Internat. J. Math. & Math. Sci.
VOL. 16 NO. 3 (1993) 503-510

FINITE ELEMENT ESTIMATES FOR A CLASS OF NONLINEAR VARIATIONAL INEQUALITIES

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(Received November 5, 1990 and in revised form March 16, 1991)

ABSTRACT. It is well known that a wide class of obstacle and unilateral problems arising in pure and applied sciences can be studied in a general and unifield framework of variational inequalities. In this paper, we derive the error estimates for the finite element approximate solution for a class of highly nonlinear variational inequalities encountered in the field of elasticity and glaciology in terms of $W^{1,p}(\Omega)$ and $L_p(\Omega)$-norms. As a special case, we obtain the well-known error estimates for the corresponding linear obstacle problem and nonlinear problems.

KEY WORDS AND PHRASES. Finite element techniques, error estimates, variational inequalities, obstacle problems.
1991 AMS SUBJECT CLASSIFICATION CODES. 65K10, 29A10, 73C20.

1. INTRODUCTION.

Variational inequality theory is an interesting branch of applicable mathematics, which not only provides us with a uniform framework for studying a large number of problems occurring in different branches of pure and applied sciences, but also gives us powerful and new numerical methods of solving them. In this paper, we consider a broad class of highly nonlinear elliptic boundary value problems having some extra constrained conditions. A much used approach with any elliptic problem is to reformulate it in a weak or variational form and then to approximate these. In the presence of a constraint, this approach leads to a variational inequality, which is the weak formulation.

In recent years, the finite element techniques are being applied to compute the approximate solutions of various classes of variational inequalities. Relative to the linear variational inequalities, little is known about the accuracy and convergance properties of finite element approximation of nonlinear variational inequalities associated with nonlinear elliptic boundary value problems. The nonlinear problems are much more complicated, since each problem has to be treated individually. This is one of the reasons that there is no unified and general theory for the nonlinear problems. An error analysis of finite element method for the boundary value problem having nonlinear operator $-\nabla(|\nabla u|^{p-2}\nabla)$ was derived by Glowinski and Marroco [1], which was an improvement of the results of Oden [2]. For piecewise linear finite element approximations, they obtained error estimates in the $W^{1,p}$-norm of order $h^{1/p-1}$, which were extended by Noor [3] for strongly nonlinear problems. Babuska [4] also obtained the same type of estimate for the finite element approximation of second order quasilinear elliptic problems.

Error estimates for various types of variational inequalities involving second order linear and nonlinear elliptic operators have been derived by many workers including Falk [5], Mosco and

Strang [6], Janovsky and Whiteman [7] and Noor ([8], [9]), under sufficient regular solutions. Oden and Reddy [10] obtained some general results for a class of highly nonlinear variational inequalities involving certain psuedo-monotone operators under the assumption that all the solutions (exact and approximate ones) of these variational inequalities are in the interior of a closed convex set in $W^{1,p}(\Omega)$. This assumption converts the variational inequalities into variational equations, which makes the error analysis a standard one as in the uncontrained case. The most important and difficult part of the problem is when the solutions are not in the interior of a closed convex set, a case not covered by their analysis. It is also known that in the presence of the constraints, the approximate solution is no longer a projection of the exact solution as in the unconstrained case. This represents a major difficulty in obtaining the error estimates for the finite element approximation of nonlinear variational inequalities.

In the present study, our analysis is based on the existence theory of nonlinear operator equations put forward by Glowinski and Marroco [1]. We extend their results for a class of nonlinear obstacle problems arising in elasticity and glaciology in Section 2. Section 3 is devoted to an analysis of error estimates in finite element approximation for our model problem. Here we derive error estimates in the $W^{1,p}(\Omega)$ and L_p-norms using the ideas and technique of Mosco and Strang [6]. Our results represent a substantial generalization and improvement of the error analysis of finite element approximation of strongly nonlinear monotone operators and variational inequalities contributed by Glowinski and Marroco [1], Oden and Reddy [10] and Noor [9].

2. VARIATIONAL INEQUALITY FORMULATION.

The mathematical model discussed in this paper arises in the field of elasticity and Oceanography, see [11]. We consider the problem of finding the velocity of the glacier, which is required to satisfy the nonlinear obstacle problem of the type $1 < p < \infty$.

$$\left.\begin{array}{rl} -\nabla(|\nabla u|^{p-2}\nabla u)-\nabla^2 u \geq f & \text{in } \Omega \\ u \geq \psi & \text{in } \Omega \\ (-\nabla(|\nabla u|^{p-2}\nabla u)-\nabla^2 u-f)(u-\psi)=0 & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{array}\right\}, \tag{2.1}$$

where Ω is the cross-section of the glacier and ψ is the given function, known as the obstacle. The presence of f and $-\nabla^2 u$ may be interpreted as body heating terms, these arises from resistivity and are local Joule heating effects. Also, in elasticity, the problem of torsional stiffness of a prismatic bar with a simply connected convex cross-section Ω and subject to steady creep, which is characterized by a power law, can be described by (2.1) and p is the exponent of the creep law. The case $p = 1$ and $\nabla^2 u = 0$ is related to the problem of capillarity and minimal surfaces, see Finn [12].

The problem (2.1) is a generalization of the nonlinear problem of finding u such that

$$\left.\begin{array}{rl} -\nabla(|\nabla u|^{p-2}\nabla u)=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{array}\right\}, \tag{2.2}$$

for which the error estimates have been derived by using the finite element approximation by Glowinski and Marroco [1]. The presence of the obstacle needs a different approach for deriving the error estimates and this is the main motivation of this paper.

Let $\Omega \subset R^n$ be a bounded open domain with smooth boundary $\partial\Omega$. We consider $W_o^{1,p}(\Omega)$ a reflexive Banach space with norm

$$\|v\| = \left(\int_\Omega |\nabla v|^p\right)^{1/p}$$

and the dual space $W^{-1,q}(\Omega)$. $\frac{1}{p}+\frac{1}{q}=1$. The pairing between $W_o^{1,p}(\Omega)$ and $W^{-1,q}(\Omega)$ is denoted by $<\cdot,\cdot>$. For more details and notation, see Kikuchi and Oden [13].

We here study the problem (2.1) in the framework of variational inequalities. To do so, we consider that set K defined by

$$K=\{v\in W_o^{1,p}(\Omega):\ v\geq\psi \quad \text{on } \Omega\}, \tag{2.3}$$

which is a closed convex set in $W_o^{1,p}(\Omega)$.

The energy (potential) functional $I[v]$ associated with the obstacle problem (2.1) is given by

$$I[v]=\frac{1}{p}\int_\Omega |\nabla v|^p\,dx+\int_\Omega (\nabla v)^2\,dx-2\int_\Omega f\,dx$$

$$=J(v)+b(v,v)-2<f,v>, \tag{2.4}$$

where

$$J(v)=\frac{1}{p}\int_\Omega |\nabla v|^p\,dx,\quad b(u,v)=\int_\Omega \nabla u.\nabla v\,dx,\quad \textit{a bilinear form}$$

and

$$<f,u>=\int_\Omega fv\,dx.$$

Following the techniques of Noor [8] and Kikuchi and Oden [13], one can show that the minimum of $I[v]$, defined by (2.4), can be characterized by a class of variational inequalities of the type

$$<Tu,v-u>+b(u,v-u)\geq<f,v-u>,\quad \text{for all } v\in K, \tag{2.5}$$

which is known as the weak formulation of the obstacle problem (2.1) with

$$<Tu,v>=<J'(u),v>=\int_\Omega |\nabla u|^{p-2}\,\nabla u\ \nabla v\,dx. \tag{2.6}$$

We here consider the variational inequality (2.5) to obtain the error estimate for $u-u_h$ in both $W^{1,p}(\Omega)$ and L_p-norms. In order to derive the main results, we need the following results which are due to Glowinski and Marroco [1].

LEMMA 2.1 For all $u,v\in W_o^{1,p}(\Omega)$, we have

$$<Tu-Tv,\ u-v>\ \geq\ \alpha\|u-v\|^p,\qquad p\geq 2 \tag{2.7}$$

$$<Tu-Tv,\ u-v>\ \leq\ \beta\|u-v\|(\|u\|+\|v\|)^{p-2},\qquad p\geq 2, \tag{2.8}$$

$$<Tu-Tv,\ u-v>\ \geq\ \alpha\|u-v\|^2(\|u\|+\|v\|)^{p-2},\qquad 1<p\leq 2, \tag{2.9}$$

$$\|Tu-Tv\|\ \geq\ \beta\|u-v\|^{p-1}\qquad 1<p\leq 2. \tag{2.10}$$

We also remark that if the operator T satisfies the relations (2.7)-(2.10) and the bilinear form $b(u,v)$ is positive continuous, then, using the techniques of Noor [14] and Kikuchi and Oden [13], we can prove the existence of a unique solution of (2.5). Furthermore, concerning the regularity of the solution $u\in K$ satisfying (2.5), we assume the following hypothesis:

(A) {For $\psi\in W_o^{1,p}(\Omega)\cap W^{2,p}(\Omega)$, $u\in K$ satisfying (2.5) also lies in $W^{2,p}(\Omega)$}.

3. FINITE ELEMENT APPROXIMATIONS.

In this section, we derive the error estimates for the finite element approximation of variational inequalities of type (2.5). To do so, we consider a finite dimensional subspace $S_h\subset W_o^{1,p}(\Omega)$ of continuous piecewise linear functions on the triangulation of the polygonal domain Ω vanishing on its boundary $\partial\Omega$. Let ψ_h be the interpolant of ψ such that ψ_h agrees at all the vertices of the triangulation. For our purpose, it is enough to choose the finite dimensional convex subset $K_h=S_h\cap\{v_h\geq\psi_h$ only at the vertices of the triangulation}, as in Berger and Falk [15]. For other choices of convex subsets, see ([5], [7], [8], [13]).

The variational inequality (2.6) can in practice seldom be solved, and so, approximation u_h to u from a finite dimensional convex subset K_h are sought. Thus the finite element approximation u_h of u is:

Find $u_h \in K_h$ such that

$$< Tu_h, v_h - u_h > + b(u_h, v_h - u_h) \geq < f, v_h - u_h >, \text{ for all } v_h \in K_h. \tag{3.1}$$

We also note that in certain cases, the equality holds instead of inequality in (2.6). This happens when v, together $2u - v$, also lies in K. In this case, we get

$$< Tu, v - u > + b(u, v - u) = < f, v - u >. \tag{3.2}$$

Furthermore, if $\tilde{u}$ is the interpolant of u, which agrees at every vertex of Ω, then $\tilde{u}$ lies in K_h. It is well known from approximation theory, see Ciarlet [16] that

$$\| u - \tilde{u} \| \leq ch \| u \|_2. \tag{3.3}$$

Finally, let M and M_h be the cones composed of non-negative functions on $W_o^{1,p}(\Omega)$ and its subspace S_h. Thus, it is clear that

$$U = u - \psi \qquad \text{is in } M$$

$$U_h = u_h - \psi_h \qquad \text{is in } M_h.$$

From these relations, it follows that

$$u - u_h = U - U_h + \psi - \psi_h. \tag{3.4}$$

We also need the following result of Mosco and Strang [6], which is known as the one-sided approximation result.

LEMMA 3.1. Suppose that $U \geq 0$ in the polygon (plane) Ω and U lies in $W_o^{1,p}(\Omega) \cap W^{2,p}(\Omega)$. Then, there exists a V_h in S_h such that

$$0 \leq V_h \leq U \qquad \text{in } \Omega$$

and

$$\| U - V_h \| \leq ch \| U \|_2 \tag{3.5}$$

We now state and prove the main result of this paper.

THEOREM 3.1. Let the nonlinear operator $T: W_o^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ satisfy the relations (2.7)-(2.10) and $b(u,v)$ be a positive continuous bilinear form. If $V_h \in M_h$ and $2U - V_h \in M$, then

$$\| u - u_h \|_{W_o^{1,p}(\Omega)} = \begin{cases} 0(h^{\frac{1}{p-1}}), & p \geq 2, \quad (3.6) \\ 0(h^{\frac{1}{3-p}}), & 1 < p \leq 2, \quad (3.7) \end{cases}$$

where $u \in K$ and $u_h \in K_h$ are the solutions of (2.6) and (3.1) respectively and the hypothesis (A) holds.

PROOF. Since both $v = \psi + V_h$ and $2u - v = \psi + (2U - V)$ are in K, we have from (2.5) and (3.2) that

$$< Tu, V_h - U > + b(u, V_h - U) = < f, V_h - U >. \tag{3.8}$$

Letting $v_h = \psi_h + V_h$ and $u_h = \psi_h + U_h$ in (3.1), we obtain

$$< Tu_h, V_h - U_h > + b(u_h, V_h - U_h) \geq < f, V_h - U_h >, \tag{3.9}$$

and taking $v = \psi + U_h$ in (2.5), we have

$$< Tu, U_h - U > + b(u, U_h - U) \geq < f, U_h - U >. \tag{3.10}$$

Subtracting (3.8) from (3.10), we get

$$< Tu, U_h - V_h > + b(u, U_h - V_h) \geq < f, U_h - V_h > \tag{3.11}$$

From (3.9) and (3.11), it follows that

$$< Tu - Tu_h, U_h - V_h > + b(u - u_h, U_h - V_h) \geq 0,$$

which can be written as

$$< Tu - Tu_h,\ U - U_h > + b(u - u_h, U - U_h) \ \leq\ < Tu - Tu_h, U - V_h > + b(u - u_h, U - V_h > \qquad (3.12)$$

Since $b(u,v)$ is a positive bilinear, so far $p \geq 2$ and using (2.7), we have

$$\begin{aligned}
\alpha \| u - u_h \|^p &\leq\ < Tu - Tu_h, u - u_h > \\
&\leq\ < Tu - Tu_h, u - u_h > + b(u - u_h, u - u_h > \\
&=\ < Tu - Tu_h, \psi - \psi_h > + b(u - u_h, \psi - \psi_h > + < Tu - Tu_h, U - U_h > + b(u - u_h, U - U_h). \\
&\leq\ < Tu - Tu_h, \psi - \psi_h > + b(u - u_h, \psi - \psi_h) \\
&\qquad + < Tu - Tu_h, U - V_h > + b(u - u_h, U - V_h), \text{ by using (3.12)}. \\
&\leq \| u - u_h \| \{ \beta(\| u \| + \| v \|)^p + \gamma \} \{ \| \psi - \psi_h \| + \| U - V_h \| \}, \qquad (3.13)
\end{aligned}$$

by using (2.8) and the continuity of $b(u,v)$.

Without loss of generality, we assume that

$$\| u_h \| \leq \| u \| . \qquad (3.14)$$

Combining (3.3), (3.8), (3.13) and (3.14), we have, for $p \geq 2$,

$$\| u - u_h \|_{W_o^{1,p}(\Omega)} = 0(h^{1/p - 1}),$$

which is the required result (3.6).

Similarly, we can show that,

$$\| u - u_h \|_{W_o^{1,p}(\Omega)} = 0(h^{\frac{1}{3-p}}), \text{ for } 1 < p \leq 2.$$

REMARK 3.1.

(1) For $p = 2$, the results obtained in this paper are exactly those of Falk [5] and Mosco and Strang [6].

(2) In the absence of the constraints, our results reduce to the well known results of Glowinski and Marroco [1] and Babuska [4].

(3) For $p = 4$, we have $\| u - u_h \|_{W_o^{1,4}(\Omega)} = 0(h^{1/3})$, which is proved by Oden and Reddy [10] in finite elasticity under the assumption that the solution lies in the interior of the convex set K. Thus our results represents an improvement of the previous results For $1 < p \leq 2$, there is no counterpart in the linear theory and our results appear to be new ones.

Using the one-sided approximation result of Mosco and Strang [6] and Aubin-Nitsche trick [16], and the techniques of Noor [17] and Mosco [18], we can derive the following error estimate for the finite element approximation of variational inequality (2.6) in the L_p-norm.

THEOREM 3.2. If $u \in K$ and $u_h \in K_h$ are solutions of (2.6) and (3.1) respectively and hypothesis (A) holds, then

$$\| (u - u_h)^+ \|_{L_p(\Omega)} = \begin{cases} 0(h^{\frac{p}{p-1}}), & p \geq 2 \\ 0(h^{\frac{p-4}{p-3}}), & 1 < p \leq 2 \end{cases}$$

and

$$\| (u - u_h)^- \|_{L_p(\Omega)} = \begin{cases} 0(h^{\frac{p}{p-1}}), & p \geq 2 \\ 0(h^{\frac{p-4}{p-3}}), & 1 < p \leq 2 \end{cases}$$

where $(u - u_h)^+ = Sup\ (u - u_h, 0)$ and $(u - u_h)^- = Inf\ (u - u_h, 0)$

REMARK 3.2. For piecewise linear elements and $p = 4$, we obtain $\| u - u_h \|_{0,4} = 0(h^{4/3})$, a result obtained by Oden and Reddy [10] under the assumption that all the solutions lie in the which can be written as

$$< Tu - Tu_h,\ U - U_h > + b(u - u_h, U - U_h) \ \le\ < Tu - Tu_h, U - V_h > + b(u - u_h, U - V_h > \qquad (3.12)$$

Since $b(u,v)$ is a positive bilinear, so far $p \ge 2$ and using (2.7), we have

$$\begin{aligned} \alpha \| u - u_h \|^p &\le\ < Tu - Tu_h, u - u_h > \\ &\le\ < Tu - Tu_h, u - u_h > + b(u - u_h, u - u_h > \\ &=\ < Tu - Tu_h, \psi - \psi_h > + b(u - u_h, \psi - \psi_h > +\ < Tu - Tu_h, U - U_h > + b(u - u_h, U - U_h). \\ &\le\ < Tu - Tu_h, \psi - \psi_h > + b(u - u_h, \psi - \psi_h) \\ &\qquad +\ < Tu - Tu_h, U - V_h > + b(u - u_h, U - V_h), \text{ by using (3.12).} \\ &\le \| u - u_h \| \{\beta(\| u \| + \| v \|)^p + \gamma\}\{\| \psi - \psi_h \| + \| U - V_h \|\}, \end{aligned} \qquad (3.13)$$

by using (2.8) and the continuity of $b(u,v)$.

Without loss of generality, we assume that

$$\| u_h \| \le \| u \|. \qquad (3.14)$$

Combining (3.3), (3.8), (3.13) and (3.14), we have, for $p \ge 2$,

$$\| u - u_h \|_{W_o^{1,p}(\Omega)} = 0(h^{1/p-1}),$$

which is the required result (3.6).

Similarly, we can show that,

$$\| u - u_h \|_{W_o^{1,p}(\Omega)} = 0(h^{\frac{1}{3-p}}), \text{ for } 1 < p \le 2.$$

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(3) For $p = 4$, we have $\| u - u_h \|_{W_o^{1,4}(\Omega)} = 0(h^{1/3})$, which is proved by Oden and Reddy [10] in finite elasticity under the assumption that the solution lies in the interior of the convex set K. Thus our results represents an improvement of the previous results. For $1 < p \le 2$, there is no counterpart in the linear theory and our results appear to be new ones.

Using the one-sided approximation result of Mosco and Strang [6] and Aubin-Nitsche trick [16], and the techniques of Noor [17] and Mosco [18], we can derive the following error estimate for the finite element approximation of variational inequality (2.6) in the L_p-norm.

THEOREM 3.2. If $u \in K$ and $u_h \in K_h$ are solutions of (2.6) and (3.1) respectively and hypothesis (A) holds, then

$$\| (u - u_h)^+ \|_{L_p(\Omega)} = \begin{cases} 0(h^{\frac{p}{p-1}}), & p \ge 2 \\ 0(h^{\frac{p-4}{p-3}}), & 1 < p \le 2 \end{cases}$$

and

$$\| (u - u_h)^- \|_{L_p(\Omega)} = \begin{cases} 0(h^{\frac{p}{p-1}}), & p \ge 2 \\ 0(h^{\frac{p-4}{p-3}}), & 1 < p \le 2 \end{cases}$$

where $(u - u_h)^+ = Sup\ (u - u_h, 0)$ and $(u - u_h)^- = Inf\ (u - u_h, 0)$

REMARK 3.2. For piecewise linear elements and $p = 4$, we obtain $\| u - u_h \|_{0,4} = 0(h^{4/3})$, a result obtained by Oden and Reddy [10] under the assumption that all the solutions lie in the interior of the closed convex set K in $W^{1,p}$-space. In this way, our results represent an improvement of their result. For $1 < p \leq 2$, our results appear to be new ones and there is no counterpart in the linear theory.

4. CONCLUSION.

In this paper, we have obtained the error estimates of the finite element approximations of the solutions of a class of highly nonlinear variational inequalities in the $W^{1,p}$ and L_p-norms, which appear to be new ones. These estimates are distinctly nonlinear in character. In particular, for $p = 2$, corresponding to the linear elliptic theory, we obtain an error of order h, which agrees with the recent results.

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