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ON RESOLVABLE AND IRRESOLVABLE SPACES

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ABSTRACT. In this paper some properties of open hereditarily irresolvable spaces are obtained and the topology for a minimal irresolvable space is specified. Maximal resolvable spaces are characterized in the last section.

KEY WORDS AND PHRASES. Resolvable, irresolvable, minimal topologies, maximal topologies.
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1. INTRODUCTION.

Let (X, τ) be a topological space. For any subset A of X we denote the closure of A respectively the interior of A with respect to τ by $cl_{\tau} A$ and $int_{\tau} A$. The relative topology on a subset A of (X, τ) is denoted by τ/A. If $B \subset A \subset X$, the closure of B respectively the interior of B with respect to τ/A is denoted by $cl_{\tau/A}B$ respectively $int_{\tau/A}B$.

A space (X, τ) is called irresolvable if each pair of dense subsets has a nonempty intersection; otherwise (X, τ), is called resolvable [6]. A subset A of X is resolvable if the subspace $(A, \tau/A)$ is resolvable. (X, τ) is said to be hereditarily irresolvable if it does not contain a nonempty resolvable subset.

E. Hewitt has studied resolvable and irresolvable spaces in [6] where he proved the following theorem.

THEOREM 1.1 [6] Every topological space (X, τ) can be represented uniquely as a disjoint union $X = F \cup G$ where F is closed and resolvable and G is open and hereditarily irresolvable.

This representation is called Hewitt representation of (X, τ).

In [5], M. Ganster has established some equivalences on a subclass of the class of irresolvable spaces in which open subspaces of each member are irresolvable. In particular, he has proved that each open subspace of a space (X, τ) is irresolvable if and only if $int_{\tau} D$ in dense for each dense subset D of (X, τ). In section 3 of this paper, some results are provided to illustrate the behavior of such a subclass of the class of irresolvable spaces.

Formally we give the following definition.

DEFINITION 1.2 A space (X, τ) is said to be open hereditarily irresolvable (simply o.h.i.) if each open subspace of (X, τ) is irresolvable.

A space (X, τ) with property R is said to be minimal R (maximal R) if for any topology τ', strictly coarser (finer) than τ, (X, τ') does not have the property R [3].

It clearly follows that for any coarser (finer) topology τ' of τ, if (X, τ) is resolvable (irresolvable) then (X, τ') is also resolvable (irresolvable) and this result may not hold for finer (coarser) topologies. Naturally the problems of characterizing maximal resolvable spaces and minimal irresolvable spaces arise. In this paper we have characterized maximal resolvable spaces and specified the topology for a minimal irresolvable space. Incidentally, we have also shown that, although the class of o.h.i. spaces is a subclass of the class of irresolvable spaces, the class of minimal irresolvable spaces coincides with the class of minimal o.h.i. spaces.

2. SOME DEFINITIONS AND PRELIMINARIES.

DEFINITION 2.1 A subset A of a space (X, τ) is called

i) an α-set [10], if $A \subset \text{int}_\tau \text{cl}_\tau \text{int}_\tau A$,

ii) a semi-open set [7], if $A \subset \text{cl}_\tau \text{int}_\tau A$,

iii) a pre-open set [9], if $A \subset \text{int}_\tau \text{cl}_\tau A$,

iv) a locally closed set [2], if $A = U \cap F$ where U is open and F is closed in (X, τ).

Let us denote by τ^α, $PO(X, \tau)$ and $LC(X, \tau)$, the collections of α-sets, pre-open sets and locally closed sets in (X, τ) respectively. In [10], Njastad has proved that τ^α forms a topology on X and $\tau^\alpha \supset \tau$.

We denote by $[\tau]$, the equivalence class of all topologies on a set X which have the same collection of semi-open sets as those of (X, τ). In [4], Crossley and Hildebrand have established that $[\tau]$ is a sub semi-lattice of the lattice of all topologies on X with the greatest element τ^α with respect to the usual join operation on topologies.

A topological property preserved under semi-homeomorphisms, which are bijections so that the image of semi-open sets are semi-open and inverse image of semi-open sets are also semi-open, is called a semi-topological property [4].

3. SOME PROPERTIES OF O.H.I. SPACES.

As a consequence of Theorem 2 and Theorem 4 in [5], it readily follows that a space (X, τ) is o.h.i. if and only if $\tau^\alpha = PO(X, \tau)$. The following theorem establishes the relationship between an o.h.i. space (X, τ) and the locally closed sets in (X, τ^α).

THEOREM 3.1. A space (X, τ) is o.h.i. if and only if for every subset A of X, $A \in LC(X, \tau^\alpha)$.

PROOF. Suppose that (X, τ) is o.h.i. Let $A \subset X$. We first show that A is the union of an open set and a nowhere dense set in (X, τ). Now $A = (A \cap \text{int}_\tau \text{cl}_\tau A) \cup (A \sim \text{int}_\tau \text{cl}_\tau A) = B \cup C$, say, where $B (= A \cap \text{int}_\tau \text{cl}_\tau A) \in PO(X, \tau)$ and $C (= A \sim \text{int}_\tau \text{cl}_\tau A)$ is nowhere dense in (X, τ). If B is empty we are done. If not, then $B \in \tau^\alpha$ and hence $B = U \cup N_1$, where $U (\neq \emptyset)$ is open and N_1 is nowehre dense in (X, τ). Thus $A = U \cup N$, where U is open and N is nowhere dense in (X, τ).

Now we show that $A \in LC(X, \tau^{\alpha})$, clearly $X - A = F \cap V$, where F is closed in (X,τ) and V is open in (X, τ^{α}). Now $F = \text{int}_{\tau} F \cup (F - \text{int}_{\tau} F) = U' \cup N'$, say, where U' is open in (X, τ) and N' is nowwhere dense in (X, τ). Also $V = U'' \cup N''$, where U'' is open in (X,τ) and N'' is nowhere dense in (X, τ). Thus $X - A = (U' \cup N') \cap (U'' \cup N'') = U^* \cup N^*$, say, where U^* is open in (X,τ) and N^* is nowhere dense in (X,τ). Now $A = (X - U^*) \cap (X - N^*)$ implies that $A \in LC(X, \tau^{\alpha})$. Conversely, suppose that for every subset A of X, $A \in LC(X, \tau^{\alpha})$. It suffices to prove that $\tau^{\alpha} = PO(X, \tau)$. Clearly $\tau^{\alpha} \subset PO(X,\tau)$. Let $A \in PO(X, \tau)$. Then $A \subset \text{int}_{\tau}\, \text{cl}_{\tau} A$. Since $A \in LC(X, \tau^{\alpha})$, $A = U \cap F$, where U is open in (X, τ^{α}) and F is closed in (X, τ^{α}). Let $x \in A$. Then $x \in \text{int}_{\tau}\, \text{cl}_{\tau}(U \cap F)$ and there exists a τ-open set G_x containing x such that $G_x \subset \text{int}_{\tau}\, \text{cl}_{\tau} (U \cap F)$. Consequently, $G_x \subset \text{cl}_{\tau} U \cap \text{int}_{\tau}\, \text{cl}_{\tau} F$. Since $U \subset \text{int}_{\tau}\, \text{cl}_{\tau}\, \text{int}_{\tau} U$ and $F \supset \text{cl}_{\tau}\, \text{int}_{\tau}\, \text{cl}_{\tau} F$, it follows that $G_x \subset \text{cl}_{\tau}\, \text{int}_{\tau} U \cap F$. Now it can be easily verified that $\text{int}_{\tau} U \cap \text{int}_{\tau} F$ is dense in G_x. Hence $x \in \text{int}_{\tau}\, \text{cl}_{\tau}\, \text{int}_{\tau} A$. Thus $A \in \tau^{\alpha}$ and consequently, $\tau^{\alpha} = PO(X, \tau)$. This completes the proof of the theorem.

Recall that for a space (X, τ) and for any subset A of X with $A \notin \tau$, the simple extension [8] of τ by A is the topology $\tau(A) = \{U \cup (V \cap A) : U, V \in \tau\}$.

It is obvious that any simple extension of an irresolvable space (X, τ) is irresolvable. But a simple extension of an o.h.i. space may not, in general, be o.h.i., as shown by the following example.

EXAMPLE 3.2. Let $X = \{a, b, c, d, e, f\}$

Let $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, d, e, f\}, \{a, b, d, e, f\}, \{a, c, d, e, f\}\}$.

Choose $A = \{c, e, f\}$,

Then $\tau/A = \{\emptyset, A, \{c\}, \{e, f\}\}$,

and $\tau(A) = \tau \cup \tau/A \cup \{\{a, e, f\}, \{a, b, e, f\}, \{a, c, e, f\}, \{a, b, c, e, f\}\}$.

Now it is clear that (X, τ) is o.h.i., since $\text{int}_{\tau} D$ is dense in (X, τ) for any dense subset D in (X, τ). But $(X, \tau(A))$ is not o.h.i., for, choose the dense subset $D = \{a, c, f\}$ in $(X, \tau(A))$. Clearly $\text{int}_{\tau(A)} D = \{a, c\}$, which is not dense in $(X, \tau(A))$.

However, the following theorem holds.

THEOREM 3.3 If a space (X, τ) and the subspace $(A, \tau/A)$ are o.h.i. then $(X, \tau(A))$ is o.h.i.

PROOF. Let $D \subset X$ be dense in $(X, \tau(A))$. Then D is dense in (X, τ) and since (X,τ) is o.h.i., $\text{int}_{\tau} D$ is dense in (X,τ). Take any nonempty open set $U \cup (V \cap A)$ in $(X, \tau(A))$. We are to show that $(U \cup (V \cap A)) \cap \text{int}_{\tau(A)} D \neq \emptyset$. If $U \neq \emptyset$, we are done. Suppose $U = \emptyset$. Then $V \cap A \neq \emptyset$. Now $A \cap D$ is dense in $(A, \tau(A)/A)$; which implies that $A \cap D$ is dense in $(A, \tau/A)$, since $(A, \tau/A) = (A, \tau(A)/A)$. Therefore $\text{int}_{\tau/A}(A \cap D)$ is dense in $(A, \tau/A)$, since $(A, \tau/A)$ is o.h.i. Hence $\text{int}_{\tau(A)/A} (A \cap D)$ is dense in $(A, \tau(A)/A)$. i.e., $(V \cap A) \cap \text{int}_{\tau(A)} D \neq \emptyset$. This completes the proof of the theorem.

One can easily verify that if (X, τ) $(|X| > 1)$ is o.h.i. then for some coarser topology τ' of τ, (X, τ') fails to be o.h.i., where $|X|$ denotes the cardinality of X. However, we shall prove that (X, τ) is o.h.i. if and only if (X, τ^{α}) is o.h.i. and using this result we shall show that the property of being o.h.i. is a semi-topological property.

THEOREM 3.4 If a space (X, τ) is o.h.i. then for any $\tau^* \in [\tau]$, (X, τ^*) is also o.h.i.

PROOF. Let $D \subset X$ be dense in (X, τ^*). Then D is dense in (X, τ) and consequently $\mathrm{int}_{\tau} D$ is dense in (X, τ). Clearly $\mathrm{int}_{\tau^*} D \neq \emptyset$. Consider any nonempty open set V in (X, τ^*). Then since $\mathrm{int}_{\tau} V \neq \emptyset$, $\mathrm{int}_{\tau} V \cap \mathrm{int}_{\tau} D = H$ (say) $\neq \emptyset$. Now $\mathrm{int}_{\tau} (\mathrm{int}_{\tau^*} H) \cap \mathrm{int}_{\tau} D = F$ (say) $\neq \emptyset$. Therefore $\mathrm{int}_{\tau^*} F \neq \emptyset$ and $\mathrm{int}_{\tau^*} F \subset \mathrm{int}_{\tau^*} D \cap V$. Hence $\mathrm{int}_{\tau^*} D$ is dense in (X, τ^*). Thus (X, τ^*) is o.h.i.

COROLLARY 3.5 A space (X, τ) is o.h.i. if and only if (X, τ^{α}) is o.h.i.

THEOREM 3.6 The property of being o.h.i. is a semi-topological property.

PROOF. Obviously this is a topological property. Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a semi-homeomorphism and let (X, τ) be o.h.i. Then by corollary 3.5 it follows that (X, τ^{α}) is o.h.i. Now it is proved in [4] that $f : (X, \tau^{\alpha}) \longrightarrow (Y, \sigma^{\alpha})$ is a homeomorphism. Hence (Y, σ^{α}) is o.h.i. and by corollary 3.5 it follows that (Y, σ) is o.h.i.

4. MINIMAL IRRESOLVABLE SPACES.

Note that if (X, τ) $(|X| > 1)$ contains an isolated point, then it is minimal irresolvable if and only if $\tau = \{\emptyset, \{p\}, X\}$ for some $p \in X$. But we shall prove that a minimal irresolvable space must contain an isolated point. To prove this we require the following two Lemmas.

LEMMA 4.1 If a space (X, τ) is irresolvable and $|X| > 1$, then τ is not the indiscrete topology.

We omit the easy proof.

LEMMA 4.2 Let (X, τ) be irresolvable and let $X = F \cup G$ be the Hewitt representation of (X, τ). If W is a nonempty open subset of G then $\sigma(W) = \{X\} \cup \{U \in \tau : U \subseteq W\}$ is a coarser irresolvable topology on X.

PROOF OF LEMMA 4.2 It is clear that $\sigma(W)$ is a topology on X with $\sigma(W) \subsetneq \tau$. Now suppose that $(X, \sigma(W))$ is resolvable, i.e., there exist disjoint subsets D and E of X which are dense in $(X, \sigma(W))$. If $D^* = D \cap W$ and $E^* = E \cap W$ then D^* and E^* are nonempty. Since G is hereditarily irresolvable, W is an irresolvable subspace of (X, τ). Hence either D^* or E^* fails to be dense in $(W, \tau/W)$, say D^*. So W is not a subset of $\mathrm{cl}_{\tau} (W \cap D)$. Hence there exists a nonempty τ-open set U with $U \subseteq W$ and $U \cap W \cap D = U \cap D = \emptyset$. But $U \in \sigma(W)$. So we have a contradiction. Consequently, $(X, \sigma(W))$ is irresolvable.

THEOREM 4.3 Let (X, τ) be irresolvable with $|X| > 1$. Then (X, τ) is minimal irresolvable if and only if there exists $p \in X$ such that $\tau = \{\emptyset, \{p\}, X\}$.

PROOF. Clearly, if $\tau = \{\emptyset, \{p\}, X\}$ for some $p \in X$, then (X, τ) is minimal irresolvable. Now suppose that (X, τ) is minimal irresolvable. If $X = F \cup G$ be the Hewitt representation of (X, τ) then G is nonempty. If $G = \{p\}$ for some

$p \in X$ then by Lemma 4.2, $\sigma(G) = \{\emptyset, \{p\}, X\} = \tau$ and we are done. Otherwise, $|G| > 1$ and by Lemma 4.1 there exists a proper nonempty τ-open subset W of G(G being irresolvable and τ-open). If $|W| > 1$ by the same argument, there exists a proper nonempty τ-open subset V of W. By Lemma 4.2, $\sigma(V) = \tau$, and thus $W \in \sigma(V)$. According to the definition of $\sigma(V)$ it follows that $W = X$. But this is a contradiction to the fact that W is a proper subset of G. Consequently, we have $W = \{p\}$ for some $p \in X$, and by Lemma 4.2, $\tau = \sigma(W) = \{\emptyset, \{p\}, X\}$.

As a straightforward consequence of Lemma 4.2 and Theorem 4.3 we get

THEOREM 4.4 Let (X, τ) be a space with $|X| > 1$. Then (X, τ) is minimal o.h.i. if and only if (X, τ) is minimal irresolvable.

REMARK 4.5 D.R. Anderson [1] has demonstrated the existence of a large class of connected irresolvable spaces which have no isolated points. Theorem 4.3 now indicates the existence of irresolvables spaces whose topologies have no minimal irresolvable subtopology.

5. MAXIMAL RESOLVABLE SPACES.

It is not difficult to see that maximal resolvable spaces exist. One of the simple examples is the following.

EXAMPLE 5.1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X\}$. Then (X, τ) maximal resolvable.

In this section we investigate necessary and sufficient conditions for a space to be maximal resolvable. We first require the following Lemmas.

LEMMA 5.2 Let (X, τ) be resolvable and $A \subset X$ be a resolvable subspace. Then $(X, \tau(A))$ is resolvable.

PROOF OF LEMMA 5.2 There exists a subset D of A such that D is dense in A and $\operatorname{int}_{\tau/A} D = \emptyset$. Two cases arises :

Case I. $X - \operatorname{cl}_\tau A = \emptyset$.

Then D is dense in (X, τ) and $\operatorname{int}_\tau D = \emptyset$. Consider the topology $\tau(A)$. Clearly D is dense in $(X, \tau(A))$ and $\operatorname{int}_{\tau(A)} D = \emptyset$. So $(X, \tau(A))$ is resolvable in this case.

Case II. $X - \operatorname{cl}_\tau A \neq \emptyset$.

Since open subspace of a resolvable space is resolvable, $X - \operatorname{cl}_\tau A$ is resolvable. Choose $D^* \subset X - \operatorname{cl}_\tau A$ such that D^* is dense in $X - \operatorname{cl}_\tau A$ with $\operatorname{int}_{\tau/X-\operatorname{cl}_\tau A} D^* = \emptyset$. Then $D \cup D^*$ is dense in (X, τ) and $\operatorname{int}_\tau (D \cup D^*) = \emptyset$; for, if there exists a nonempty τ-open set $0 \subset D \cup D^*$, then $0 \cap D^* \neq \emptyset \Rightarrow 0 \cap (X - \operatorname{cl}_\tau A) = 0'$(say) $\neq \emptyset$ $\Rightarrow \operatorname{int}_{\tau/X-\operatorname{cl}_\tau A} D^* \neq \emptyset$ (since $0' \cap A = \emptyset$) ; a contradiction to the choice of D^*. Now consider the topology $\tau(A)$. Clearly $D \cup D^*$ is dense in $(X, \tau(A))$ and $\operatorname{int}_{\tau(A)}(D \cup D^*) = \emptyset$; for if $\emptyset \neq U \cup (V \cap A) \subset D \cup D^*$, for some $U, V \in \tau$ then $U = \emptyset$ and $V \cap A \subset D \cup D^* \Rightarrow V \cap A \subset D$; a contradiction, since $V \cap A$ is nonempty open in $(A, \tau/A)$ and $\operatorname{int}_{\tau/A} D = \emptyset$. Hence $(X, \tau(A))$ is resolvable in this case also.

Now we come to the main theorem of this section.

THEOREM 5.3 For a space (X, τ) the following are equivalent :

(i) (X, τ) is maximal resolvable,

(ii) The set of all open subsets of X = the set of all resolvable subsets of X,

(iii) any continuous bijection f from a resolvable space (Y, σ) onto (X, τ) is a homeomorphism.

PROOF. (i) $\Rightarrow$ (ii) : Clearly every open subset is resolvable. Now suppose $A \subset X$ is resolvable but not open. Then by Lemma 5.2 $(X, \tau(A))$ is resolvable. Hence (X, τ) cannot be maximal resolvable since $\tau(A) \supsetneq \tau$.

(ii) $\Rightarrow$ (i) : Suppose (X, τ) is not maximal resolvable. Then there exists a topology τ' containing τ properly such that (X, τ') is resolvable. Let $U \in \tau'$ such that $U \notin \tau$. Then U is resolvable in (X, τ') and hence resolvable in (X, τ). This contradicts (ii).

(i) $\Rightarrow$ (iii): If $f : (Y, \sigma) \longrightarrow (X, \tau)$ is a continuous bijection then for $\tau' = \{f(G) : G \in \sigma\}$, $f : (Y, \sigma) \longrightarrow (X, \tau')$ is a homeomorphism and (X, τ') is resolvable (since the property of being resolvable is a topological property). Since $\tau' \supset \tau$ and (X, τ) is maximal resolvable, it follows that $\tau' = \tau$.

(iii) $\Rightarrow$ (i) : If (X, τ) is resolvable but not maximal resolvable, then there exists a topology $\tau' \supsetneq \tau$ such that (X, τ') is resolvable. The identity map $I : (X, \tau') \longrightarrow (X, \tau)$ is a continuous bijection which is not a homeomorphism.

NOTE 5.4 By repeated application of simple extension, from any given resolvable space one can arrive at a maximal resolvable space by using Lemma 5.2 and Theorem 5.3.

COROLLARY 5.5 Let (X, τ) be maximal resolvable. Then

(i) (X, τ) is extremally disconnected,

(ii) Semi-open sets are open in (X, τ)

PROOF. (i) Let G be open and $x \in cl_\tau G - G$. Since G is resolvable, so is $G \cup \{x\}$ and by theorem 5.3 $G \cup \{x\}$ is open.

Thus $cl_\tau G = \bigcup_{x \in cl_\tau G} (G \cup \{x\})$ is open. Hence (X, τ) is extremally disconnected.

(ii) Proof of (ii) follows as in (i).

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