make the implementation difficult. A typical example is given in [4]; and for the purpose of overcoming this difficulty, Li proposed in [4] a rigorous numerical procedure which can be implemented on a computer with negligible round-off error problem. The piecewise constant approximation is used to reduce the original infinite-dimensional fixed point problem to the fixed point problems of stochastic matrices, thus solving a conjecture of Ulam's [8].

The numerical procedure proposed by Li has two features. The first one is that it actually belongs to the category of Galerkin's projection method, although originally it was obtained by the probability analysis. The second one is that it uses a sequence of finite rank Markov operators to approximate the Frobenius-Perron operator. Since Frobenius-Perron operators are also Markov operators, this kind of approach is natural from the viewpoint of both theory and practice. In this paper, we propose the first order and second order piecewise polynomial approximation schemes for the computation of fixed points of Frobenius-Perron operators, based on the Galerkin's projection method. The numerical experiments of Kohda and Murao [1] based on a general piecewise polynomial Galerkin approximation scheme show that the first order and second order piecewise polynomial approximation methods are much more efficient than the piecewise constant approximation in [4]. But in order to prove the convergence of their methods, the boundedness as well as the uniqueness of the invariant density are assumed which makes the convergence analysis much easier. In [1] the convergence theorem is stated and its proof is based on the variation analysis of best piecewise polynomial approximations to functions of bounded variation under the L^2-norm along with the fact that every orthogonal projection in a Hilbert space has the operator norm 1. Without the assumption of boundedness of invariant densities, we show in this paper with different approach that our scheme is convergent in L^1, which is not a Hilbert space, for the class of nonsingular measurable transformations satisfying the condition of the Lasota-Yorke theorem. The method of finite Markov approximations will be developed in further studies.

The paper is organized as follows. In Section 2, we outline some background material. Sections 3 and 4 are devoted to the first order and second order piecewise polynomial projection approximation methods and their convergence analysis, respectively.

2. FROBENIUS-PERRON OPERATORS AND PROJECTION METHOD.

Let $I=[0,1]$ and S be a transformation from I into itself. For $A \subset [0,1]$, we write $S^{-1}(A)$ for $\{x: S(x) \in A\}$. The Lebesgue measure on $[0,1]$ will be denoted by m. Denote by $L^1(0,1)$ the space of all integrable functions defined on $[0,1]$ with respect to the Lebesgue measure m. $L^1(0,1)$ is a Banach space with norm $\|f\| = \int_0^1 |f(x)|\,dx$. Let $S:[0,1]\to[0,1]$ be a nonsingular measurable transformation, i.e., for any measurable subset A of $[0,1]$ with $m(A)=0$, we have $m(S^{-1}(A))=0$.

The operators $P_S: L^1(0,1)\to L^1(0,1)$ defined by

$$\int_A (P_S f)(x)dx = \int_{S^{-1}(A)} f(x)dx$$

is called the Frobenius-Perron operator associated with S. If there is no ambiguity, we shall write P for P_S. By Radon-Nikodym Theorem, the Frobenius-Perron operator is well defined when S is nonsingular [2]. For $f \in L^1(0,1)$, the measure $\mu(A) \equiv \mu_f(A) = \int_A f(x)dx$ is invariant under S if and only if $P_S f = f$. Here the invariance of measure μ (under S) means $\mu(S^{-1}(A)) = \mu(A)$ for every measurable set A. Thus, to calculate the invariant measure for S, we may calculate instead the fixed point of the Frobenius-Perron operator. More precisely, we need $f \in L^1(0,1)$ which satisfies $P_S f = f$.

We list some basic properties of Frobenius-Perron operator P_S without proof. For more detailed discussion of P_S, see [2].

(1) P_S is a Markov operator, i.e., P_S is linear and maps (positive) density functions to (positive) density functions. Thus $\| P_S \| = 1$.

(2) For the n-th iterate $S^n, P_{S^n} = (P_S)^n$.

We give here a brief description of Galerkin's projection method in Banach spaces (see [6] for more details). Let X be a Banach space. Suppose M and N are both closed subspaces of X. If $X = M + N$ and $M \cap N = \{0\}$, then we say X is a direct sum of M and N and M and N are complementary to each other. In this case, we may define a linear operator $Q: X \to X$ as follows,

$$Qx = u \text{ if } x = u + v, u \in M, v \in N.$$

This operator is continuous and satisfies $Q^2 = Q$. We call Q the projection of X onto M along N.

Now, let X and Y be two Banach spaces, $T: X \to Y$ be a bounded linear operator from X to Y, and $y \in Y$. We want to solve the operator equation

$$Tx = y$$

The general idea of projection method is as follows. Choose two sequences of finite-dimensional subspaces X_n and Y_n of X and Y, respectively. Let $\{Q_n\}$ be a sequence of projections from Y to Y_n. In X_n we want to find $x^{(n)}$ such that $Q_n(Tx^{(n)} - y) = 0$, or

$$Q_n T x^{(n)} = Q_n y.$$

If we choose a basis of X_n and a basis of Y_n, then the above approximate operator equation of finite rank can be written as a system of algebraic equations. Thus we can use the usual numerical methods to solve the linear equation and obtain the approximate solutions to the original problem. This numerical procedure is referred to as the projection method. In particular, if $X = Y$ and if we choose $X_n = Y_n$ and the same basis in Y_n as in X_n, then the corresponding projection method is called the Galerkin's method.

3. PIECEWISE LINEAR PROJECTION APPROXIMATION OF FROBENIUS-PERRON OPERATOR.

A transformation $S:[0,1] \to [0,1]$ will be called piecewise C^2, if there exists a partition $0 = a_0 < a_1 < \cdots < a_r = 1$ of the unit interval such that for each integer $k = 1, \ldots, r$, the restriction S_k of S on the open interval (a_{k-1}, a_k) is a C^2-function which can be extended to the closed interval $[a_{k-1}, a_k]$ as a C^2-function. S need not be continuous at the point a_k.

Assume $S:[0,1] \to [0,1]$ satisfies the condition of Lasota-Yorke theorem. That is, S is piecewise C^2 satisfying $inf\,|S'(x)| > 1$. In this section, we look for the approximate solutions of Frobenius-Perron operator equation $P_S f = f$ in the space of piecewise linear functions.

Divide $I = [0,1]$ into n subintervals $I_1, I_2, \ldots I_n$. For $i = 1, \ldots, n$, let $I_i = [x_{i-1}, x_i]$ and $1_i = \chi_{I_i}/m(I_i)$. Denote by Δ_n the $2n$-dimensional subspace of $L^1(0,1)$ spanned by the basis $\{1_i, x1_i\}_{i=1}^n$, i.e., $\Delta_n \subset L^1(0,1)$ is the set of all functions which are linear on each subinterval I_i.

Define $Q_n: L^1(0,1) \to \Delta_n$ by requiring that for $i = 1, \ldots, n$,

$$< f - Q_n f, 1_i > = 0$$

and

$$< f - Q_n f, x1_i > = 0.$$

Here for $g \in L^1(0,1)$ and $h \in L^\infty(0,1) = [L^1(0,1)]^*$, $< g,h > = \int_0^1 g(x)h(x)dx$. The following lemma shows that these requirements uniquely defines Q_n and implies that Q_n is a projection from $L^1(0,1)$ to Δ_n along ${}^\perp\Delta_n \equiv \{g \in L^1(0,1): < g,h > = 0 \text{ for all } h \in \Delta_n\}$. Because of the similarity in the "orthogonality condition" with the L^2-space case, we may call $Q_n: L^1(0,1) \to \Delta_n$ the orthogonal projection, even though its norm may not be 1.

LEMMA 3.1. Let $\tilde{x}_i = (x_{i-1} + x_i)/2, i = 1,...,n$. For any $f \in L^1(0,1)$, we have

$$Q_n f = \sum_{i=1}^n (c_i + d_i x)1_i$$

where for $i = 1,...,n$,

$$\begin{cases} c_i = \int_{I_i} f(x)dx - \dfrac{12\tilde{x}_i}{m(I_i)^2} \int_{I_i} (x - \tilde{x}_i)f(x)dx \\ d_i = \dfrac{12}{m(I_i)^2} \int_{I_i} (x - \tilde{x}_i)f(x)dx. \end{cases} \tag{1}$$

PROOF. Let $Q_n f = \sum_{i=1}^n (c_i + d_i x)\, 1_i$, then

$$< Q_n f,\ 1_i > = c_i < 1_i,\ 1_i > + d_i < x1_i,\ 1_i > = \frac{1}{m(I_i)}c_i + \frac{\tilde{x}_i}{m(I_i)}d_i$$

$$< Q_n f,\ x1_i > = c_i < 1_i,\ x1_i > + d_i < x1_i,\ x1_i >$$

$$= \frac{\tilde{x}_i}{m(I_i)}\, c_i + \frac{x_i^2 + x_i x_{i-1} + x_{i-1}^2}{3m(I_i)}\, d_i.$$

From the condition of the orthogonal projection, we have

$$\begin{cases} \dfrac{1}{m(I_i)}\, c_i + \dfrac{\tilde{x}_i}{m(I_i)}\, d_i = \dfrac{1}{m(I_i)} \int_{I_i} f(x)dx \\ \dfrac{\tilde{x}_i}{m(I_i)}\, c_i + \dfrac{x_i^2 + x_i x_{i-1} + x_{i-1}^2}{3m(I_i)}\, d_i = \dfrac{1}{m(I_i)} \int_{I_i} xf(x)dx. \end{cases} \tag{2}$$

The equation (2) has a unique solution

$$\begin{cases} c_i = \int_{I_i} f(x)dx - \dfrac{12\tilde{x}_i}{m(I_i)^2} \int_{I_i} (x - \tilde{x}_i)f(x)dx \\ d_i = \dfrac{12}{m(I_i)^2} \int_{I_i}(x - \tilde{x}_i)f(x)dx. \end{cases} \qquad \text{(Q.E.D.)}$$

The next lemma establishes the uniform boundedness of the sequence Q_n.

LEMMA 3.2. For all n, $\| Q_n \| \le 2$.

PROOF. Given n and $f \in L^1(0,1)$,

$$\| Q_n f \| = \int_0^1 |(Q_n f)(x)|\, dx = \int_0^1 \sum_{i=1}^n |(c_i + d_i x)1_i(x)|\, dx$$

$$= \sum_{i=1}^n \int_{I_i} \frac{1}{m(I_i)} |c_i + d_i x|\, dx.$$

By (1), on the subinterval I_i, $Q_n f$ only depends on the value of f on I_i. Hence it is enough to estimate one of $\frac{1}{m(I_i)}\int_{I_i} |c_i + d_i x|\, dx$. Without loss of generality, assume $d_i \ne 0$. For simplicity, let $I = I_i = [a,b]$, $\tilde{x} = \tilde{x}_i$, $c = c_i$, $d = d_i$ and f be defined on I. Let $\varphi(x) = (c + dx)/m(I)$.

First of all, let us assume $f \geq 0$. If $\varphi \geq 0$, then from the first equality of (2),

$$\int_I |\varphi(x)|dx = \int_I \varphi(x)dx = \frac{1}{m(I)}\int_a^b (c+dx)dx$$

$$= \frac{1}{m(I)} \frac{(c+dx)^2}{2d}\Big|_a^b = \frac{1}{2dm(I)} [(c+db)^2-(c+da)^2]$$

$$= \frac{1}{2dm(I)} [2cdm(I) + d^2m(I)\cdot 2\tilde{x}] = c + d\tilde{x}$$

$$= \int_I f(x)dx = \int_I |f(x)|dx.$$

If $\varphi \not\geq 0$, then from the fact that φ is the best approximation to f among all linear functions on $[a,b]$ under L^2-norm if $f \in L^2(0,1)$, we see that φ cannot be non-positive. Therefore φ must have a zero $z = -\frac{c}{d}$ in (a,b). We assume $\varphi(b) > 0$ and $\varphi(a) < 0$. The other case can be treated similarly. Thus we have

$$\int_I |\varphi(x)|dx = \frac{1}{2}[(z-a)|\varphi(a)| + (b-z)|\varphi(b)|]$$

$$= \frac{1}{2}[(b+\frac{c}{d})\varphi(b) + (a+\frac{c}{d})\varphi(a)],$$

and,

$$b + \frac{c}{d} = \frac{m(I)^2\int_I f(x)dx}{12\int_I (x-\tilde{x})f(x)dx} + \frac{m(I)}{2},$$

$$a + \frac{c}{d} = \frac{m(I)^2\int_I f(x)dx}{12\int_I (x-\tilde{x})f(x)dx} - \frac{m(I)}{2},$$

$$\varphi(b) = \frac{1}{m(I)}(c+db) = \frac{1}{m(I)}[\int_I f(x)dx + \frac{6}{m(I)}\int_I (x-\tilde{x})f(x)dx],$$

$$\varphi(b) = \frac{1}{m(I)}(c+da) = \frac{1}{m(I)}[\int_I f(x)dx - \frac{6}{m(I)}\int_I (x-\tilde{x})f(x)dx].$$

Hence,

$$\int_I |\varphi(x)|dx = \frac{1}{2}\{(\frac{m(I)^2\int_I f(x)dx}{12\int_I (x-\tilde{x})f(x)dx} + \frac{m(I)}{2})$$

$$\cdot\frac{1}{m(I)}(\int_I f(x)dx + \frac{6}{m(I)}\int_I (x-\tilde{x})f(x)dx)$$

$$+(\frac{m(I)^2\int_I f(x)dx}{12\int_I (x-\tilde{x})f(x)dx} - \frac{m(I)}{2})$$

$$\cdot\frac{1}{m(I)}(\int_I f(x)dx - \frac{6}{m(I)}\int_I (x-\tilde{x})f(x)dx)\}$$

$$= \frac{m(I)[\int_I f(x)dx]^2}{12\int_I (x-\tilde{x})f(x)dx} + \frac{3}{m(I)}\int_I (x-\tilde{x})f(x)dx$$

$$\leq \frac{m(I)[\int_I f(x)dx]^2}{12\int_I (x-\tilde{x})f(x)dx} + \frac{3}{2}\int_I f(x)dx.$$

Note that $z = -\frac{c}{d} \in (a,b)$, we have

$$a < \tilde{x} - \frac{m(I)^2 \int_I f(x)dx}{12\int_I (x-\tilde{x})f(x)dx} < b.$$

It follows that

$$\frac{m(I)^2 \int_I f(x)dx}{12\int_I (x-\tilde{x})f(x)dx} < \tilde{x} - a = \frac{m(I)}{2}.$$

Therefore

$$\int_I |\varphi(x)|\,dx < \tfrac{1}{2}\int_I f(x)dx + \tfrac{3}{2}\int_I f(x)dx = 2\int_I f(x)dx.$$

For general $f \in L^1(I)$, write $f = f^+ - f^-$ where $f^+ = max\{f,0\}$ and $f^- = max\{-f,0\}$, and we have

$$\begin{aligned}\int_I |\varphi(x)|\,dx &= \int_I |Qf|\,dx = \int_I |Qf^+ - Qf^-|\,dx \\ &\le \int_I |Qf^+|\,dx + \int_I |Qf^-|\,dx \\ &\le 2\int_I f^+dx + 2\int_I f^-dx = 2\int_I |f|\,dx\end{aligned}$$

where $Q: L^1(I) \to Span\{1,x\}$ is the above mentioned orthogonal projection.

From the above estimate, we obtain

$$\begin{aligned}\|Q_nf\| &= \int_0^1 |(Q_nf)(x)|\,dx = \sum_{i=1}^n \int_{I_i} \frac{1}{m(I_i)}|c_i + d_ix|\,dx \\ &\le \sum_{i=1}^n 2\int_{I_i} |f(x)|\,dx = 2\int_0^1 |f(x)|\,dx = 2\|f\|,\end{aligned}$$

i.e., for all n, $\|Q_n\| \le 2$. (Q.E.D.)

LEMMA 3.3. When mesh $(\Delta_n) \equiv max\{m(I_i): 1 \le i \le n\} \to 0, Q_nf \to f$ for all $f \in L^1(0,1)$.

PROOF. Given $f \in L^1(0,1)$ and $\varepsilon > 0$, there exists a continuous function g such that $\|f - g\| < \varepsilon$. Now

$$\begin{aligned}\|Q_ng - g\| &= \sum_{i=1}^n \int_{I_i} |(Q_ng)(y) - g(y)|\,dy \\ &= \sum_{i=1}^n \int_{I_i} \left|\frac{1}{m(I_i)}(c_i + d_iy) - g(y)\right| dy \\ &= \sum_{i=1}^n \int_{I_i} \left|\frac{1}{m(I_i)}\int_{I_i} g(x)dx - \frac{12\tilde{x}_i}{m(I_i)^3}\int_{I_i}(x-\tilde{x}_i)g(x)dx \right. \\ &\qquad \left. + \left(\frac{12}{m(I_i)^3}\int_{I_i}(x-\tilde{x}_i)g(x)dx\right)y - g(y)\right| dy\end{aligned}$$

$$\leq \sum_{i=1}^{n} \int_{I_i} \left| \frac{1}{m(I_i)} \int_{I_i} g(x)dx - g(y) \right| dy$$

$$+ \sum_{i=1}^{n} \int_{I_i} \left| \frac{12}{m(I_i)^3} \left(\int_{I_i} (x-\tilde{x}_i)g(x)dx \right)(y-\tilde{x}_i) \right| dy$$

$$\leq \sum_{i=1}^{n} \int_{I_i} \frac{1}{m(I_i)} \left(\int_{I_i} |g(x)-g(y)|\, dx \right) dy$$

$$+ \sum_{i=1}^{n} \frac{12}{m(I_i)^3} \int_{I_i} |(x-\tilde{x}_i)g(x)|\, dx \cdot \int_{I_i} |y-\tilde{x}_i|\, dy.$$

Since g is uniformly continuous on $[0,1]$, when mesh (Δ_n) is sufficiently small, for any $x, y \in I_i, i = 1,\dots,n$, we have $|g(x)-g(y)| < \varepsilon$. Applying Hölder's inequality, we get

$$\|Q_n g - g\| \leq \sum_{i=1}^{n} \int_{I_i} \frac{1}{m(I_i)} \cdot m(I_i) \frac{\varepsilon}{2} dy$$

$$+ \sum_{i=1}^{n} \frac{12}{m(I_i)^3} \left[\int_{I_i} (x-\tilde{x}_i)^2 dx \right]^{1/2}$$

$$\cdot \left[\int_{I_i} g(x)^2 dx \right]^{1/2} \cdot \int_{I_i} |y-\tilde{x}_i|\, dy$$

$$\leq \sum_{i=1}^{n} \frac{\varepsilon}{2} m(I_i) + \sum_{i=1}^{n} \frac{12}{m(I_i)^3} \left\{ \left[\frac{(x-\tilde{x}_i)^3}{3} \right]_{x_{i-1}}^{x_i} \right\}^{1/2}$$

$$\cdot \left[\int_{I_i} g(x)^2 dx \right]^{1/2} \cdot \left[\frac{(m(I_i)/2)^2}{2} + \frac{(m(I_i)/2)^2}{2} \right]$$

$$= \frac{\varepsilon}{2} + \frac{\sqrt{3}}{2} \sum_{i=1}^{n} m(I_i)^{1/2} \left[\int_{I_i} g(x)^2 dx \right]^{1/2}.$$

For n sufficiently large, $(\int_{I_i} g(x)^2 dx)^{1/2} < \frac{\varepsilon}{\sqrt{3}}, i = 1,\dots,n$, hence,

$$\|Q_n g - g\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \sum_{i=1}^{n} m(I_i)^{1/2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Together with Lemma 3.2, for n sufficiently large,

$$\|Q_n f - f\| \leq \|Q_n f - Q_n g\| + \|Q_n g - g\| + \|g - f\|$$
$$\leq 2\|f-g\| + \varepsilon + \|f-g\| \leq 4\varepsilon.$$

This proves $lim_{n\to\infty} Q_n f = f$. (Q.E.D.)

The following result is the key to our convergence analysis.

LEMMA 3.4. For any $f \in L^1(0,1)$ of bounded variation and for all n

$$\bigvee_0^1 Q_n f \leq 13 \bigvee_0^1 f,$$

where $\vee_0^1 f$ is the variation of f on $[0,1]$ (for the definition of variation, see [7]).

PROOF. By definition, $Q_nf = \sum_{i=1}^n (c_i + d_i x)1_i$, where $\{c_i, d_i\}$ are given by (1). Since Q_nf is piecewise linear, its variation is given by

$$\bigvee_0^1 Q_nf = \sum_{i=1}^n 1\frac{1}{m(I_i)} |(c_i + d_i x_i) - (c_i + d_i x_{i-1})|$$

$$+ \sum_{i=1}^{n-1} \left| \frac{c_i + d_i x_i}{m(I_i)} - \frac{c_{i+1} + d_{i+1} x_i}{m(I_{i+1})} \right|$$

$$= \sum_{i=1}^n |d_i| + \sum_{i=1}^{n-1} \left| \frac{c_i}{m(I_i)} - \frac{c_{i+1}}{m(I_{i+1})} + \left(\frac{d_i}{m(I_i)} - \frac{d_{i+1}}{m(I_{i+1})} \right) x_i \right|$$

$$= \sum_{i=1}^n |d_i| + \sum_{i=1}^{n-1} \left| \frac{1}{m(I_i)} \int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})} \int_{I_{i+1}} f(x)dx \right.$$

$$+ \frac{12\tilde{x}_i}{m(I_{i+1})^3} \int_{I_{i+1}} (x - \tilde{x}_{i+1}) f(x)dx - \frac{12\tilde{x}_i}{m(I_i)^3} \int_{I_i} (x - \tilde{x}_i) f(x)dx$$

$$\left. + \frac{12x_i}{m(I_i)^3} \int_{I_i} (x - \tilde{x}_i) f(x)dx - \frac{12x_i}{m(I_{i+1})^3} \int_{I_{i+1}} (x - \tilde{x}_{i+1}) f(x)dx \right|$$

$$= \sum_{i=1}^n |d_i| + \sum_{i=1}^{n-1} \left| \frac{1}{m(I_i)} \int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})} \int_{I_{i+1}} f(x)dx \right.$$

$$+ \frac{12(\tilde{x}_{i+1} - x_i)}{m(I_{i+1})^3} \int_{I_{i+1}} (x - \tilde{x}_{i+1}) f(x)dx$$

$$\left. + \frac{12(x_i - \tilde{x}_i)}{m(I_i)^3} \int_{I_i} (x - \tilde{x}_i) f(x)dx \right|$$

$$= \sum_{i=1}^n |d_i| + \sum_{i=1}^{n-1} \left| \frac{1}{m(I_i)} \int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})} \int_{I_{i+1}} f(x)dx \right.$$

$$\left. + \frac{6}{m(I_{i+1})^2} \int_{I_{i+1}} (x - \tilde{x}_{i+1}) f(x)dx + \frac{6}{m(I_i)^2} \int_{I_i} (x - \tilde{x}_i) f(x)dx \right|.$$

From the definition of d_i we have

$$\bigvee_0^1 Q_nf \le \sum_{i=1}^n |d_i| + \sum_{i=1}^{n-1} \left| \frac{1}{m(I_i)} \int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})} \int_{I_{i+1}} f(x)dx \right|$$

$$+ \sum_{i=1}^{n-1} \left| \frac{1}{2} d_{i+1} + \frac{1}{2} d_i \right|$$

$$\le \sum_{i=1}^n |d_i| + \sum_{i=1}^{n-1} \left| \frac{1}{m(I_i)} \int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})} \int_{I_{i+1}} f(x)dx \right|$$

$$+ \sum_{i=1}^n |d_i|.$$

It is easy to see that the middle summation of the above inequality is not greater than $\bigvee_0^1 f$ (for a proof, see [4]). Hence,

$$\bigvee_0^1 Q_n f \leq 2\sum_{i=1}^{n} |d_i| + \bigvee_0^1 f.$$

Now we estimate $\sum_{i=1}^{n} |d_i|$. Let $F_i(x) = \int_{x_{i-1}}^{x} f(t)dt$, then the formula of integration by parts for the Stieljes-Lebesgue integral [6] gives

$$d_i = \frac{12}{m(I_i)^2}\int_{I_i}(x-\tilde{x}_i)f(x)dx = \frac{12}{m(I_i)^2}\int_{I_i}(x-\tilde{x}_i)dF_i(x)$$

$$= \frac{12}{m(I_i)^2}\left[(x-\tilde{x}_i)F_i(x)\Big|_{x_{i-1}}^{x_i} - \int_{I_i} F_i(x)d(x-\tilde{x}_i)\right]$$

$$= \frac{12}{m(I_i)^2}\left[\frac{m(I_i)}{2}F_i(x_i) - \int_{I_i} F_i(x)dx\right]$$

$$= \frac{6}{m(I_i)}\int_{I_i} f(t)dt - \frac{12}{m(I_i)^2}\int_{I_i}\left(\int_{x_{i-1}}^{x} f(t)dt\right)dx$$

$$= 6\left[\frac{1}{m(I_i)}\int_{I_i} f(t)dt - \frac{1}{A_i}\int\int_{\Omega_i} f(t)dtdx\right]$$

where $\Omega_i = \{(x,t): x_{i-1} \leq x \leq x_i,\ x_{i-1} \leq t \leq x\}$ is a triangular region in the (x,t)-plane and $A_i = \frac{1}{2}m(I_i)^2$ is the area of Ω_i. With the same reason described in [4], we obtain

$$\sum_{i=1}^{n} |d_i| = 6\sum_{i=1}^{n}\left|\frac{1}{m(I_i)}\int_{I_i} f(t)dt - \frac{1}{A_i}\int\int_{\Omega_i} f(t)dtdx\right| \leq 6\bigvee_0^1 f.$$

Therefore,

$$\bigvee_0^1 Q_n f \leq 13\bigvee_0^1 f. \qquad \text{(Q.E.D.)}$$

Let $P_n = Q_n \circ P_S|_{\Delta_n}$, where $P_S|_{\Delta_n}$ is the restriction of P_S on Δ_n, then $P_n: \Delta_n \to \Delta_n$ is linear. We want to find the fixed points of P_n in Δ_n. For this purpose, we first investigate the representation of P_n under the basis $\{1_i, x1_i\}_{i=1}^{n}$.

LEMMA 3.5. For $i = 1,\ldots,n$,

$$P_n 1_i = \sum_{j=1}^{n} c_j(1_i)1_j + \sum_{j=1}^{n} d_j(1_i)x1_j$$

$$P_n(x1_i) = \sum_{j=1}^{n} c_j(x1_i)1_j + \sum_{j=1}^{n} d_j(x1_i)x1_j,$$

where

$$c_j(1_i) = \frac{m(S^{-1}(I_j)\cap I_i)}{m(I_i)} - \frac{12\tilde{x}_j}{m(I_j)^2}\int_{I_j}(x-\tilde{x}_j)(P1_i(x)dx,$$

$$d_j(1_i) = \frac{12}{m(I_j)^2}\int_{I_j}(x-\tilde{x}_j)(P1_i)(x)dx,$$

$$c_j(x1_i) = \int_{I_j}(P(x1_i))(x)dx - \frac{12\tilde{x}_j}{m(I_j)^2}\int_{I_j}(x-\tilde{x}_j)(P(x1_i))(x)dx,$$

$$d_j(x1_i) = \frac{12}{m(I_j)^2}\int_{I_j}(x-\tilde{x}_j)(P(x1_i))(x)dx.$$

PROOF. By definition, $P_n 1_i = Q_n \circ P_S 1_i, P_n(x1_i) = Q_n \circ P_S(x1_i)$. From the definition of $P_S, \int_{I_j}(P_S 1_i)(x)dx = \frac{m(S^{-1}(I_j)\cap I_i)}{m(I_i)}$. Combined with Lemma 3.1, we have the result. (Q.E.D.)

LEMMA 3.6. P_n has a nontrivial fixed point f_n in Δ_n.

PROOF. Let $C_1 = (c^1_{ji}) = (c_j(1_i))$, $C_2 = (c^2_{ji} = (c_j(x1_i))$, $D_1 = (d^1_{ji}) = (d_j(1_i))$, $D_2 = (d^2_{ji}) = (d_j(x1_i))$, where $c_j(1_i)$, $d_j(1_i)$, $c_j(x1_i)$ and $d_j(x1_i)$ are as in Lemma 3.5. Then the function $f_n(x) = \sum_{i=1}^n c_i 1_i + \sum_{i=1}^n d_i x 1_i$ is a fixed point of P_n if and only if the column vector $(c_i,...,c_n,d_1,...,d_n)^T$ is a fixed point of the matrix

$$\tilde{P}_n = \begin{bmatrix} C_1 & C_2 \\ D_1 & D_2 \end{bmatrix}.$$

We first prove that the row vector $l = (1,...1,\tilde{x}_1,...,\tilde{x}_n)$ satisfies $l = l\tilde{P}_n$. In fact, from the first equality of (2),

$$\sum_{j=1}^n (c_j(1_i) + \tilde{x}_j d_j(1_i)) = \sum_{j=1}^n \int_{I_j}(P1_i)(x)dx = \sum_{j=1}^n \int_{S^{-1}(I_j)} 1_i(x)dx$$

$$= \sum_{j=1}^n \frac{m(S^{-1}(I_j)\cap I_i)}{m(I_i)} = 1,$$

$$\sum_{j=1}^n (c_j(x1_i) + \tilde{x}_j d_j(x1_i)) = \sum_{j=1}^n \int_{I_j}(P(x1_i))(x)dx$$

$$= \sum_{j=1}^n \int_{S^{-1}(I_j)} x1_i(x)dx = \int_0^1 x1_i(x)dx$$

$$= \frac{1}{m(I_i)}\int_{I_i} xdx = \frac{1}{m(I_i)}\,\frac{x_i^2 - x_{i-1}^2}{2} = \tilde{x}_i.$$

Hence the matrix $\tilde{P}_n$ has eigenvalue 1 and it follows that $\tilde{P}_n\mu = \mu$ has a nontrivial solution. (Q.E.D.)

In [3], Lasota and Yorke prove that, if $S:[0,1]\to[0,1]$ is a piecewise C^2-function satisfying $M = \inf|S'| > 2$, then for any $f\in L^1(0,1)$ of bounded variation, $\vee_0^1 P_S f \le \alpha\|f\| + \beta\vee_0^1 f$ with $\alpha > 0$ and $\beta = \frac{2}{M} < 1$. We shall call this inequality the Lasota-Yorke inequality. With this result, we can prove the following

LEMMA 3.7. Suppose $S:[0,1]\to[0,1]$ is piecewise C^2 and $M \equiv \inf|S'| > 26$. Then the sequence $\{\vee_0^1 f_n\}$ is bounded with f_n the fixed points of P_n satisfying $\|f_n\| = 1$.

PROOF. Since f_n is piecewise linear, it has bounded variation. From the Lasota-Yorke inequality, Pf_n is a function of bounded variation. From the same inequality and the fact that $f_n = P_n f_n = Q_n \circ Pf_n$, using Lemma 3.4, we obtain

$$\bigvee_0^1 f_n = \bigvee_0^1 Q_n \circ Pf_n \le 13\bigvee_0^1 Pf_n \le 13(\alpha\|f_n\| + \beta\bigvee_0^1 f_n)$$

$$= 13\alpha + 13\beta\bigvee_0^1 f_n = 13\alpha + \frac{26}{M}\bigvee_0^1 f_n.$$

By assumption, $M > 26$, therefore for all n

$$\bigvee_0^1 f_n \le \frac{13\alpha}{1 - 26/M} < \infty. \qquad \text{(Q.E.D.)}$$

Now we can prove our convergence theorem for the first-order piecewise polynomial Galerkin approximation scheme for Frobenius-Perron operator equations.

THEOREM 3.1. Suppose $S:[0,1]\to[0,1]$ is piecewise C^2 and $M = inf\,|S'| > 26$. Then for any n, P_n has a fixed point f_n with $\|f_n\| = 1$ in Δ_n and when mesh $(\Delta_n)\to 0$, there exists a subsequence $\{f_{n_i}\} \subset \{f_n\}$ such that f_{n_i} converges to a fixed point of P_S.

PROOF. By Lemma 3.7 and the Helly Theorem [7], there is a subsequence $\{f_{n_i}\} \subset \{f_n\}$ which converges to some $f \in L^1(0,1)$. Now

$$\|P_S f - f\| \le \|f - f_{n_i}\| + \|f_{n_i} - Q_{n_i} \circ P_S f_{n_i}\|$$
$$+ \|Q_{n_i} \circ P_S f_{n_i} - Q_{n_i} \circ P_S f\| + \|Q_{n_i} \circ P_S f - P_S f\|.$$

Since $\{\|Q_{n_i} \circ P_S\|\}$ is uniformly bounded and $Q_{n_i} \circ P_S f_{n_i} = f_{n_i}$, Lemma 3.3 implies that the right hand side of the above inequality approaches zero as $i\to\infty$. Thus $P_S f = f$. (Q.E.D.)

COROLLARY 3.1. Suppose $S:[0,1]\to[0,1]$ is piecewise C^2 and $M = inf\,|S'| > 1$, then a sequence of functions can be constructed from piecewise linear functions which converge to a fixed point of P_S.

PROOF. Choose $k > 0$ such that $M^k > 26$. Let $\varphi = S^k$. Then $P_n(\varphi)$ has a fixed point $f_n^{(\varphi)}$ of unit length in Δ_n. Define

$$g_i = \frac{1}{k}\sum_{j=0}^{k-1} (P_S)^i f_{n_i}^{(\varphi)},$$

where f_{n_i} is a convergent subsequence of $\{f_n\}$ from the above theorem. Then g_i converges, by Theorem 3.1, to

$$g = \frac{1}{k}\sum_{j=1}^{k-1} (P_S)^j f^{(\varphi)},$$

where $f^{(\varphi)}$ is a fixed point of $\varphi = S^k$. This g is a fixed point of P_S. In fact, since $(P_S)^k f^{(\varphi)} = P_{S^k}$ $f^{(\varphi)} = P_\varphi\, f^{(\varphi)} = f^{(\varphi)}$,

$$P_S g = \frac{1}{k}\{P_S f^{(\varphi)} + \cdots + (P_S)^k f^{(\varphi)}\} = g. \qquad \text{(Q.E.D.)}$$

4. PIECEWISE QUADRATIC PROJECTION APPROXIMATION OF FROBENIUS-PERRON OPERATOR.

In this section, we shall generalize the piecewise linear approximation of the previous section to the piecewise quadratic one, that is, we look for the approximate solutions of Frobenius-Perron operator equation in the space of piecewise quadratic functions. Let $x_0 = 0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a finite partition of the interval $[0,1]$ as before. For $i = 1, \ldots, n$, $I_i = [x_{i-1}, x_i]$, $\tilde{x}_i = \frac{x_{i-1} + x_i}{2}$. Let mesh $(\Delta_n) = max_i\; m(I_i)$ and $\Delta_n = span\{1_i, x1_i, x^2 1_i\}_{i=1}^n$ where $1_i = \frac{1}{m(I_i)}\chi_{I_i}$. $\Delta_n \subset L^1(0,1)$ is a subspace of dimension $3n$.

Define the projection $Q_n: L^1(0,1)\to\Delta_n$ by the orthogonal conditions, for $i = 1, \ldots, n$

$$\langle f - Q_n f, 1_i\rangle = 0, \langle f - Q_n f, x1_i\rangle = 0, \langle f - Q_n f, x^2 1_i\rangle = 0.$$

Let $Q_nf = \sum_{j=1}^{n}(c_j + d_jx + e_jx^2)1_j$. We show that $\{c_j, d_j, e_j\}_{j=1}^{n}$ are uniquely determined by the above conditions.

For $i = 1, \dots, n$, straightforward calculation gives

$$\left\langle Q_nf, 1_i \right\rangle = \frac{1}{m(I_i)}c_i + \frac{\tilde{x}_i}{m(I_i)}d_i + \frac{x_1^2 + x_ix_{i-1} + x_{i-1}^2}{3m(I_i)}e_i,$$

$$\left\langle Q_nf, x1_i \right\rangle = \frac{\tilde{x}_i}{m(I_i)}c_i + \frac{x_i^2 + x_ix_{i-1} + x_{i-1}^2}{3m(I_i)} + \frac{\tilde{x}_i(x_i^2 + x_{i-1}^2)}{2m(I_i)}e_i,$$

$$\left\langle Q_nf, x^21_i \right\rangle = \frac{x_i^2 + x_ix_{i-1} + x_{i-1}^2}{3m(I_i)}c_i + \frac{\tilde{x}_i(x_i^2 + x_{i-1}^2)}{2m(I_i)}d_i$$

$$+ \frac{1}{5m(I_i)}(x_i^4 + x_i^3x_{i-1} + x_i^2x_{i-1}^2 + x_{i-1}^4)e_i.$$

By the orthogonal condition, we have the following equations

$$\begin{cases} c_i + \tilde{x}_id_i + \frac{1}{3}(x_i^2 + x_ix_{i-1} + x_{i-1}^2)e_i = \int_{I_i} f(x)dx \\ \tilde{x}_ic_i + \frac{1}{3}(x_i^2 + x_ix_{i-1} + x_{i-1}^2)d_i + \frac{\tilde{x}_i}{2}(x_i^2 + x_{i-1}^2) = \int_{I_i} xf(x)dx \\ \frac{1}{3}(x_i^2 + x_ix_{i-1} + x_{i-1}^2)c_i + \frac{\tilde{x}_i}{2}(x_i^2 + x_{i-1}^2)d_i + \\ \frac{1}{5}(x_1^4 + x_i^3x_{i-1} + x_i^2x_{i-1}^2 + x_ix_{i-1}^3 + x_{i-1}^4)e_i = \int_{I_i} x^2f(x)dx. \end{cases} \tag{3}$$

Eliminate c_i from the above system, we have

$$\begin{cases} \frac{1}{12}m(I_i)^2d_i + \frac{1}{6}m(I_i)^2\tilde{x}_ie_i = \int_{I_i}(x - \tilde{x}_i)f(x)dx \\ \frac{1}{6}m(I_i)^2\tilde{x}_id_i + \frac{1}{45}m(I_i)^2[4x_i^2 + 7x_ix_{i-1} + 4x_{i-1}^2]e_i = \\ \qquad \int_{I_i} x^2f(x)dx - \frac{1}{3}(x_i^2 + x_ix_{i-1} + x_{i-1}^2)\int_{I_i} f(x)dx. \end{cases} \tag{4}$$

The solutions are given by

$$\begin{cases} c_i = \frac{3}{2}\int_{I_i} f(x)dx - \frac{12\tilde{x}_i}{m(I_i)^2}\int_{I_i} xf(x)dx - \frac{3x_ix_{i-1}}{m(I_i)^2}\int_{I_i} f(x)dx + \\ \qquad \frac{60}{m(I_i)^4}[2\tilde{x}_i^2 + x_ix_{i-1}]\int_{I_i}(x - \tilde{x}_i)f(x)dx \\ d_i = \frac{12}{m(I_i)^2}\int_{I_i} xf(x)dx + \frac{18\tilde{x}_i}{m(I_i)^2}\int_{I_i} f(x)dx - \\ \qquad \frac{360}{m(I_i)^4}\tilde{x}_i\int_{I_i}(x - \tilde{x}_i)^2f(x)dx \\ e_i = \frac{180}{m(I_i)^4}\int_{I_i}[(x - \tilde{x}_i)^2 - \frac{1}{12}m(I_i)^2]f(x)dx. \end{cases} \tag{5}$$

LEMMA 4.1. $\| Q_n \| \leq 62$ for all n.

PROOF. The value of Q_nf in the subinterval I_i depends only on the value of f on I_i. So it is enough to estimate the integral $\int_{I_i} |(Q_nf)(x)|\, dx$. Let $I = I_i, I = [a, b], \tilde{x} = \frac{a+b}{2}$ and let $\varphi(x) = \frac{1}{m(I)}(c + dx + ex^2)$ be the orthogonal projection of f to $Span\{1_I, x1_I, x^21_I\}$. First of all, assume $f \geq 0$. We consider different cases.

(i) $\varphi \geq 0$. Then from the first equality of (3),

$$\int_I |\varphi(x)|\,dx = \int_I \varphi(x)dx = \frac{1}{m(I)} \int_I (c + dx + ex^2)dx$$

$$= \frac{1}{m(I)} [cx + \tfrac{d}{2}x^2 + \tfrac{e}{3}x^3]_a^b = c + d\tilde{x} + \tfrac{1}{3}(a^2 + ab + b^2)e$$

$$= \int_I f(x)dx = \int_I |f(x)|\,dx.$$

(ii) $\varphi \not\geq 0$. Then φ has distinct zeros on the real axis. Without loss of generality, we may assume $e > 0$. Consider different distribution of the zeros. Let ζ_1 and ζ_2 be zeros of φ with $\zeta_1 < \zeta_2$.

Firstly, assume $\zeta_1 \in (a,b)$ and $\zeta_2 \in (a,b)$. Then $\zeta_1 + \zeta_2 = -\frac{d}{e}, \zeta_1 \cdot \zeta_2 = \frac{c}{e}$, and

$$\int_I |\varphi(x)|\,dx = \frac{1}{m(I)}\Big[\int_a^{\zeta_1} (c + dx + ex^2)dx - \int_{\zeta_1}^{\zeta_2} (c + dx + ex^2)dx$$

$$+ \int_{\zeta_2}^{b} (c + dx + ex^2)dx\Big]$$

$$= \frac{1}{m(I)}\Big\{\Big[cx + \tfrac{d}{2}x^2 + \tfrac{e}{3}x^3\Big]_a^{\zeta_1} - \Big[cx + \tfrac{d}{2}x^2 + \tfrac{3}{3}x^3\Big]_{\zeta_1}^{\zeta_2}$$

$$+\Big[cx + \tfrac{d}{2}x^2 + \tfrac{e}{3}x^3\Big]_{\zeta_2}^{b}\Big\}$$

$$= \frac{1}{m(I)}\Big[c(b-a) + \tfrac{d}{2}(b^2 - a^2) + \tfrac{e}{3}(b^3 - a^3)\Big]$$

$$+ \frac{2}{m(I)}\Big[c(\zeta_1 - \zeta_2) + \tfrac{d}{2}(\zeta_1^2 - \zeta_2^2) + \tfrac{e}{3}(\zeta_1^3 - \zeta_2^3)\Big]$$

$$= \Big[c + d\tilde{x} + \tfrac{e}{3}(a^2 + ab + b^2)\Big]$$

$$- \frac{2(\zeta_2 - \zeta_1)}{m(I)}\Big[c + \tfrac{d}{2}(\zeta_1 + \zeta_2) + \tfrac{e}{3}((\zeta_1 + \zeta_2)^2 - \zeta_1\zeta_2\Big]$$

$$= \int_I f(x)dx - \frac{2(\zeta_2 - \zeta_1)}{m(I)}\left[c + \frac{d}{2}\left(-\frac{d}{e}\right) + \frac{e}{3}\left(\left(-\frac{d}{e}\right)^2 - \frac{c}{e}\right]\right.$$

$$= \int_I f(x)dx - \frac{2(\zeta_2 - \zeta_1)}{m(I)}\left[\frac{2c}{3} - \frac{d^2}{6e}\right]$$

$$= \int_I f(x)dx + \frac{\zeta_2 - \zeta_1}{m(I)} \cdot \frac{d^2 - 4ec}{3e}.$$

Since $\zeta_1 = \frac{-d - \sqrt{d^2 - 4ec}}{2e}$ and $\zeta_2 = \frac{-d + \sqrt{d^2 - 4ec}}{2e}$, $\zeta_2 - \zeta_1 = \frac{\sqrt{d^2 - 44ec}}{e} \leq m(I)$. Hence $(d^2 - 4ec)^{\frac{3}{2}} \leq e^3 m(I)^3$. So,

$$\frac{\zeta_2 - \zeta_1}{m(I)} \frac{d^2 - 4ec}{3e} = \frac{1}{m(I)} \frac{(d^2 - 4ec)^{3/2}}{3e^2} \leq \frac{1}{m(I)} \frac{e^3 m(I)^3}{3e^2} = \tfrac{1}{3}em(I)^2.$$

From the last equality of (5), it is easy to see that

$$\tfrac{1}{3}em(I)^2 = \frac{60}{m(I)^2} \int_I (x - \tilde{x})^2 f(x)dx - 5 \int_I f(x)dx$$

$$\leq \frac{60}{m(I)^2} \int_I \tfrac{1}{4}m(I)^2 \cdot f(x)dx - 5 \int_I f(x)dx$$

$$= 10 \int_I f(x)dx.$$

Therefore,

$$\int_I |\varphi(x)|\,dx \le \int_I f(x)dx + 10\int_I f(x)dx = 11\int_I f(x)dx = 11\int_I |f(x)|\,dx.$$

Secondly, assume there is only one zero of φ in (a,b), say $\zeta_1 \in (a,b)$ and $\zeta_2 \notin (a,b)$. In this case,

$$\int_I |\varphi(x)|\,dx = \frac{1}{m(I)}\left[\int_a^{\zeta_1} (c+dx+ex^2)dx + \int_b^{\zeta_1} (c+dx+ex^2)dx\right]$$

$$= \frac{1}{m(I)}\left\{\left[c(\zeta_1 - a) + \frac{d}{2}(\zeta_1^2 - a^2) + \frac{e}{3}(\zeta_1^3 - a^3)\right]\right.$$

$$\left. + \left[c(\zeta_1 - b) + \frac{d}{2}(\zeta_1^2 - b^2) + \frac{e}{3}(\zeta_1^3 - b^3)\right]\right\}$$

$$= \frac{\zeta_1 - a}{m(I)}\left[c + \frac{d}{2}(\zeta_1 + a) + \frac{e}{3}(\zeta_1^2 + \zeta_1 a + a^2)\right]$$

$$= \frac{(b-\zeta_1)}{m(I)}\left[c + \frac{d}{2}(\zeta_1 + b) + \frac{e}{3}(\zeta_1^2 + \zeta_1 b + b^2)\right]$$

$$= \frac{\zeta_1 - a}{m(I)}\left[c + \frac{d}{2}(\zeta_1 + a) + \frac{e}{3}(\zeta_1^2 + \zeta_1 a + a^2)\right]$$

$$+ \frac{b - \zeta_1}{m(I)}\left[c + \frac{d}{2}(\zeta_1 + b) + \frac{e}{3}(\zeta_1^2 + \zeta_1 b + b^2)\right]$$

$$- \frac{2(b-\zeta_1)}{m(I)}\left[c + \frac{d}{2}(\zeta_1 + b) + \frac{e}{3}(\zeta_1^2 + \zeta_1 b + b^2)\right]$$

$$= \left[c + \tilde{x}d + \frac{e}{3}(a^2 + ab + b^2)\right]$$

$$- \frac{2(b-\zeta_1)}{m(I)}\left[c + \frac{d}{2}(\zeta_1 + b) + \frac{e}{3}(\zeta_1^2 + \zeta_1 b + b^2)\right]$$

$$= \int_I f(x)dx + \frac{2(\zeta_1 - b)}{m(I)}\left[c + \frac{d}{2}b + \frac{d}{2}\zeta_1 + \frac{1}{3}e\zeta_1^2 + \frac{1}{3}eb\zeta_1 + \frac{1}{3}eb^2\right]$$

$$= \int_I f(x)dx + \frac{2(\zeta_1 - b)}{m(I)}\left[\frac{1}{3}(e\zeta_1^2 + d\zeta_1 + c) + \frac{1}{3}(eb - d)\zeta_1\right.$$

$$\left. + \frac{2}{3}c + \frac{d}{2}b + \frac{d}{2}\zeta_1 + \frac{1}{3}eb^2\right]$$

$$= \int_I f(x)dx + \frac{2(\zeta_1 - b)}{m(I)}\left[\frac{1}{3}eb^2 + \frac{1}{2}db + \frac{1}{6}d\zeta_1 + \frac{1}{3}eb\zeta_1 + \frac{2}{3}c\right]$$

$$= \int_I f(x)dx + \frac{(\zeta_1 - b)}{3m(I)}\left[2eb^2 + 3db + d\zeta_1 + 2eb\zeta_1 + 4c\right]$$

$$= \int_I f(x)dx + \frac{(\zeta_1 - b)}{3m(I)}\left[2(e\zeta_1^2 + d\zeta_1 + c) + 2e(b^2 - \zeta_1^2)\right.$$

$$\left. + 2d(b - \zeta_1) + d(b + \zeta_1) + 2eb\zeta_1 + 2c\right]$$

$$= \int_I f(x)dx + \frac{\zeta_1 - b}{3m(I)}\left[2e(b^2 - \zeta_1^2) + d(b - \zeta_1) + 2(db + eb\zeta_1 + c)\right]$$

$$= \int_I f(x)dx + \frac{\zeta_1 - b}{3m(I)}\left[2e(b^2 - \zeta_1^2) + d(b - \zeta_1) + 2(b - \zeta_1)(e\zeta_1 + d)\right]$$

$$= \int_I f(x)dx - \frac{(b-\zeta_1)^2}{3m(I)}\left[2e(b + \zeta_1) + d + 2(e\zeta_1 + d)\right]$$

$$= \int_I f(x)dx - \frac{(b-\zeta_1)^2}{3m(I)}\left[4e(\zeta_1 - b) + 3(2eb + d)\right].$$

From (5) we have

$$2eb + d = \frac{360b}{m(I)^4}\int_I (x-\bar{x})^2 f(x)dx - \frac{30b}{m(I)^2}\int_I f(x)dx$$
$$+\frac{12}{m(I)^2}\int_I xf(x)dx + \frac{18\bar{x}}{m(I)^2}\int_I f(x)dx - \frac{360\bar{x}}{m(I)^4}\int_I (x-\bar{x})^2 f(x)dx$$
$$= \frac{180}{m(I)^3}\int_I (x-\bar{x})^2 f(x)dx - \frac{12b}{m(I)^2}\int_I f(x)dx - \frac{18b}{m(I)^2}\int_I f(x)dx$$
$$+\frac{12}{m(I)^2}\int_I xf(x)dx + \frac{18\bar{x}}{m(I)^2}\int_I f(x)dx$$
$$= \frac{180}{m(I)^3}\int_I (x-\bar{x})^2 f(x)dx - \frac{12}{m(I)^2}\int_I (b-x)f(x)dx - \frac{9}{m(I)}\int_I f(x)dx.$$

Hence,

$$\int_I |\varphi(x)|\,dx = \int_I f(x)dx + \tfrac{4}{3}e\,\frac{(b-\zeta_1)^3}{m(I)} - \frac{(b-\zeta_1)^2}{m(I)}$$
$$\cdot\left[\frac{180}{m(I)^3}\int_I (x-\bar{x})^2 f(x)dx - \frac{12}{m(I)^2}\int_I (b-x)f(x)dx\right.$$
$$\left. - \frac{9}{m(I)}\int_I f(x)dx\right]$$
$$\le \int_I f(x)dx + \tfrac{4}{3}em(I)^2 + \frac{12}{m(I)}\int_I (b-\zeta_1)\,f(x)dx + 9\int_I f(x)dx$$
$$\le [1 + 4\cdot 10 + 12 + 9]\int_I f(x)dx = 62\int_I f(x)dx$$
$$= 62\int_I |f(x)|\,dx.$$

Therefore for $f \ge 0$, $\|\varphi\| \le 62\|f\|$. For general $f \in L^1(0,1)$, consider f^+ and f^-, respectively, the same inequality can be achieved. (Q.E.D.)

LEMMA 4.2. For any $f \in L^1(0,1)$, if mesh $(\Delta_n)\to 0$, then

$$Q_n f \to f \qquad \text{under the} \qquad L^1-norm.$$

PROOF. First assume $f \in L^2(0,1) \subset L^1(0,1)$. From the way $Q_n f$ is defined, obviously $\|f - Q_n f\|_2 = min\{\|f-g\|_2 : g \in \Delta_n\}$ where $\|\cdot\|_2$ is the L^2-norm. Hence, when mesh $(\Delta_n)\to 0$, $\|f - Q_n f\|_2 \to 0$, and the Cauchy inequality $\|Q_n f - f\| \le \|Q_n f - f\|_2$ gives $\|Q_n f\| \to 0$.

Now for $f \in L^1(0,1)$ and $\varepsilon > 0$, there exists $g \in L^2(0,1)$ such that $\|f-g\| < \varepsilon$. From

$$\|Q_n f - f\| \le \|Q_n f - Q_n g\| + \|Q_n g - g\| + \|g - f\|$$
$$\le 62\|f-g\| + \|Q_n g\| + \|g-f\|$$

and $\|Q_n g - g\| \to 0$ we obtain $\|Q_n f - f\| \to 0$. (Q.E.D.)

LEMMA 4.3. If $f \in L^1(0,1)$ is of bounded variation, then for all n, $\bigvee_0^1 Q_n f \le 121 \bigvee_0^1 f$.

PROOF. Since $Q_n f$ is piecewise quadratic,

$$\bigvee_0^1 Q_n f = \bigvee_0^1 \sum_{i=1}^{n} (c_i + d_i x + e_i x^2)1_i$$
$$= \sum_{i=1}^{n} \frac{1}{m(I_i)}\int_{I_i} |d_i + 2e_i x|\,dx$$

$$+\sum_{i=1}^{n-1}\left|\frac{c_i+d_ix_i+e_ix_i^2}{m(I_i)}-\frac{c_{i+1}+d_{i+1}x_i+e_{i+1}x_i^2}{m(I_{i+1})}\right|$$

$$=\sum_{i=1}^{n}\frac{1}{m(I_i)}\int_{I_i}|d_i+2e_ix|\,dx$$

$$+\sum_{i=1}^{n-1}\left|\left(\frac{c_i}{m(I_i)}-\frac{c_{i+1}}{m(I_{i+1})}\right)+\left(\frac{d_i}{m(I_i)}-\frac{d_{i+1}}{m(I_{i+1})}\right)x_i\right.$$

$$\left.+\left(\frac{e_i}{m(I_i)}-\frac{e_{i+1}}{m(I_{i+1})}\right)x_i^2\right|$$

$$=\sum_{i=1}^{n}\frac{1}{m(I_i)}\int_{I_i}|d_i+2e_ix|\,dx$$

$$+\sum_{i=1}^{n-1}\left|\frac{1}{m(I_{i+1})^3}\int_{I_{i+1}}\left[12\tilde{x}_{i+1}x+3x_{i+1}x_i-12x_i\left(x+\frac{3}{2}\tilde{x}_{i+1}\right)\right.\right.$$

$$\left.+15x_i^2\right]f(x)dx$$

$$-\frac{1}{m(I_i)^3}\int_{I_i}\left[12\tilde{x}_ix+3x_ix_{i-1}-12x_i\left(x+\frac{3}{2}\tilde{x}_i\right)+15x_i^2\right]f(x)dx \tag{6}$$

$$+\frac{1}{m(I_{i+1})^5}\int_{I_{i+1}}\left[360x_i\tilde{x}_{i+1}-60\,(2\tilde{x}_{i+1}^2+x_{i+1}x_i)-180x_i^2\right]$$

$$\cdot(x-\tilde{x}_{i+1})^2f(x)dx$$

$$-\frac{1}{m(I_i)^5}\int_{I_i}\left[360x_i\tilde{x}_i-60\,(2\tilde{x}_i^2+x_ix_{i-1})-180x_i^2\right](x-\tilde{x}_i)^2f(x)dx$$

$$\left.+\frac{3}{2}\left(\frac{1}{m(I_i)}\int_{I_i}f(x)dx-\frac{1}{m(I_{i+1})}\int_{I_{i+1}}f(x)dx\right)\right|$$

$$=\sum_{i=1}^{m}\frac{1}{m(I_i)}|d_i+2e_ix|+\sum_{i=1}^{n-1}\left|\frac{6}{m(I_{i+1})^2}\int_{I_{i+1}}(x-\tilde{x}_i)f(x)dx\right.$$

$$+\frac{6}{m(I_i)^2}\int_{I_i}(x-\tilde{x}_i)f(x)dx+\frac{30}{m(I_i)^3}\int_{I_i}(x-\tilde{x}_i)^2f(x)dx$$

$$-\frac{30}{m(I_{i+1})^3}\int_{I_{i+1}}(x-\tilde{x}_{i+1})^2f(x)dx+\frac{3}{2}\frac{1}{m(I_i)}\int_{I_i}f(x)dx$$

$$\left.-\frac{3}{2}\frac{1}{m(I_{i+1})}\int_{I_{i+1}}f(x)dx\right|.$$

Let $\Omega_i=\{(x,t):x\in I_i,x_{i-1}\le t\le x\},V_i=\{(x,t,x):x\in I_ix_{i-1}\le t\le x,x_{i-1}\le x\le t\}$. Again, using the integration by parts formula for functions of bounded variation yields

$$\frac{1}{m(I_{i+1})^2}\int_{I_{i+1}}(x-x_i)f(x)dx=\frac{1}{m(I_{i+1})}\int_{I_{i+1}}f(x)dx$$

$$-\frac{1}{2A(\Omega_{i+1})}\int\int_{\Omega_{i+1}}f(t)dtdt,$$

$$\frac{1}{m(I_i)^2}\int_{I_i}(x-x_i)f(x)dx=-\frac{1}{2A(\Omega_i)}\int\int_{\Omega_i}f(t)dtdx,$$

$$\frac{1}{m(I_{i+1})^2}\int_{I_{i+1}}(x-\tilde{x}_{i+1})^2 f(x)dx = \frac{1}{4m(I_{i+1})}\int_{I_{i+1}} f(x)dx - \frac{1}{2A(\Omega_{i+1})}\int\int_{V_{i-1}} f(t)dtdx + \frac{1}{3V(V_{i+1})}\int\int\int_{V_{i-1}} f(s)dxdtdx,$$

$$\frac{1}{m(I_i)^3}\int_{I_i}(x-\tilde{x}_i)^2 f(x)dx = \frac{1}{4m(I_i)}\int_{I_i} f(x)dx - \frac{1}{2A(\Omega_i)}\int\int_{\Omega_i} f(t)dtdx + \frac{1}{3V(V_i)}\int\int\int_{V_i} f(s)dsdtdx,$$

where $A(\Omega_i) = \frac{1}{2}m(I_i)^2$ is the area of Ω_i and $V(V_i) = \frac{1}{6}m(I_i)^3$ is the volume of V_i. Substituting into (6), we have

$$\begin{aligned}
\bigvee_0^1 Q_n f = {} & \sum_{i=1}^{n}\frac{1}{m(I_i)}\int_{I_i}|d_i + 2e_i x|\,dx \\
& + \sum_{i=1}^{n-1}\left|\frac{6}{m(I_{i+1})}\int_{I_{i+1}} f(x)dx - \frac{3}{A(\Omega_{i+1})}\int\int_{\Omega_{i+1}} f(t)dtdx\right. \\
& - \frac{3}{A(\Omega_i)}\int\int_{\Omega_i} f(t)dtdx + \frac{15}{2m(I_i)}\int_{I_i} f(x)dx \\
& - \frac{15}{A(\Omega_i)}\int\int_{\Omega_i} f(t)dtdx + \frac{10}{V(V_i)}\int\int\int_{V_i} f(s)dsdtdx \\
& - \frac{15}{2m(I_{i+1})}\int_{I_{i+1}} f(x)dx + \frac{15}{A(\Omega_{i+1})}\int\int_{\Omega_{i+1}} f(t)dtdx \\
& - \frac{10}{V(V_{i+1})}\int\int\int_{V_{i+1}} f(s)dsdtdx + \frac{3}{2m(I_i)}\int_{I_i} f(x)dx \\
& \left. - \frac{3}{2m(I_{i+1})}\int_{I_{i+1}} f(x)dx\right|
\end{aligned} \tag{7}$$

$$\begin{aligned}
= {} & \sum_{i=1}^{n}\frac{1}{m(I_i)}\int_{I_i}|d_i + 2e_i x|\,dx \\
& + \sum_{i=1}^{n-1}\left|\frac{9}{m(I_i)}\int_{I_i} f(x)dx - \frac{3}{2m(I_{i+1})}\int_{I_{i+1}} f(x)dx\right. \\
& + \frac{12}{A(\Omega_{i+1})}\int\int_{\Omega_{i+1}} f(t)dtdx - \frac{18}{A(\Omega_i)}\int\int_{\Omega_i} f(t)dtdx \\
& \left. + \frac{10}{V(V_i)}\int\int\int_{V_i} f(s)dsdtdx - \frac{10}{V(V_{i+1})}\int\int\int_{V_{i+1}} f(s)dsdtdx\right|
\end{aligned}$$

$$\begin{aligned}
\le {} & \sum_{i=1}^{n}\frac{1}{m(I_i)}\int_{I_i}|d_i + 2e_i x|\,dx \\
& + 3\sum_{i=1}^{n-1}\left|\frac{1}{m(I_i)}\int_{I_i} f(x)dx - \frac{1}{m(I_{i+1})}\int_{I_{i+1}} f(x)dx\right| \\
& + 6\sum_{i=1}^{n-1}\left|\frac{1}{m(I_i)}\int_{I_i} f(x)dx - \frac{1}{A(\Omega_i)}\int\int_{\Omega_i} f(t)dtdx\right| \\
& + 12\sum_{i=1}^{n-1}\left|\frac{1}{A(\Omega_{i+1})}\int\int_{\Omega_{i+1}} f(t)dtdx - \frac{1}{A(\Omega_i)}\int\int_{\Omega_i} f(t)dtdx\right|
\end{aligned}$$

$$+10\sum_{i=1}^{n-1}\left|\frac{1}{V(V_i)}\int\int\int_{V_i} f(s)dsdtdx - \frac{1}{V(V_{i+1})}\int\int\int_{V_{i+1}} f(s)dsdtdx\right|$$

$$\leq \sum_{i=1}^{n} \frac{1}{m(I_i)}\int_{I_i} |d_i + 2e_i x|\,dx + 3\bigvee_0^1 f + 6\bigvee_0^1 f + 12\bigvee_0^1 f + 10\bigvee_0^1 f$$

$$= \sum_{i=1}^{n} \frac{1}{m(I_i)}\int_{I_i} |d_i + 2e_i x|\,dx + 31\bigvee_0^1 f.$$

Let's estimate the first term of (7). For $i = 1,...,n, \frac{1}{m(I_i)}\int_{I_i} |d_i + 2e_i x|\,dx$ is the variation of $Q_n f$ on I_i. For simplicity, we omit the subscript. Let $\varphi(x) = \frac{1}{m(I)}(c + dx + ex^2), I = [a,b]$. Without loss of generality, assume $e > 0$. Then $\zeta = -\frac{d}{2e}$ is the minimal point of φ and $\varphi\left(-\frac{d}{2e}\right) = \frac{1}{m(I)}\left(c - \frac{d^2}{4e}\right)$ is the minimal value of φ. If $a < \zeta < b$, then

$$\bigvee_I \varphi = \frac{1}{m(I)}\left(\varphi(a) + \varphi(b) - 2\varphi(\zeta)\right)$$

$$= \frac{1}{m(I)}\left(c + da + ea^2 + c + db + eb^2 - 2c + \frac{d^2}{2e}\right)$$

$$= \frac{1}{m(I)}\left[d(a+b) + e(a^2+b^2) + \frac{d^2}{2e}\right]$$

$$= \left[-2e\zeta(a+b) + e(a^2+b^2) + 2e\zeta^2\right]/m(I)$$

$$= e\left[2\zeta^2 - 2\zeta(a+b) + a^2 + b^2\right]/m(I)$$

$$= e\left[(a-b)^2 + 2ab - 2\zeta(a+b) + 2\zeta^2\right]/m(I)$$

$$= \frac{1}{m(I)}\left[em(I)^2 + 2e(a-\zeta)(b-\zeta)\right] \leq em(I)$$

$$= \frac{15}{m(I)}\left[\frac{12}{m(I)^2}\int_I (x-\tilde{x})^2 f(x)dx - \int_I f(x)dx\right]$$

$$\leq 30\left|\frac{1}{m(I)}\int_I f(x)dx - \frac{3}{A(\Omega)}\int\int_\Omega f(t)dtdx + \frac{2}{V(V)}\int\int\int_V f(s)dsdtdx\right|$$

$$\leq 90\bigvee_I f.$$

If $\zeta \notin (a,b)$, then

$$\bigvee_I \varphi = \frac{1}{m(I)}|\varphi(a) = \varphi(b)| = \frac{1}{m(I)}|d(b-a) + e(b^2-a^2)|$$

$$= |d + 2e\tilde{x}| = \frac{12}{m(I)}\left|\int_I (x-\tilde{x})f(x)dx\right|$$

$$\leq 6\left|\frac{1}{m(I)}\int_I f(x)dx - \frac{1}{A(\Omega)}\int\int_\Omega f(t)dtdx\right|$$

$$\leq 6\bigvee_I f < 90\bigvee_I f.$$

Substituting into (7), we have

$$\bigvee_0^1 Q_n f \leq 90\sum_{i=1}^{n}\bigvee_{I_i} f + 31\bigvee_0^1 f = 121\bigvee_0^1 f. \qquad \text{(Q.E.D.)}$$

LEMMA 4.4. Let $P_n = Q_n \circ P_S|_{\Delta_n}$, where P_S is the Frobenius-Perron operator associated with $S:[0,1]\to[0,1]$. Then P_n has a nontrivial fixed point in Δ_n.

PROOF. Denote by $\tilde{P}_n$ the representations of $P_n:\Delta_n\to\Delta_n$ under the basis $\{1_i, x1_i, x^2 1_i\}_{i=1}^n$. Let $\zeta = (1, \tilde{x}_1, \tilde{y}_1, 1, \tilde{x}_2, \tilde{y}_2, \cdots, 1, \tilde{x}_n, \tilde{y}_n)$ where $\tilde{y}_i = \frac{1}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2)$. Then for $i = 1, \dots, n$,

$$
\begin{aligned}
(\zeta\tilde{P}_n)_{3(i-1)+1} &= \sum_{j=1}^{n} (cj(1_i) + \tilde{x}_j d_j(1_i) + \tilde{y}_j e_j(1_i)) \\
&= \sum_{j=1}^{n} \int_{I_j} (P_S 1_i)(x)dx = \int_0^1 (P_S 1_i)(x)dx \\
&= \int_0^1 1_i(x)dx = 1, \\
(\zeta\tilde{P}_n)_{3(i-1)+2} &= \sum_{j=1}^{n} (c_j(x2_i) + \tilde{x}_j d_j(x1_i) + \tilde{y}_j e_j(x1_i)) \\
&= \sum_{j=1}^{n} \int_{I_j} (P_S(x1_i))(x)dx = \int_0^1 (P_S(x1_i))(x)dx \\
&= \int_0^1 x1_i(x)dx = \tilde{x}, \\
(\zeta\tilde{P}_n)_{3i} &= \sum_{j=1}^{n} (c_j(x^2 1_i) + \tilde{x}_j d_j(x^2 1_i) + \tilde{y}_j e_j(x^2 1_i)) \\
&= \sum_{j=1}^{n} \int_{I_j} (P_S(x^2 1_i))(x)dx = \int_0^1 (P_S(x^2 1_i))(x)dx \\
&= \int_0^1 x^2 1_i(x)dx = \tilde{y}_i.
\end{aligned}
$$

That is, ζ is a left eigenvector of the matrix $\tilde{P}_n$ corresponding to the eigenvalue 1. Therefore, there is a nonzero $c \in \Re^{3n}$ such that $\tilde{P}_n c = c$. Thus P_n has a nonzero fixed point in Δ_n. (Q.E.D.)

THEOREM 4.1. Let $S:[0,1]\to[0,1]$ be piecewise C^2 and $M = \inf|S'| > 242$. Let $\{f_n\}$ be a sequence of fixed points of P_n in Δ_n with $\|f_n\| = 1$. Then there exists a subsequence $\{f_{n_i}\} \subset \{f_n\}$ convergent to a fixed point of P_S.

PROOF. By the Lasota-Yorke inequality in the previous section, we have for any n

$$
\begin{aligned}
\bigvee_0^1 f_n = \bigvee_0^1 P_n f_n = \bigvee_0^1 Q_n \circ P_S f_n \le 121 \bigvee_0^1 P_S f_n \\
\le 121\left(\alpha\|f_n\| + \frac{2}{M}\bigvee_0^1 f_n\right) = 121\alpha + \beta\bigvee_0^1 f_n
\end{aligned}
$$

with $\beta = \frac{242}{M} < 1$. Hence

$$
\bigvee_0^1 f_n \le \frac{121\alpha}{1-\beta} < +\infty.
$$

From the Helly theorem there is a subsequence $\{f_{n_i}\} \subset \{f_n\}$ which converges to some $f \in L^1(0,1)$. From

$$
\begin{aligned}
\|P_S f - f\| \le \|f - f_{n_i}\| &+ \|f_{n_i} - Q_{n_i}\circ P_S f_{n_i}\| \\
&+ \|Q_{n_i}\circ P_S f_{n_i} - Q_{n_i}\circ P_S f\| \\
&+ \|Q_{n_i}\circ P_S f - P_S f\|
\end{aligned}
$$

it is obvious that $P_S f = f$. (Q.E.D.)

The proof of the following corollary is the same as that of Corollary 3.1.

COROLLARY 4.1. Let $S:[0,1]\to[0,1]$ be piecewise C^2 and $inf\,|S'|>1$. Then, a sequence of functions can be constructed from piecewise quadratic functions which converge to a nontrivial fixed point of P_S.

5. CONCLUSIONS.

In this paper, the piecewise linear and piecewise quadratic polynomial projection methods are proposed for the computation of invariant densities of the Frobenius-Perron operator. The convergence of the methods is proved for a class of measurable nonsingular transformations of the unit interval into itself which satisfy the condition of the Lasota-Yorke theorem. Our proof is based on the following observation: The projections $Q_n: L^1(0,1)\to L^1(0,1)$ defined in the previous sections satisfy

(1) $\|Q_n\| \le M$, M is a constant.

(2) $Q_n f \to f$ strongly for any $f \in L^1(0,1)$.

(3) $\vee_0^1 Q_n f \le C \vee_0^1 f$ for any $f \in L^1(0,1)$ of bounded variation, where C is a constant.

(4) $Q_n \circ P$ has a nontrivial fixed point f_n for each n.

In general, the projection method for the Frobenius-Perron operator equation $Pf - f = 0$ is convergent if the "discretization" operators Q_n satisfy the above four requirements, as the following theorem shows.

THEOREM 5.1. Suppose the sequence of operators Q_n satisfy the conditions (1) through (4) above. Then a sequence of approximate functions can be constructed which converge to a nontrivial fixed point of P_S when $S:[0,1]\to[0,1]$ is piecewise C^2 and $inf\,|S'|>1$.

The proof of this theorem follows exactly the same line of arguments in Theorem 3.1 and Theorem 4.1, in which $M=2$, $C=13$, and $M=62$, $C=121$, respectively.

Based on the convergence analysis of the piecewise linear and piecewise quadratic polynomial approximation methods, we believe that for general higher order piecewise polynomial projection method, the convergence can also be established, although it will be more tedious and complicated.

It is important to estimate the convergence rate for a convergent numerical method. The further research will be focused on this aspect for our projection methods for Frobenius-Perron operator equation or more general Markov operator equation. For solving noncompact operator equations in nonreflexive Banach spaces, this field is not fully developed, although is essential to physical sciences.

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