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# INTEGRAL MEANS OF CERTAIN CLASS OF ANALYTIC FUNCTIONS

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ABSTRACT. In this paper we discuss the following class of functions

$$S_\lambda(\alpha,\beta)=\{f(z):\left|\frac{f(z)}{g(z)}-1\right|<\beta\left|\lambda\frac{f(z)}{g(z)}+1\right|, z\in D\} \text{ where } 0\le\lambda\le 1,\ 0<\beta\le 1,\ 0\le\alpha<1,$$

and $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ is analytic in $D=\{z: |z|<1\}$, $g(z)$ is a starlike function of order $\alpha$. A subordination about this class is obtained, the integral means of functions in $S_\lambda(\alpha,\beta)$ and some extremal properties are studied.

KEY WORDS AND PHRASES. Analytic function, subordination, integral mean, distortion, coefficient inequality.
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## 1. INTRODUCTION.

Let $A$ be the class consisting of all functions $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ which are analytic in $D=\{z: |z|<1\}$. Owa [1] has introduced the class $\tilde{S}_\lambda(\alpha,\beta)$. If $f(z)\in A$ and there exists $g(z)=z-\sum_{n=2}^{\infty}|b_n|z^n\in S^*(\alpha)$ $(0\le\alpha<1)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<\beta\left|\lambda\frac{f(z)}{g(z)}+1\right| \qquad (0\le\lambda\le 1,\ 0<\beta\le 1,\ z\in D), \tag{1.1}$$

we say $f(z)\in\tilde{S}_\lambda(\alpha,\beta)$. Owa [1] discussed the coefficient estimates of functions in $\tilde{S}_\lambda(\alpha,\beta)$. In this paper, we discuss the general case, i.e., the class $S_\lambda(\alpha,\beta)$ which is generated by a function

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n\in S^*(\alpha).$$

We first gave a subordinate about this class, then we discuss the integral means of functions in $S_\lambda(\alpha,\beta)$, from this we can get some extremal properties about $S_\lambda(\alpha,\beta)$. We also discuss a subclass of $S_\lambda(\alpha,\beta)$.

## 2. A SUBORDINATION ABOUT $S_\lambda(\alpha,\beta)$.

We say that $g(z)$ is subordinate to $f(z)$ if there exists a function $\omega(z)$ analytic in $D$ satisfying $\omega(0)=0$ and $|\omega(z)|<1$ such that $g(z)=f(\omega(z))$ $(|z|<1)$. This subordination is denoted by $g(z)\prec f(z)$. About the class $S_\lambda(\alpha,\beta)$, we have the following:

THEOREM 2.1. If $f(z)\in S_\lambda(\alpha,\beta)$, i.e., there exists a function $g(z)\in S^*(\alpha)$ such that the inequality (1.1) holds, then we have

$$\frac{f(z)}{g(z)}\prec\frac{1+\beta z}{1-\beta\lambda z}=p_{\beta,\lambda}(z). \tag{2.1}$$

PROOF. Let $p(z)=\frac{f(z)}{g(z)}$, then $p(0)=1$. Now we divide the proof into three cases.

CASE (a). Let $\lambda \neq 0$, $\beta$ and $\lambda$ are not equal to 1 at the same time. Now the inequality (1.1) can be written as $|p(z)-1| < |\beta\lambda p(z)+\beta|$, that is,
$|p(z)|^2 - 2Re p(z) + 1 < \beta^2\lambda^2 |p(z)|^2 + 2\beta^2 \lambda Re p(z) + \beta^2$. From this we can get

$$\left| p(z) - \frac{1-\beta}{1+\beta\lambda} - \frac{\beta(1+\lambda)}{1-\beta^2\lambda^2} \right| < \frac{\beta(1+\lambda)}{1-\beta^2\lambda^2}.$$

Because univalent function $p_{\beta,\lambda}(z) = \frac{1+\beta z}{1-\beta\lambda z}$ maps $D$ onto the disk

$$\left\{ w: \left| w - \frac{1-\beta}{1+\beta\lambda} - \frac{\beta(1+\lambda)}{1-\beta^2\lambda^2} \right| < \frac{\beta(1+\lambda)}{1-\beta^2\lambda^2} \right\},$$

so $p(D) \subset p_{\beta,\lambda}(D)$ and $p(0) = p_{\beta,\lambda}(0) = 1$. From the principle of subordination of univalent functions, we have $p(z) \prec p_{\beta,\lambda}(z)$, that is (2.1).

CASE (b). Let $\lambda = 0$. Now the inequality (1.1) becomes $|p(z)-1| < \beta$.
Because univalent function $p_{\beta,o}(z) = 1+\beta z$ maps $D$ onto the disk $\{w: |w-1| < \beta\}$, so $p(D) \subset p_{\beta,o}(D)$ and $p(0) = p_{\beta,o}(0) = 1$. Thus $p(z) \prec p_{\beta,o}(z)$.

CASE (c). Let $\lambda = \beta = 1$. The inequality (1.1) becomes $|p(z)-1| < |p(z)+1|$, that is $Re.\, p(z) > 0$. Because $p(0) = 1$, so $p(z) \prec \frac{1+z}{1-z} = p_{1,1}(z)$.

Thus for any $0 \le \lambda \le 1, 0 < \beta \le 1$, we have proved (2.1).

## 3. THE INTEGRAL MEANS OF FUNCTIONS IN $S_\lambda(\alpha,\beta)$.

We first state some lemmas.

LEMMA 3.1 [2]. For any $g,h \in L^1[-\pi,\pi]$, the following statements are equivalent:

(a) For every convex non-decreasing function $\Phi$ on $(-\infty,\infty)$,

$$\int_{-\pi}^{\pi} \Phi\,(g(x))\,dx \le \int_{-\pi}^{\pi} \Phi\,(h(x))\,dx.$$

(b) For every $t \in (-\infty,\infty)$,

$$\int_{-\pi}^{\pi} (g(x)-t)^{+}\,dx \le \int_{-\pi}^{\pi} (h(x)-t)^{+}\,dx.$$

(c) $g^*(\theta) \le h^*(\theta)$, $(0 \le \theta \le \pi)$.

LEMMA 3.2 [2]. If $g,h$ are real integrable functions on $[-\pi,\pi]$, then $(g+h)^*(\theta) \le g^*(\theta) + h^*(\theta)$ $(0 \le \theta \le \pi)$, with equality holding if and only if $g$,$h$ are symmetric decreasing arrangement functions.

The definitions of $u^*(x)$ and the symmetric decreasing arrangement function can be found in [2].

LEMMA 3.3 [3]. Let $\Phi(t)$ be a convex increasing function, if $g(z) \prec f(z)$ in $D$, then

$$\int_{-\pi}^{\pi} \Phi\,(|g(re^{i\theta})|)\,d\theta \le \int_{-\pi}^{\pi} \Phi\,(|f(re^{i\theta})|)\,d\theta \qquad (0<r<1) \tag{3.1}$$

and if $u(z)$ is a harmonic function in $D$, $v(z) = u(\omega(z))$, where $\omega(z)$ is analytic in $D$, $\omega(0) = 0$, $|\omega(z)| < 1$, then

$$\int_{-\pi}^{\pi} \Phi(\pm v(re^{i\theta}))\,d\theta \le \int_{-\pi}^{\pi} \Phi\,(\pm u(re^{i\theta}))\,d\theta \qquad (0<r<1). \tag{3.2}$$

When $f(z)$ is not a constant, the equality in (3.1) holds if and only if $\omega(z) = e^{i\theta}z$ or $\Phi(u) = a\,log u + b$ $(a<0)$.

Let

$$k_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}},$$

it is well known that $k_\alpha(z) \in S^*(\alpha)$. For any $g(z) \in S^*(\alpha)$, we have

$$g(z) = z\, exp\left\{2(1-\alpha) \int_{|x|=1} log \frac{1}{1-xz}\, d\mu(x)\right\},$$

so we can easily obtain

$$\frac{g(z)}{z} \prec \frac{1}{(1-z)^{2(1-\alpha)}} = \frac{k_\alpha(z)}{z}. \tag{3.3}$$

THEOREM 3.1. If $f(z) \in S_\lambda(\alpha,\beta)$, $F_x(z) = e^{-ix}\, k_\alpha\,(e^{ix}z) \cdot p_{\beta,\lambda}\,(e^{ix}z)$, $\Phi(t)$ is a convex non-decreasing function on $(-\infty,\infty)$, then

$$\int_{-\pi}^{\pi} \Phi(\pm\, log \frac{|f(re^{i\theta})|}{r})\, d\theta \le \int_{-\pi}^{\pi} \Phi(\pm\, log \frac{|F_o(re^{i\theta})|}{r})\, d\theta \qquad (0 < r < 1). \tag{3.4}$$

For a strictly convex function $\Phi$, the equality holds only for $f(z) = F_x(z)$.

PROOF. From the definition of $S_\lambda(\alpha,\beta)$, we know there exists a function $g(z) \in S^*(\alpha)$ such that the inequality (1.1) holds. So we have, from Theorem 2.1

$$p(z) = \frac{f(z)}{g(z)} \prec \frac{1+\beta z}{1-\beta\lambda z} = P_{\beta,\lambda}(z)$$

Thus

$$\int_{-\pi}^{\pi} \Phi(\pm\, log\,|\,p(re^{i\theta})\,|)\, d\theta \le \int_{-\pi}^{\pi} \Phi(\pm\, log\,|\,p_{\beta,\lambda}(re^{i\theta})\,|)\, d\theta, \qquad \text{by Lemma 3.3.}$$

Then from Lemma 3.1 we have

$$(log\,|\,p(re^{i\theta})\,|)^* \le (log\,|\,p_{\beta,\lambda}(re^{i\theta})\,|)^*\,.$$

On the other hand, because $\frac{f(z)}{z} = p(z) \cdot \frac{g(z)}{z}$, we have, by Lemma 3.2,

$$\left(log \frac{|f(re^{i\theta})|}{r}\right)^* \le (log\,|\,p(re^{i\theta})\,|)^* + \left(log \frac{|g(re^{i\theta})|}{r}\right)^*$$

Using (3.3) and Lemmas 3.3 and 3.1, we can easily get

$$\left(log \frac{|g(re^{i\theta})|}{r}\right)^* \le \left(log \frac{|k_\alpha(re^{i\theta})|}{r}\right)^*.$$

So we obtain

$$\left(log \frac{|f(re^{i\theta})|}{r}\right)^* \le (log\,|\,p_{\beta,\lambda}(re^{i\theta})\,|)^* + \left(log \frac{|k_\alpha(re^{i\theta})|}{r}\right)^*.$$

By evaluation we know $log\,|\,p_{\beta,\lambda}(re^{i\theta})\,|$ and $log\frac{|k_\alpha(re^{i\theta})|}{r}$ are symmetric decreasing arrangement functions, so again from Lemma 3.2 we have

$$\left(log\frac{|f(re^{i\theta})|}{r}\right)^* \le \left(log \left|p_{\beta,\lambda}(re^{i\theta}) \cdot \frac{k_\alpha(re^{i\theta})}{r}\right|\right)^* = \left(log\frac{|F_o(re^{i\theta})|}{r}\right)^*$$

Finally we obtain, by Lemma 3.1,

$$\int_{-\pi}^{\pi} \Phi\left(log \frac{|f(re^{i\theta})|}{r}\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(log \frac{|F_o(re^{i\theta})|}{r}\right) d\theta$$

We can similarly prove the case of negative sign. The condition of the equality can easily be obtained.

THEOREM 3.2. Let $f(z) \in S_\lambda(\alpha,\beta)$, then for $p > 0$ we have

$$\int_{-\pi}^{\pi} \left|f\,(re^{i\theta})\right|^p d\theta \le \int_{-\pi}^{\pi} \left|k_\alpha\,(re^{i\theta})\; p_{\beta,\lambda}\,(re^{i\theta})\right|^p d\theta \qquad (0 < r < 1) \tag{3.5}$$

and

$$\int_{-\pi}^{\pi} \left| f(re^{i\theta}) \right|^{-p} d\theta \le \int_{-\pi}^{\pi} \left| k_\alpha(re^{i\theta})\, p_{\beta,\lambda}(re^{i\theta}) \right|^{-p} d\theta \qquad (0 < r < 1) \tag{3.6}$$

where the equality holds only for $f(z) = \frac{1}{x} k_\alpha(xz)\, p_{\beta,\lambda}(xz)$, $|x| = 1$.

PROOF. We only need let $\Phi(t) = e^{pt}$ in Theorem 3.1.

COROLLARY 3.1. If $f(z) \in S_\lambda(\alpha,\beta)$, then we have the following sharp inequality:

$$\frac{r(1-\beta r)}{(1+r)^{2(1-\alpha)}(1+\beta\lambda r)} \le |f(z)| \le \frac{r(1+\beta r)}{(1-r)^{2(1-\alpha)}(1-\beta\lambda r)} \qquad (|z| = r). \tag{3.7}$$

PROOF. Take $p$-th root in both sides of (3.5) and (3.6), and let $p \to \infty$, we can get inequality (3.7).

COROLLARY 3.2. If $f(z) \in S_\lambda(\alpha,\beta)$, then we have $f(D) \supset \{w\colon |w| < d(\alpha,\beta,\lambda)\}$, where

$$d(\alpha,\beta,\lambda) = \frac{1-\beta}{2^{2(1-\alpha)}(1+\beta\lambda)}$$

cannot be replaced by any larger number.

PROOF. We can easily know $f(z)$ is univalent in $D$ from the definition of $S_\lambda(\alpha,\beta)$, so

$$dist\,(0,\partial f(D)) = \lim_{|z|\to 1} inf\, |f(z)| \ge \lim_{|z|\to 1} \frac{|z|(1-\beta|z|)}{(1+|z|)^{2(1-\alpha)}(1+\beta\lambda|z|)} = \frac{1-\beta}{2^{2(1-\alpha)}(1+\beta\lambda)}.$$

The sharpness can bee seen from the function $\frac{z(1+\beta z)}{(1-z)^{2(1-\alpha)}(1-\beta\lambda z)} \in S_\lambda(\alpha,\beta)$.

## 4. A SUBCLASS $\in S_\lambda(\alpha,\beta)$.

Let $g(z) = z$, we obtain a subclass $\in S_\lambda(\alpha,\beta)$, we denote it by $S_\lambda(\beta)$. Corresponding to (2.1), for the class $S_\lambda(\beta)$, we have the following subordination:

$$\frac{f(z)}{z} \prec \frac{1+\beta z}{1-\beta\lambda z} = p_{\beta,\lambda}(z). \tag{4.1}$$

Thus for $S_\lambda(\beta)$ we have

THEOREM 4.1. Let $f(z) \in S_\lambda(\beta), \Phi(t)$ is a convex non-decreasing function on $(-\infty,\infty)$, then

$$\int_{-\pi}^{\pi} \Phi\left(\pm \log \frac{|f(re^{i\theta})|}{r}\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\pm \log \left|\frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}}\right|\right) d\theta \qquad (0 < r < 1). \tag{4.2}$$

For a strictly convex function $\Phi$, the equality holds only for function $f(z) = zp_{\beta,\lambda}(xz)$, $|x| = 1$

If we use subordination (4.1) and Lemma 3.3, we can obtain the following:

THEOREM 4.2. Let $f(z) \in S_\lambda(\beta), \Phi(t)$ is a convex non-decreasing function on $(-\infty,\infty)$, then

(a) $$\int_{-\pi}^{\pi} \Phi\left(\frac{|f(re^{i\theta})|}{r}\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\left|\frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}}\right|\right) d\theta, \tag{4.3}$$

(b) $$\int_{-\pi}^{\pi} \Phi\left(\left|\log \frac{f(re^{i\theta})}{re^{i\theta}}\right|\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\left|\log \frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}}\right|\right) d\theta, \tag{4.4}$$

(c) $$\int_{-\pi}^{\pi} \Phi\left(\pm \arg \frac{f(re^{i\theta})}{re^{i\theta}}\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\arg \frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}}\right) d\theta. \tag{4.5}$$

For a strictly convex function $\Phi$, the equality holds only for $f(z) = zp_{\beta,\lambda}(xz)$, $|x| = 1$. From (4.5) we obtain the rotation theorem of $S_\lambda(\beta)$.

COROLLARY 4.1. Let $f(z) \in S_\lambda(\beta)$, then for $|z| = r < 1$ we have the following sharp inequality:

$$\left|\arg \frac{f(z)}{z}\right| \le \arcsin \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}.$$

PROOF. If we take

$$\Phi(t) = \begin{cases} t^{2n} & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases}$$

in (4.5), we have

$$\int_o^{2\pi} \left| arg^+ \frac{f(re^{i\theta})}{re^{i\theta}} \right|^{2n} d\theta \leq \int_o^{2\pi} \left| arg^+ \frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}} \right|^{2n} d\theta.$$

Take the $2n$-th root in both sides of this inequality and let $n \to \infty$, we get

$$\max_{-\pi \leq \theta \leq \pi} arg^+ \frac{f(re^{i\theta})}{re^{i\theta}} \leq \max_{-\pi \leq \theta \leq \pi} arg^+ \frac{1+\beta re^{i\theta}}{1-\beta\lambda re^{i\theta}} = arcsin \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}.$$

This implies

$$arg^+ \frac{f(re^{i\theta})}{re^{i\theta}} \leq arcsin \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}.$$

Similarly, we have

$$arg^- \frac{f(re^{i\theta})}{re^{i\theta}} \leq arcsin \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}.$$

So for any $f(z) \in S_\lambda(\beta)$, we have

$$\left| arg \frac{f(z)}{z} \right| \leq arcsin \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2} \quad (|z| = r < 1),$$

where they equality holds only for $f(z) = zp_{\beta,\lambda}(xz), |x| = 1$. The proof of Corollary 4.1 is complete.

From the univalence of $f(z)$ we know $\frac{f(z)}{z} \neq 0$, so we can define a single-valued and analytic branch of $log \frac{f(z)}{z}$. Let

$$g(z) = log \frac{f(z)}{z} = \sum_{n=1}^{\infty} \gamma_n z^n,$$

then we have:

COROLLARY 4.2. Let $f(z) \in S_\lambda(\beta)$, then we have

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{(\beta^n\lambda^n + (-1)^{n-1}\beta^n)^2}{n^2}, \tag{4.6}$$

where the inequality holds only for $f(z) = zp_{\beta,\lambda}(xz), |x| = 1$.

PROOF. Let

$$G(z) = log \frac{1+\beta z}{1-\beta\lambda z} = \sum_{n=1}^{\infty} \frac{\beta^n\lambda^n + (-1)^{n-1}\beta^n}{n} z^n.$$

Take $\Phi(t) = t^2$ in (4.4), we have

$$\frac{1}{2\pi} \int_o^{2\pi} |g(re^{i\theta})|^2 d\theta \leq \frac{1}{2\pi} \int_o^{2\pi} |G(re^{i\theta})|^2 d\theta,$$

that is,

$$\sum_{n=1}^{\infty} |\lambda_n|^2 r^{2n} \leq \sum_{n=1}^{\infty} \frac{(\beta^n\lambda^n + (-1)^{n-1}\beta^n)^2}{n^2} r^{2n},$$

let $r \to 1$, we obtain the inequality we need to prove.

REMARK. Let $\lambda = \beta = 1$ in Corollary 4.2, that is $f(z) \in S_1(1)$, i.e., $Re(f(z)/z) > 0$. Inequality (4.6) becomes

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2} = \frac{\pi^2}{2}.$$

This inequality is sharp.

Finally, we consider the initial coefficients of $f(z) \in S_\lambda(\beta)$.

LEMMA 4.1. If $f'(0) = F'(0) = 1$ and they satisfy the following equality

$$\frac{(1-\beta)-(1+\beta\lambda)f(z)/z}{(1-\beta\lambda)f(z)/z-(1+\beta)} = \frac{F(z)}{z}, \tag{4.7}$$

then $f(z) \in S_\lambda(\beta)$ if and only if $F(z) \in S_1(1)$, i.e., $Re(F(z)/z) > 0$.

PROOF. Let $f(z) \in S_\lambda(\beta)$, then $p(z) = \frac{f(z)}{z} \prec \frac{1+\beta z}{1-\beta\lambda z} = p_{\beta,\lambda}(z)$, so $p(D) \subset p_{\beta,\lambda}(D) = D_1$ where $D_1$ is a disk which diameter is $(\frac{1-\beta}{1+\beta\lambda}, \frac{1+\beta}{1-\beta\lambda})$, thus

$$(1-\beta\lambda)p(z)-(1+\beta) \neq 0.$$

From this we know $F(z)$ is analytic in $D$. And $F(0) = 0$ because of $p(0) = 1$. On the other hand, the function

$$\frac{(1-\beta)-(1+\beta\lambda)w}{(1-\beta\lambda)w-(1+\beta)}$$

maps $D_1$ onto the right half plane, so we have $Re(F(z)/z) > 0$ $(z \in D)$, i.e., $F(z) \in S_1(1)$.

We can prove the opposite result similarly.

THEOREM 4.3. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_\lambda(\beta)$, then for real number $\mu$ we have the sharp estimates:

$$|a_2| \le \beta(1+\lambda) \tag{4.8}$$

$$|a_3 - \mu a_2^2| \le \begin{cases} \beta(1+\lambda)(\beta\lambda - \mu(\beta+\beta\lambda)), & \mu \le -\frac{1-\beta\lambda}{\beta+\beta\lambda}, \quad (4.9) \\ \beta(1+\lambda), & -\frac{1-\beta\lambda}{\beta+\beta\lambda} < \mu < \frac{1+\beta\lambda}{\beta+\beta\lambda}, \quad (4.10) \\ \beta(1+\lambda)(\mu(\beta+\beta\lambda)-\beta\lambda), & \mu > \frac{1+\beta\lambda}{\beta+\beta\lambda}. \quad (4.11) \end{cases}$$

PROOF. Because $f(z) \in S_\lambda(\beta)$, then $F(z)$ defined by (4.7) belongs to $S_1(1)$, i.e., $Re(F(z)/z) > 0$, so there exists an analytic function $p(z)$ satisfying $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, $Re\,p(z) > 0$ such that

$$\frac{F(z)}{z} = p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Substituting it into (4.7) and comparing the coefficients of both sides of (4.7), we have

$$a_2 = \tfrac{1}{2}(\beta+\beta\lambda)p_1, \quad a_3 - \mu a_2^2 = \tfrac{1}{2}(\beta+\beta\lambda)\left\{ p_2 - \tfrac{1}{2}((1-\beta\lambda)+\mu(\beta+\beta\lambda))\, p_1^2 \right\}.$$

It is well known that $|p_n| \le 2$ $(n = 1, 2, \cdots\cdots)$

$$|p_2 - \mu p_1^2| \le \begin{cases} 2(1-2\mu), & \mu \le 0 \\ 2, & 0 < \mu < 1 \\ 2(2\mu-1), & \mu \ge 1 \end{cases}.$$

So we proved the results. Its easy to know the function $f(z) = \frac{z(1+\beta\varepsilon z)}{1-\beta\lambda\varepsilon z}, (|\varepsilon| = 1)$ attains the equalities in (4.8), (4.9) and (4.11), and the function $f(z) = \frac{z(1+\beta\varepsilon z^2)}{1-\beta\lambda\varepsilon z^2}, (|\varepsilon| = 1)$ attains the inequality in (4.10).

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