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THE NACHBIN COMPACTIFICATION VIA CONVERGENCE ORDERED SPACES

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ABSTRACT. We construct the Nachbin compactification for a $T_{3.5}$-ordered topological ordered space by taking a quotient of an ordered convergence space compactification. A variation of this quotient construction leads to a compactification functor on the category of $T_{3.5}$-ordered convergence ordered spaces.

KEY WORDS AND PHRASES: topological ordered space, convergence ordered space, T_2-ordered space, $T_{3.5}$-ordered space, Nachbin compactification.
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0. INTRODUCTION.

The Nachbin (or Stone-Cech-ordered) compactification (see [1], [6]) is the largest T_2-ordered topological ordered compactification of a $T_{3.5}$-ordered topological ordered space. In [4], one of the authors and G.D. Richardson constructed an ordered compactification (X^*, φ) for an arbitrary convergence ordered space X. This latter compactification exhibits essentially the same universal property as the Nachbin compactification, but behaves poorly relative to separation properties (see Example 1.4).

Starting with an arbitrary convergence ordered space X, we introduce an equivalence relation $\mathcal{R}$ on the set $|X^*|$ which underlies X^*, and obtain an ordered quotient space $X^*/\mathcal{R}$ which is both compact and T_2-ordered. We next give two conditions C and O which are necessary and sufficient to make the natural map from X into $X^*/\mathcal{R}$ both an order embedding and a homeomorphic embedding, so that $X^*/\mathcal{R}$ becomes a T_2-ordered convergence ordered compactification of X. For ordered convergence spaces X satisfying conditions C and O, it turns out that the topological modification λX of X is a $T_{3.5}$-ordered topological ordered space, and $\lambda(X^*/\mathcal{R})$ is the Nachbin compactification of λX. In particular, if X is assumed to be a $T_{3.5}$-ordered topological ordered space, then $\lambda(X^*/\mathcal{R})$ is the Nachbin compactification of X.

In addition to giving an alternate construction for the Nachbin compactification, we obtain some interesting results pertaining to convergence ordered compactifications. In Section 3, we define a regular convergence ordered space satisfying conditions C and O to be a $T_{3.5}$*-ordered convergence ordered space*, and we show that for such a space X, the regular modification $r(X^*/\mathcal{R})$ of the quotient $X^*/\mathcal{R}$ is a regular, T_2-ordered convergence ordered compactification of X. Relative to this compactification functor, the regular, T_2-ordered, compact convergence spaces (with increasing, continuous maps as morphisms) form an epireflective subcategory of the category of all $T_{3.5}$-ordered convergence ordered spaces (with increasing, continuous maps as morphisms).

1. PRELIMINARIES.

We introduce some basic notation and terminology and summarize some results from [4]. If $(X, \leq)$ is a poset, and $A \subseteq X$, we denote by $i(A), d(A)$, and $A^\wedge$ the *increasing, decreasing,* and *convex* hulls, respectively, of A; note that $A^\wedge = i(A) \cap d(A)$. Similarly, if $\mathbf{F}(X)$ is the set of all (proper) filters on X and $\mathcal{F} \in \mathbf{F}(X)$, let $i(\mathcal{F})$, the filter generated by $\{i(F) : F \in \mathcal{F}\}$, be the *increasing hull* of $\mathcal{F}$; the decreasing hull $d(\mathcal{F})$ and convex hull $\hat{\mathcal{F}}$ are defined analagously. A filter $\mathcal{F}$ is said to be *convex* if $\mathcal{F} = \hat{\mathcal{F}}$. Note that $\hat{\mathcal{F}} = i(\mathcal{F}) \vee d(\mathcal{F})$.

If $(X, \leq, \rightarrow)$ is a poset $(X, \leq)$ equipped with a convergence structure $\rightarrow$ which is *locally convex* (i.e., $\hat{\mathcal{F}} \rightarrow x$ whenever $\mathcal{F} \rightarrow x$), then $(X, \leq, \rightarrow)$ is called a *convergence ordered space*; we usually write X rather than $(X, \leq, \rightarrow)$ when there is no danger of ambiguity. A convergence ordered space is T_1 *-ordered* if the sets $i(x)$ and $d(x)$ are closed for all $x \in X$, and T_2*-ordered* if the order $\leq$ is a closed subset of $X \times X$. For any convergence ordered space X, let $CI^*(X)$ (respectively, $CD^*(X)$) denote the set of all continuous, increasing (respectively, decreasing) maps from X into $[0,1]$.

A convergence ordered space whose convergence structure is a topology is called a *topological ordered space*. Such a space is said to be *convex* if the open monotone (i.e., increasing or decreasing) sets form a subbase for the topology. For the remainder of this paper, we shall adopt the notational abbreviation used in [4] and write "t.o.s" instead of "topological ordered space" and "c.o.s." in place of "convergence ordered space".

A t.o.s. X is said to be $T_{3.5}$-ordered if it satisfies the following conditions: (1) If $x \in X, A$ is a closed subset of X, and $x \notin A$, then there is $f \in CI^*(X)$ and $g \in CD^*(X)$ such that $f(x) = g(x) = 0$ and $f(y) \vee g(y) = 1$, for all $y \in A$; (2) If $x \not\leq y$ in X, there is $f \in CI^*(X)$ such that $f(y) = 0$ and $f(x) = 1$. The $T_{3.5}$-ordered spaces are precisely those which allow T_2-ordered t.o.s. compactifications, and all $T_{3.5}$-ordered spaces are convex.

If X is a $T_{3.5}$-ordered t.o.s., then the *Nachbin compactification* of X (see [1], [6]) is obtained by embedding X into an "ordered cube", whose component intervals are indexed by $CI^*(X)$. The Nachbin compactification $\beta_0 X$ is characterized by the following well-known result.

PROPOSITION 1.1. If X is a $T_{3.5}$-ordered t.o.s., then $\beta_0 X$ is T_2-ordered. Furthermore, if $f : X \rightarrow Y$ is an increasing, continuous map and Y is a compact, T_2-ordered t.o.s., then f has a unique, increasing, continuous extension $f' : \beta_0 X \rightarrow Y$.

We next describe briefly the construction of the convergence ordered compactification X^* of an arbitrary c.o.s. X described in [4], which has essentially the same lifting property as $\beta_0 X$. Given a c.o.s. X, let X' be the set of all non-convergent maximal convex filters on X, and let $X^* = \{\dot{x} : x \in X\} \cup X'$. Before proceeding further, it will be useful to establish the following proposition about maximal convex filters.

PROPOSITION 1.2. The maximal convex filters on a poset X are precisely the set $\{\hat{\mathcal{F}} : \mathcal{F}$ is an ultrafilter on $X\}$.

PROOF. Clearly every maximal convex filter is the convex hull of every finer ultrafilter. Conversely, suppose $\mathcal{F}$ is an ultrafilter on X and $\mathcal{G}$ is a convex filter such that $\hat{\mathcal{F}} \leq \mathcal{G}$. Then for any convex set $G \in \mathcal{G}$, the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ generated by $\{i(G) \cap F : F \in \mathcal{F}\}$ and $\{d(G) \cap F : F \in \mathcal{F}\}$, respectively, are well-defined filters finer than, and hence equal to, $\mathcal{F}$. Thus $i(G) \in \mathcal{F}$ and $d(G) \in \mathcal{F}$ implies $i(G) \cap d(G) = G \in \mathcal{F}$; therefore $\mathcal{G} = \hat{\mathcal{F}}$. ∎

Again assuming that X is an arbitrary c.o.s., let $\varphi : X \to X^{\bullet}$ be defined by $\varphi(x) = \dot{x}$, for all $x \in X$. A partial order $\leq^{\bullet}$ is defined on $X^{\bullet}$ as follows: $\mathcal{F} \leq^{\bullet} \mathcal{G}$ iff $i(\mathcal{F}) \leq \mathcal{G}$ (or, equivalently, $d(\mathcal{G}) \leq \mathcal{F}$). Since $x \leq y$ iff $\dot{x} \leq^{\bullet} \dot{y}$, $\varphi : (X, \leq) \to (X^{\bullet}, \leq^{\bullet})$ is an order embedding.

If $A \subseteq X$, let $A^{\bullet} = \{\mathcal{F} \in X^{\bullet} : A \in \mathcal{F}\}$; if $\mathcal{F} \in \boldsymbol{F}(X)$, let $\mathcal{F}^{\bullet}$ denote the filter in $\boldsymbol{F}(X^{\bullet})$ generated by $\{F^{\bullet} : F \in \mathcal{F}\}$. A convergence structure $\xrightarrow{*}$ on $(X^{\bullet}, \leq^{\bullet})$ is defined as follows: For $\mathcal{A} \in \boldsymbol{F}(X^{\bullet})$,

$$\mathcal{A} \xrightarrow{*} \dot{x} \in \varphi(X) \text{ iff there is } \mathcal{F} \to x \text{ such that } \mathcal{F}^{\bullet} \leq \mathcal{A};$$
$$\mathcal{A} \xrightarrow{*} \mathcal{G} \in X' \text{ iff } \mathcal{G}^{\bullet} \leq \mathcal{A}.$$

Writing $X^{\bullet}$ in place of $(X^{\bullet}, \leq^{\bullet}, \xrightarrow{*})$, we state the following result which is proved in [4].

PROPOSITION 1.3. If X is a c.o.s., then $(X^{\bullet}, \varphi)$ is a convergence ordered compactification of X. If $f : X \to Y$ is a continuous, increasing map and Y a compact, regular, T_2-ordered c.o.s., then f has a unique, increasing, continuous extension $f_{\bullet} : X^{\bullet} \to Y$.

Recall that a convergence space Y is *regular* if $cl_Y \mathcal{F} \to x$ whenever $\mathcal{F} \to x$. Here "cl_Y" is the closure operator for Y, and $cl_Y \mathcal{F}$ is the filter on Y generated by $\{cl_Y F : F \in \mathcal{F}\}$.

In [4], a c.o.s. X is defined to be *strongly T_2-ordered* if X is T_2 (i.e., convergent filters have unique limits) and the following conditions hold: (S_1) if $\mathcal{F} \to x, \mathcal{G} \in X'$, and $i(\mathcal{F}) \leq \mathcal{G}$, then $d(\mathcal{G}) \leq \dot{x}$; (S_2) if $\mathcal{F} \to x, \mathcal{G} \in X'$, and $d(\mathcal{F}) \leq \mathcal{G}$, then $i(\mathcal{G}) \leq \dot{x}$. In Proposition 2.8, [4], it is shown that $X^{\bullet}$ is T_2-ordered iff X is strongly T_2-ordered. As we see in the next example, very nice c.o.s.'s may fail to be strongly T_2-ordered.

EXAMPLE 1.4. Let X be the Euclidean plane with its usual (product) order and topology. Let $\mathcal{F}$ be the filter on X generated by sets of the form $F_n = \{(a, b) \in X : -\frac{1}{n} < a < 0, \ \ b = 0\}$ for each natural number n, and let $x = (0, 0)$. Let $\mathcal{G}$ be the convex hull of any ultrafilter containing the set $S = \{(a, b) \in X : a = -b^{-1}\}$ and coarser than the filter $\mathcal{H}$ generated by sets of the form $H_n = \{(a, b) \in X : b \geq n\}$ for $n = 1, 2, 3, \ldots$. Note that (S_1) is violated by $\mathcal{F}, \mathcal{G}$ and x; thus the compactification $X^{\bullet}$ of X is not T_2-ordered. ∎

2. $\beta_0 X$ AS A QUOTIENT OF $X^{\bullet}$.

Let $(X, \leq, \to)$ be any c.o.s., and let $(X^{\bullet}, \varphi)$ be the convergence ordered compactification of X constructed in the last section. By Proposition 1.3 there is, for any $f \in CI^{\bullet}(X)$, a unique, continuous, increasing extension $f_{\bullet} : X^{\bullet} \to [0, 1]$.

We define an equivalence relation $\mathcal{R}$ on $X^{\bullet}$ as follows: $\mathcal{R} = \{(\mathcal{F}, \mathcal{G}) \in X^{\bullet} \times X^{\bullet} : f_{\bullet}(\mathcal{F}) = f_{\bullet}(\mathcal{G}),$ for all $f \in CI^{\bullet}(X)\}$. Let σ be the projection map of $X^{\bullet}$ onto $X^{\bullet}/\mathcal{R}$ (i.e., for each $\mathcal{F} \in X^{\bullet}$, $\sigma(\mathcal{F}) = [\mathcal{F}]$, where $[\mathcal{F}]$ is the $\mathcal{R}$-equivalence class containing $\mathcal{F}$). A partial order $\leq_{\mathcal{R}}$ on $X^{\bullet}/\mathcal{R}$ is defined as follows:

$$[\mathcal{F}] \leq_{\mathcal{R}} [\mathcal{G}] \text{ iff } f_{\bullet}(\mathcal{F}) \leq f_{\bullet}(\mathcal{G}) \text{ in } R \text{ for all } f \in CI^{\bullet}(X).$$

We also impose on $X^{\bullet}/\mathcal{R}$ the quotient convergence structure which is described (see [2]) as follows: If $\Phi \in \boldsymbol{F}(X^{\bullet}/\mathcal{R})$ and $[\mathcal{F}] \in X^{\bullet}/\mathcal{R}$, then $\Phi \to [\mathcal{F}]$ in $X^{\bullet}/\mathcal{R}$ iff there is $\mathcal{F}' \in [\mathcal{F}]$ and there is a filter $\mathcal{A} \in \boldsymbol{F}(X^{\bullet})$ such that $\mathcal{A} \xrightarrow{*} \mathcal{F}'$ in $X^{\bullet}$ and $\sigma(\mathcal{A}) \leq \Phi$.

THEOREM 2.1. For any c.o.s. X, $X^{\bullet}/\mathcal{R}$ is a compact, T_2-ordered c.o.s.

PROOF. $X^{\bullet}/\mathcal{R}$ is obviously compact. To show that $X^{\bullet}/\mathcal{R}$ is T_2-ordered, it is sufficient (by Proposition 1.2, [4]) to show that if $\Phi, \Theta \in \mathbf{F}(X^{\bullet}/\mathcal{R})$, $\Phi \to [\mathcal{F}]$ and $\Theta \to [\mathcal{G}]$ in $X^{\bullet}/\mathcal{R}$, and $\Phi \times \Theta$ has a trace on the order $\leq_{\mathcal{R}}$, then $[\mathcal{F}] \leq_{\mathcal{R}} [\mathcal{G}]$.

If $f \in CI^{\bullet}(X)$, define $f_{\mathcal{R}} : X^{\bullet}/\mathcal{R} \to [0,1]$ by $f_{\mathcal{R}}([\mathcal{F}]) = f_{\bullet}(\mathcal{F})$, for all $\mathcal{F} \in X^{\bullet}$. It is easy to verify that $f_{\mathcal{R}}$ is well-defined and $f_{\mathcal{R}} \in CI^{\bullet}(X^{\bullet}/\mathcal{R})$. If $\Phi \to [\mathcal{F}]$ and $\Theta \to [\mathcal{G}]$ in $X^{\bullet}/\mathcal{R}$ and $\Phi \times \Theta$ has a trace on $\leq_{\mathcal{R}}$, it follows that $f_{\mathcal{R}}(\Phi) \times f_{\mathcal{R}}(\Theta)$ has a trace on the order of $[0,1]$; since $[0,1]$ is T_2-ordered, $f_{\mathcal{R}}([\mathcal{F}]) = f_{\bullet}(\mathcal{F}) \leq f_{\bullet}(\mathcal{G}) = f_{\mathcal{R}}([\mathcal{G}])$. The latter inequality holds for all $f \in CI^{\bullet}(X)$, and so $[\mathcal{F}] \leq_{\mathcal{R}} [\mathcal{G}]$, which establishes that $X^{\bullet}/\mathcal{R}$ is T_2-ordered. ∎

For an arbitrary c.o.s. X, we have already defined the continuous, increasing maps $\varphi : X \to X^{\bullet}$ and $\sigma : X^{\bullet} \to X^{\bullet}/\mathcal{R}$; we define $\varphi_{\mathcal{R}} : X \to X^{\bullet}/\mathcal{R}$ by $\varphi_{\mathcal{R}} = \sigma \circ \varphi$. It is clear that $\varphi_{\mathcal{R}}(X)$ is dense in the compact, T_2-ordered c.o.s. $X^{\bullet}/\mathcal{R}$. We are now interested in characterizing those spaces X for which $(X^{\bullet}/\mathcal{R}, \varphi_{\mathcal{R}})$ is a compactification. With this goal in mind, we introduce the following conditions.

CONDITION C. For any maximal convex filter $\mathcal{F}$ on X, $\mathcal{F} \to x$ in X iff $f(\mathcal{F}) \to f(x)$ in $[0,1]$ for all $f \in CI^{\bullet}(X)$.

CONDITION O. For any points x, y in X, $x \leq y$ in X iff $f(x) \leq f(y)$ in $[0,1]$, for all $f \in CI^{\bullet}(X)$.

It is easy to verify that any $T_{3.5}$-ordered t.o.s. satisfies Conditions C and O.

LEMMA 2.2. If X is a c.o.s. satisfying Conditions C and O, then $[\dot{x}] = \{\dot{x}\}$, for all $x \in X$.

PROOF. $CI^{\bullet}(X)$ separates points in X by Condition O, and so σ is one-to-one on $\varphi(X)$. This implies $\dot{y} \notin [\dot{x}]$ if $x \neq y$. Next, assume that there is $\mathcal{F} \in X' \cap [\dot{x}]$. Then $f_{\bullet}(\mathcal{F}) = f_{\bullet}(\dot{x}) = f(x)$ for all $f \in CI^{\bullet}(X)$; in other words, $f(\mathcal{F}) \to f(x)$ in R, for all $f \in CI^{\bullet}(X)$. Condition C then implies $\mathcal{F} \to x$ in X, contradicting the assumption $\mathcal{F} \in X'$. ∎

THEOREM 2.3. Let X be a c.o.s. Then $\varphi_{\mathcal{R}} : X \to X^{\bullet}/\mathcal{R}$ is an order and a homeomorphic embedding iff X satisfies Conditions C and O.

PROOF. Suppose that X satisfies Conditions C and O. Then $\varphi_{\mathcal{R}}$ is one-to-one since $CI^{\bullet}(X)$ separates points in X. Also note that $\varphi_{\mathcal{R}} = \sigma \cdot \varphi = (\sigma|_{\varphi(X)}) \circ \varphi$, and thus $\sigma|_{\varphi(X)}$ is one-to-one.

Let $\Phi \to [\dot{x}]$ in $X^{\bullet}/\mathcal{R}$. Then there is $\mathcal{A} \in \mathbf{F}(X^{\bullet})$ such that $\mathcal{A} \xrightarrow{*} \dot{x}$ in $X^{\bullet}$ and $\Phi \geq \sigma(\mathcal{A})$. By definition of $*$ convergence in $X^{\bullet}$, there is a filter $\mathcal{F}$ on X such that $\mathcal{F} \to x$ and $\mathcal{A} \geq \mathcal{F}^{\bullet}$. Therefore, $\varphi_{\mathcal{R}}^{-1}(\Phi) \geq \varphi_{\mathcal{R}}^{-1}(\sigma(\mathcal{A})) \geq \varphi_{\mathcal{R}}^{-1}(\sigma(\mathcal{F}^{\bullet})) = \varphi^{-1} \cdot (\sigma|_{\varphi(X)})^{-1}(\sigma(\mathcal{F}^{\bullet}))$. It follows by Lemma 2.2 that $(\sigma|_{\varphi(X)})^{-1}(\sigma(\mathcal{F}^{\bullet})) \geq \mathcal{F}^{\bullet}$. Consequently, $\varphi_{\mathcal{R}}^{-1}(\Phi) \geq \varphi^{-1}(\mathcal{F}^{\bullet}) = \mathcal{F} \to x = \varphi_{\mathcal{R}}^{-1}([\dot{x}])$, i.e. $\varphi_{\mathcal{R}}^{-1}(\Phi) \to \varphi_{\mathcal{R}}^{-1}([\dot{x}])$. Thus $\varphi_{\mathcal{R}}^{-1}$ is continuous.

Let $[\dot{x}] \leq_{\mathcal{R}} [\dot{y}]$ in $X^{\bullet}/\mathcal{R}$; then for any $f \in CI^{\bullet}(X)$, $f_{\bullet}(\dot{x}) \leq f_{\bullet}(\dot{y})$, i.e. $f_{\bullet}(\varphi(x)) \leq f_{\bullet}(\varphi(y))$, which implies $f(x) \leq f(y)$, for all $f \in CI^{\bullet}$(X). By Condition O, $x \leq y$. Thus $\varphi_{\mathcal{R}}^{-1}$ is increasing, and we conclude that $\varphi_{\mathcal{R}}$ is an order and homeomorphic embedding.

Conversely, assume that $\varphi_{\mathcal{R}}$ is both an order and homeomorphic embedding. Let $\mathcal{F}$ be a maximal convex filter on X such that, for some $x \in X$, $f(\mathcal{F}) \to f(x)$ for all $f \in CI^{\bullet}(X)$. Suppose $\mathcal{F} \to x$ is not true. Then we need to consider two cases.

CASE 1. $\mathcal{F} \to y$ and $y \neq x$. This implies that for each $f \in CI^{\bullet}(X)$, $f(\mathcal{F}) \to f(y)$. From this we deduce that $[\dot{x}] = [\dot{y}]$, which is a contradiction, since $\varphi_{\mathcal{R}}$ is assumed to be one-to-one.

CASE 2. $\mathcal{F} \in X'$. This leads to the conclusion that $[\mathcal{F}] = [\dot{x}]$; in other words, $\varphi_{\mathcal{R}}(\mathcal{F}) \to [\dot{x}]$ in $X^{\bullet}/\mathcal{R}$, which implies $\mathcal{F} \to x$ in X, since $\varphi_{\mathcal{R}}$ is a homeomorphic embedding. This contradicts $\mathcal{F} \in X'$. We therefore conclude that X satisfies Condition C.

Finally, let $x, y \in X$ such that $f(x) \leq f(y)$ for all $f \in CI^{\bullet}(X)$. Then $f_{\bullet}(\varphi(x)) \leq f_{\bullet}(\varphi(y))$ for all $f \in CI^{\bullet}(X)$, i.e. $f_{\bullet}(\dot{x}) \leq f_{\bullet}(\dot{y})$ for all $f \in CI^{\bullet}(X)$. This implies $[\dot{x}] \leq_{\mathcal{R}} [\dot{y}]$ in $X^{\bullet}/\mathcal{R}$, and $x \leq y$

follows since $\varphi_{\mathcal{R}}$ is an order embedding. Therefore, X satisfies Condition O. ∎

THEOREM 2.4. For every c.o.s. X satisfying Conditions C and O, $((X^*/\mathcal{R}), \varphi_{\mathcal{R}})$ is a T_2-ordered c.o.s. compactification of X. Furthermore, for any compact, regular, T_2-ordered c.o.s. Y and for any continuous, increasing map $f : X \to Y$, there is a unique, continuous, increasing extension $f_{\mathcal{R}} : X^*/\mathcal{R} \to Y$.

PROOF. The first assertion is an immediate corollary of Theorem 2.3. The second follows easily with the help of Proposition 1.3. ∎

For any c.o.s. X, let $\omega_o X$ be the t.o.s. consisting of the poset $(X, \leq)$ with the weak topology induced by $CI^*(X)$. Note that $CI^*(X) = CI^*(\omega_o X)$.

PROPOSITION 2.5. Let X be a c.o.s. satisfying Condition C. Let $i : X \to \omega_o X$ be the identity map. Then i is an order isomorphism and a homeomorphism relative to ultrafilter convergence.

PROOF. It is obvious that i is a continuous order isomorphism. Let $\mathcal{F} \to x$ in $\omega_0 X$, where $\mathcal{F}$ is an ultrafilter. By Proposition 1.2, $\hat{\mathcal{F}}$ is a maximal convex filter and $f(\mathcal{F}) \to f(x)$ implies $f(\hat{\mathcal{F}}) \to f(x)$ in $[0,1]$, for all $f \in CI^*(X)$. Condition C thus guarantees that $\hat{\mathcal{F}} \to x$ in X, and hence $\mathcal{F} \to x$ in X. ∎

PROPOSITION 2.6. If X is a c.o.s. satisfying Conditions C and O, then $\omega_o X$ is a $T_{3.5}$-ordered t,o.s.

PROOF. First observe that $\omega_o X$ also satisfies Condition C and O; O is obvious, and C follows from Proposition 2.5, since X and $\omega_o X$ have the same ultrafilter convergence and hence, by Proposition 1.2, the same convergence of maximal convex filters.

For $f \in CI^*(\omega_o X)$, let I be the closed interval $[0,1]$ indexed by f, and let $P = \Pi\{I_f : f \in CI^*(X)\}$ be equipped with the usual product order and product topology. Then P is a compact, T_2-ordered t.o.s. Define $\varphi_o : \omega_o X \to P$ by $\varphi_o(x) = \hat{x}$, where $\hat{x} : CI^*(\omega_o X) \to [0,1]$ is given by $\hat{x}(f) = f(x)$, for all $f \in CI^*(\omega_o X)$. Since $\omega_o X$ has the weak topology induced by $CI^*(\omega_o X) = CI^*(X)$, and $CI^*(\omega_o X)$ separates points in $\omega_o X$ by Condition O, φ_o is a topological embedding (see 8.12, [10]). By Condition O, φ_o is also an order embedding. ∎

Given a c.o.s. X satisfying C and O, we introduce some additional functional notation. Let e_o be the evaluation embedding of the $T_{3.5}$-ordered t.o.s. $\omega_o X$ into its Nachbin compactification $\beta_o X$, and let $e = e_o \cdot i : X \to \beta_o(\omega_o X)$. The unique extension of e to X^* (guaranteed by Proposition 1.3) is denoted by e_*, and the extension of e to $X^*/\mathcal{R}$ (guaranteed by Theorem 2.4) is denoted by $e_{\mathcal{R}}$. If $f \in CI^*(X) = CI^*(\omega_o X)$, the unique extensions of f in $CI^*(X^*)$ and $CI^*(\beta_o(\omega_o X))$ (see Proposition 1.3 and 2.4) are denoted by f_* and f^*, respectively. The following commutative diagram is helpful in keeping track of these various maps.

$$\begin{array}{ccccc} X & \overset{\varphi}{\rightarrow} & X^* & \overset{\sigma}{\rightarrow} & X^*/\mathcal{R} \\ i\downarrow & \overset{\searrow}{e} & \downarrow e_* & & \overset{\swarrow}{e_{\mathcal{R}}} \\ \omega_o X & \underset{e_o}{\rightarrow} & \beta_o(\omega_o X) & & \end{array}$$

THEOREM 2.7. If X is any c.o.s. satisfying C and O, then $e_{\mathcal{R}}$ is an order isomorphism and a homeomorphism relative to ultrafilter convergence.

PROOF. Since $[\mathcal{F}] = [\mathcal{G}]$ in $X^*/\mathcal{R}$ iff $e_*(\mathcal{F}) = e_*(\mathcal{G})$ iff $e_{\mathcal{R}}([\mathcal{F}]) = e_{\mathcal{R}}([\mathcal{G}])$, it follows that $e_{\mathcal{R}}$ is one-to-one. Furthermore, $e(X)$ is dense in $\beta_o(\omega_o X)$, which implies that the extension $e_{\mathcal{R}}$ is onto $\beta_o(\omega_o X)$. It follows from Theorem 2.4 that $e_{\mathcal{R}}$ is continuous and increasing. Finally, if $\mathcal{H}$ is an ultrafilter on $\beta_o(\omega_o X)$ and $\mathcal{H} \to a$ in $\beta_o(\omega_o X)$, then there is $\alpha \in X^*/\mathcal{R}$ such that $e_{\mathcal{R}}^{-1}(\mathcal{H}) \to \alpha$ in

$X^*/\mathcal{R}$ since the latter space is compact. It follows by uniqueness of filter limits in both spaces and the continuity of $e_{\mathcal{R}}$ that $e_{\mathcal{R}}^{-1}(a) = \alpha$. ∎

If X is any convergence space, let λX denote its *topological modification* (i.e., λX is the set $|X|$ equipped with the finest topological structure coarser than X.) If X is a c.o.s. satisfying C and O, we obtain from Proposition 2.5 and Theorem 2.7 that $\lambda X = \omega_o X$ and $\lambda(X^*/\mathcal{R})$ is a compact, T_2-ordered t.o.s. homeomorphic and order isomorphic under $e_{\mathcal{R}}$ to $\beta_o(\omega_o X)$. Let $\varphi_o : \omega_o X \to X^*/\mathcal{R}$ be defined by $\varphi_o = \sigma \circ \varphi \circ i^{-1} = \varphi_{\mathcal{R}} \circ i^{-1}$.

COROLLARY 2.8. If X is a c.o.s. satisfying C and O, then $(\lambda(X^*/\mathcal{R}), \varphi_o)$ is the Nachbin compactification of $\omega_o X = \lambda X$. If X is a $T_{3.5}$-ordered t.o.s., then $(\lambda(X^*/\mathcal{R}), \varphi_o)$ is the Nachbin compactification of X.

One question which deserves clarification is the status of $X^*/\mathcal{R}$ as a "quotient" of X^*. We have indeed equipped $X^*/\mathcal{R}$ with the quotient convergence structure, but can we interpret $\leq_{\mathcal{R}}$ as the "quotient order" relative to the order $\leq^*$ defined on X^*? Various notions of "quotient order" have been considered (for instance, see [5] and [8]), but the order $\leq_{\mathcal{R}}$ is generally different than these. Instead of regarding the order and convergence structures of $X^*/\mathcal{R}$ separately, we think that it is appropriate to consider the notion of a "quotient c.o.s.", where order and convergence structures are considered together. From this perspective, the next theorem indicates that $X^*/\mathcal{R}$ is indeed a quotient c.o.s. of X^*, at least in the category of c.o.s.'s which satisfy Conditions C and O.

THEOREM 2.10. For a c.o.s. X, let X^* and $X^*/\mathcal{R}$ be defined as before. Let Y be any c.o.s. satisfying C and O, and let $h : X^*/\mathcal{R} \to Y$. Then h is continuous and increasing iff $h \circ \sigma : X^* \to Y$ is continuous and increasing.

PROOF. If h is continuous and increasing, the same is obviously true for $h \circ \sigma$.

Conversely, suppose $h \circ \sigma$ is continuous and increasing. Let $\Phi \to [\mathcal{F}]$ in $X^*/\mathcal{R}$; then there is $\mathcal{F}' \in [\mathcal{F}]$ and a filter $\mathcal{A}$ on X^* such that $\mathcal{A} \to \mathcal{F}'$ in X^* and $\Phi \geq \sigma(\mathcal{A})$. Hence $h \circ \sigma(\mathcal{A}) \to h \circ \sigma(\mathcal{F}')$ in Y, by continuity of $h \circ \sigma$. But $\Phi \geq \sigma(\mathcal{A})$ and $\sigma(\mathcal{F}') = [\mathcal{F}]$, so $h(\Phi) \to h([\mathcal{F}])$, implying that h is continuous.

To show that h is increasing, let e_Y be the natural map from Y into $\beta_o(\omega_o Y)$ and consider $g = e_Y \circ h \circ \sigma \circ \varphi : X \to \beta_o(\omega_o Y)$. Since $g : \omega_o X \to \beta_o(\omega_o Y)$ is also continuous and increasing, there is a continuous, increasing extension $g^* : \beta_o(\omega_o X) \to \beta_o(\omega_o Y)$ which makes the diagram below commute.

$$\begin{array}{ccccccc} X & \xrightarrow{\varphi} & X^* & \xrightarrow{\sigma} & X^*/\mathcal{R} & \xrightarrow{h} & Y \\ & e\searrow & & & \swarrow e_{\mathcal{R}} & & \downarrow e_Y \\ & & \beta_o(\omega_o X) & & \underset{g^*}{\longrightarrow} & & \beta_o(\omega_o Y) \end{array}$$

Thus $e_Y \circ h \circ \sigma \circ \varphi = g^* \circ e_{\mathcal{R}} \circ \sigma \circ \varphi$, and since $\sigma \circ \varphi : X \to X^*/\mathcal{R}$ is a dense injection, $e_Y \circ h = g^* \circ e_{\mathcal{R}}$. But e_Y is an order embedding, so $h = e_Y^{-1} \circ g^* \circ e_{\mathcal{R}}$, and h is increasing. ∎

3. $T_{3.5}$-ORDERED CONVERGENCE ORDERED SPACES.

In this brief concluding section, we introduce the notion of a $T_{3.5}$-ordered c.o.s., describe the largest regular, T_2-ordered c.o.s. compactification of such a space, and interpret this compactification in the language of category theory. The necessary categorical terminology can be found in [7].

In [3], a convergence space X is defined to be *completely regular* if it allows a symmetric compactification. In [9], it is shown that the Hausdorff, completely regular convergence spaces, which we shall refer to as $T_{3.5}$ *convergence spaces* , are precisely those convergence spaces which allow a regular, Hausdorff convergence space compactification.

Given a convergence space X, let rX denote the *regular modification* of X (i.e., rX is the set $|X|$ equipped with the finest regular convergence structure coarser than the original convergence structure on X).

We define a c.o.s. X which is regular and satisfies conditions C and O to be a $T_{3.5}$-*ordered c.o.s.*. It follows by Proposition 2.5 that a $T_{3.5}$-ordered c.o.s. X has the same ultrafilter convergence as its topological modification $\lambda X = \omega_o X$.

THEOREM 3.1. Let X be a $T_{3.5}$-ordered c.o.s. and let $\eta_o X = r(X^*/\mathcal{R})$ be the regular modification of $X^*/\mathcal{R}$. Then $(\eta_o X, \varphi_{\mathcal{R}})$ is a regular, T_2-ordered c.o.s. compactification of X. If Y is a regular, T_2-ordered, compact c.o.s. and $f : X \to Y$ is continuous and increasing, then f has a unique, continuous, increasing extension $f_* : \eta_o X \to Y$.

PROOF. By Theorem 2.3, $\varphi_{\mathcal{R}} : X \to X^*/\mathcal{R}$ is an order embedding and a homeomorphic embedding. By the functorial properties of the regular modification and the fact that $rX = X$, it follows that $\varphi_{\mathcal{R}} : X \to \eta_o X$ is continuous. Because $X^*/\mathcal{R}$ and $\eta_o X$ have the same ultrafilter convergence, it is easy to verify that the regular modification of $\varphi_{\mathcal{R}}(X)$ (considered as a subspace of $X^*/\mathcal{R}$) coincides with $\varphi_{\mathcal{R}}(X)$ considered as a subspace of $\eta_o X$. From this we see that $\varphi_{\mathcal{R}}^{-1}$ is also continuous, and the first assertion is established. The second assertion is an immediate consequence of Theorem 2.4. ∎

We denote by $\boldsymbol{C}$ the category of all $T_{3.5}$ ordered c.o.s.'s, with increasing continuous maps as morphisms; let $\boldsymbol{D}$ be the full subcategory of $\boldsymbol{C}$ consisting of all regular, compact, T_2-ordered c.o.s.'s. If $\iota : \boldsymbol{D} \to \boldsymbol{C}$ is the inclusion functor, it follows by Theorem 3.1 that the functor $\eta_o : \boldsymbol{C} \to \boldsymbol{D}$, which assigns to each object X in $\boldsymbol{C}$ its compactification $\eta_o X$ and to each morphism $f : X \to Y$ in $\boldsymbol{C}$ the extension $f_* : \eta_o X \to \eta_o Y$ whose existence follows by Theorem 3.1, is the left adjoint of ι.

THEOREM 3.2. If $\boldsymbol{C}$ and $\boldsymbol{D}$ are the categories defined in the preceding paragraph, then $\boldsymbol{D}$ is an epireflective subcategory of $\boldsymbol{C}$.

If X is a $T_{3.5}$-ordered t.o.s., it is generally not true that $\beta_o X = \eta_o X$, although it is true in this case that $\beta_o X = \lambda(\eta_o X)$.

The $T_{3.5}$ convergence spaces mentioned earlier in this section are the $T_{3.5}$-ordered c.o.s.'s for which the partial order is equality. Indeed, any $T_{3.5}$ convergence space X, equipped with the trivial order (equality), satisfies Condition C and O relative to $CI^*(X) = C^*(X)$, the set of all continuous maps from X into $[0,1]$. For such a space X, $\eta_o X$ (which also has the trivial order) coincides with the largest regular, Hausdorff convergence space compactification of X constructed in [9].

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