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ON MONODROMY MAP

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ABSTRACT. Let Γ be a Fuchsian group acting on the upper half-plane U and having signature $\{p,\ n,\ 0;\ \nu_1,\ \nu_2, \cdots, \nu_n\}$; $2p-2+\sum_{j=1}^{n}(1-\frac{1}{\nu_j})>0$.

Let $T(\Gamma)$ be the Teichmüller space of Γ. Then there exists a vector bundle $\mathfrak{B}(T(\Gamma))$ of rank $3p-3+n$ over $T(\Gamma)$ whose fibre over a point $t \in T(\Gamma)$ representing Γ_t is the space of bounded quadratic differentials $B_2(\Gamma_t)$ for Γ_t. Let $Hom(\Gamma, G)$ be the set of all homomorphisms from Γ into the Moebius group G.

For a given $(t,\phi) \in \mathfrak{B}(T(\Gamma))$ we get an equivalence class of projective structures and a conjugacy class of a homomorphism $\chi \in Hom(\Gamma, G)$. Therefore there is a well defined map

$$\Phi: \mathfrak{B}(T(\Gamma)) \to Hom(\Gamma, G)/G,$$

Φ is called the monodromy map. We prove that the monodromy map is a holomorphic local homeomorphism. The case $n=0$ gives the previously known result by Earle, Hejhal and Hubbard.

KEY WORDS AND PHRASES. Quadratic differentials, Projective structures, Quasiconformal map, Tiechmüller space, Bers' fibre space, Monodromy map, Beltrami coefficient, Deformation, Cusp.

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1. INTRODUCTION.

Let Γ be a finitely generated Fuchsian group acting on the upper half plane U such that U/Γ is a Riemann surface of finite genus p with a finite number of possible punctures and ramification points n and with a finite number of possible analytic boundary curves m. Let $\{x_1, x_2, \cdots, x_n\}$ be the set of points on U/Γ that are either punctures or ramification points. let ν_j be the ramification index of $\pi^{-1}(x_j)$, where

$$\pi: U \to U/\Gamma$$

is the natural projection map, and we set $\nu_i = \infty$ for punctures. Then the sequence $\{p, n, m, \nu_1, \nu_2, \cdots, \nu_n\}$ is called the signature of the group Γ.

In this paper, we consider Γ to be a Fuchsian group acting on the upper half-plane U and having signature $\{p,\ n,\ 0,\ \nu_1,\ \nu_2, \cdots, \nu_n\}$; $2p-2+\sum_{j=1}^{n}(1-\frac{1}{\nu_j})>0$.

Let $T(\Gamma)$ be the Teichmüller space of Γ. Then there exists a vector bundle $\mathfrak{B}(T(\Gamma))$ of rank $3p-3+n$ over $T(\Gamma)$ whose fibre over a point representing Γ_t is the space of bounded quadratic differentials $B_2(\Gamma_t)$ for Γ_t. Let $Hom(\Gamma, G)$ be the set of all homomorphisms from Γ into the Moebius group G.

For a given $(t,\phi)\in\mathfrak{B}(T(\Gamma))$ we get an equivalence class of projective structures and a conjugacy class of a homomorphism $\chi_\phi\in Hom(\Gamma,G)$. Therefore there is a well defined map

$$\Phi:\mathfrak{B}(T(\Gamma))\to Hom(\Gamma,G)/G.$$

Φ is called the monodromy map. We prove that the monodromy map is a holomorphic local homeomorphism.

The case $n=0$ gives the previously known result by Earle, Hejhal and Hubbard. Falting [6], Gallo and Porter [7] have similar results for $n>0$. The monodromy map restricted on each fibre is known to be injective by Kra [11]. As a generalization of this result for a Fuchsian group Γ with signature $\{p,\ n,\ m,\ \nu_1,\ \nu_2,\cdots,\ \nu_n\}$; $n>0$, $m>0$, author has proven a uniqueness theorem in [15]. A similar result has been proven by Gallo and Porter [8].

In Section I, we discuss some well known interesting properties of Moebius transformations and with their help, we find the set of regular points in $Hom(\Gamma,G)$. This technical result is needed to prove the main result in Section II. In Section II, we prove that the monodromy map is a holomorphic local homeomorphism.

SECTION I. Let $A_1,\ B_1,\ A_2,\ B_2,\cdots,\ A_p,\ B_p,\ C_1,\ C_2,\cdots C_n$ be a fixed set of generators of Γ satisfying the relations

$$\prod_{i=1}^{p}[A_i,B_i]\prod_{j=1}^{n}C_j=I \text{ and } C_j^{\nu_j}=I,\quad j=m+1,\cdots n,$$

where $[A_i,\ B_i]=A_i\ B_i\ A_i^{-1}\ B_i^{-1}$ and $C_1,\ C_2,\cdots,\ C_m$ are the parabolic generators and C_{m+1}, $C_{m+2},\cdots C_n$ are elliptic generators with periods $\nu_{m+1},\ \nu_{m+2},\cdots,\ \nu_n$, respectively.

A homomorphism $\chi\in Hom(\Gamma,G)$ is completely determined by $2p+n$ Moebius transformations

$\chi(A_i)=S_i$

$\chi(B_i)=T_i$

$\chi(C_j)=W_j$,

$i=1,\ 2,\cdots,\ m$ and $j=m+1,\ m+2,\cdots,\ n$ satisfying the relations

$$\prod_{i=1}^{p}[S_i,T_i]\prod_{j=1}^{n}W_j=I \text{ and } W_j^{\nu_j}=I,\ j=m+1,\ m+2,\cdots n.$$

Let P be the set of all parabolic transformations and E_j be the set of all elliptic transformations with a fixed multiplier K_j^2; $K_j^{2\nu_j}=1$, $j=m+1,\ m+2,\cdots,\ n$. Let $Hom^*(\Gamma,G)$ consist of homomorphisms preserving parabolic transformations and the multipliers of the elliptic transformations. Then for $\chi\in Hom^*(\Gamma,G)$,

$$\begin{aligned}\chi(C_j)&=W_j\in P, j=1,2,\cdots,m\\ &=W_j\in E_j, j=m+1,m+2,\cdots,n.\end{aligned}$$

Hence $\{S_1,T_1,S_2,T_2,\cdots,S_p,T_p,W_1,W_2,\cdots,W_n\}$ is a point in $G^{2p}\times P^m\times E_{m+1}\times E_{m+2}\cdots\times E_n$. We denote $\{S_1,T_1,\cdots,S_p,T_p,W_1,\cdots,W_n\}$ by $\{S_i,T_i,W_j\}$ and $G^{2p}\times P^m\times E_{m+1}\times\cdots\times E_n$ by $G_{2p,n}$ for short.

Following lemma of Gardiner and Kra [9], we show that P and each E_j are two-dimensional submanifolds of G. We also determine the tangent space of P or E_j at any point.

At this point, let us introduce the adjoint representation $u\mapsto u^A$ of $SL(2,\mathbf{C})$ in $\mathfrak{g}$, the Lie algebra of $SL(2,\mathbf{C})$ (that is the tangent space of $SL(2,\mathbf{C})$ at identity I) which is defined by $u^A=AdA(u)$, $u\in\mathfrak{g}$, $A\in SL(2,\mathbf{C})$ where $AdA:\mathfrak{g}\mapsto\mathfrak{g}$ is the differential at I of the map $SL(2,\mathbf{C})\ni\chi\mapsto A^{-1}\circ\chi\circ A\in SL(2,\mathbf{C})$.

Explicitly,

$$u^A = \lim_{t\to 0} \frac{A^{-1} \circ e^{tu} \circ A - I}{t} = A^{-1} \circ u \circ A$$

A parabolic transformation with fixed point $x \neq \infty$ can be written as an element of $SL(2,\mathbb{C})$ as $\begin{pmatrix} 1+px & -px^2 \\ p & 1-px \end{pmatrix}$; $p \neq 0$, which is unique up to multiplication by -1 [14]. We consider the natural map

$$\pi: SL(2,\mathbb{C}) \mapsto G$$

which is two-to-one and unramified.

Each parabolic transformation corresponds to two matrices in $SL(2,\mathbb{C})$, one of which has trace 2 and the other has trace -2. Thus $\pi^{-1}(P)$ consists of two disjoint sets P^+ and P^-, where

P^+ = the set of elements in $SL(2,\mathbb{C})$ with trace $2\backslash\{I\}$,

P^- = the set of elements in $SL(2,\mathbb{C})$ with trace $-2\backslash\{-I\}$.

We prove the following lemma which has been proven by Gardiner and Kra in [9] in a slightly different manner. We shall adopt the calculations from [9].

LEMMA 1.1. Let $f: SL(2,\mathbb{C}) \to \mathbb{C}$ be the mapping defined by

$$f(x) = tr\ x.$$

If $u \in ker\ (df)\ (B)$ with $B \in P^+$, then there exists a $v \in \mathfrak{g}$ such that

$$u = v^B - v.$$

PROOF. f is holomorphic. Let $B \in P^+$. Then there exists an $A \in SL(2,\mathbb{C})$ such that

$$A^{-1}BA = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}.$$

We consider the function

$$SL(2,\mathbb{C}) \ni B \overset{F}{\mapsto} A^{-1}\ BA \in SL(2,\mathbb{C}).$$

Since F is a holomorphic isomorphism,

$$u \in ker\ d(f \circ F)\ (B) \Leftrightarrow (dF)\ (B)\ u \in ker\ (df)\ (FB).$$

Moreover, for $v \in \mathfrak{g}$, $B \in SL(2,\mathbb{C})$, $A \in SL(2,\mathbb{C})$

$$u = v^B - v \Leftrightarrow u^A = v^{B \circ A} - v^A = v_1^{A^{-1}BA} - v_1;\ v_1 = v^A,$$

and

$$(dF)\ (B)\ (u) = u^A.$$

Thus it suffices to assume that $B = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$. For $u = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g}$,

$$\begin{aligned}
(dF)\ (B)\ (u) &= \lim_{t\to 0} \frac{f(Be^{tu}) - f(B)}{t} \\
&= \lim_{t\to 0} \frac{f\left\{\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+at & bt \\ ct & 1-at \end{pmatrix} + \circ(t)\right\} - f\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}}{t} \\
&= \lim_{t\to 0} \frac{f\left\{\begin{pmatrix} 1+at+pct & bt+p(1-at) \\ ct & 1-at \end{pmatrix} + \circ(t)\right\} - 2}{t} \\
&= \lim_{t\to 0} \frac{2+pct-2}{t} \\
&= pc.
\end{aligned}$$

Thus if $u \in ker(df)(B), c = 0$; that is, $u = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$. We check that there exists a $v = \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} \in \mathfrak{g}$ such that

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} = B^{-1}\begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix} B - \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix}$$

since

$$B=\begin{pmatrix}1 & p\\ 0 & 1\end{pmatrix}, \; B^{-1}\begin{pmatrix}a' & b'\\ c' & -a'\end{pmatrix}B-\begin{pmatrix}a' & b'\\ c' & -a'\end{pmatrix}=\begin{pmatrix}-pc' & -c'p^2+2a'p\\ 0 & c'p\end{pmatrix}.$$

We choose $c'=-\frac{a}{p}, a'=\frac{b-ap}{2p}$, and b' arbitrarily. This completes the proof of the lemma.

In the above calculation for $(df)(B)$ with $B=\begin{pmatrix}1 & p\\ 0 & 1\end{pmatrix}$ we notice that, for $u\in \mathfrak{g}$,

$$(df)\,(B)\,(u)=pc.$$

Since $p\neq 0$, $c\neq 0$, $(df)\,(B)$ is surjective. Again the differential of the map $F\colon x\to A^{-1}xA$, $x\in SL(2,\mathbf{C})$, $A\in SL(2,\mathbf{C})$ is surjective. Hence $(df)\,(B)$ is surjective for any $B\in P^+$. Therefore, df has maximal rank at each point of P^+; that is, P^+ is the set of regular points of f in $f^{-1}(2)$ and hence P^+ is a submanifold of $SL(2,\mathbf{C})$ of dimension 2 by the implicit function theorem. Moreover, for $B\in P^+$,

$$T_B(P^+)=ker\,(df)\,(B).$$

Hence from the above Lemma we conclude that

$$T_B(P^+)=\{u\in \mathfrak{g};\; u=v^B-v \text{ for some } v\in \mathfrak{g}\}.$$

Similarly, we can show that P^- is a submanifold of $SL(2,\mathbf{C})$ of dimension 2 and for $B\in P^-$,

$$T_B(P^-)=\{u\in \mathfrak{g};\; u=v^B-v \text{ for some } v\in \mathfrak{g}\}.$$

Since P^+ and P^- project to P in G, P is a submanifold of G of dimension 2. Thus we prove the following:

COROLLARY 1. P is a submanifold of G of dimension 2. Moreover, for $g\in P$,

$$T_g(P)=\{u\in \mathfrak{g};\; u=v^g-v \text{ for some } v\in \mathfrak{g}\}.$$

An elliptic transformation g with the fixed points x and y can be written as

$$\frac{g(z)-x}{g(z)-y}=k^2\,\frac{z-x}{z-y},$$

where k^2 is the multiplier of g, $k^2\neq 1$. Choosing a positive square root of k^2, we write $k^2=\frac{k}{1/k}$. Then solving the above equation we can write in the matrix form

$$g=\frac{1}{x-y}\begin{pmatrix}x/k-yk & xy(k-1/k)\\ 1/k-k & xk-y/k\end{pmatrix}$$

which is unique up to multiplication by -1 [14]. If $k^2=-1$, the above expression for g is symmetric in x and y.

Let E be the set of all elliptic transformations with the multiplier k^2. Each elliptic transformation in E corresponds to two matrices in $SL(2,\mathbf{C})$, one of which has trace $k+1/k$, and the other has trace $-(k+1/k)$. Hence if $k^2\neq -1$, $\pi^{-1}(E)$ consists of two disjoint sets E^+ and E^-, where

E^+ = the set of elements in $SL(2,\mathbf{C})$ with trace $k+1/k$,

E^- = the set of elements in $SL(2,\mathbf{C})$ with trace $-(k+1/k)$.

If $k^2=-1, \pi^{-1}(E)$ is just one set; we denote it by E^o, where E^o = the set of elements in $SL(2,\mathbf{C})$ with trace zero. As before, we have the following:

LEMMA 1.2. Let $f\colon SL(2,\mathbf{C})\to C$ be the mapping defined by

$$f(x)=tr(x).$$

If $u\in ker(df)(B)$, with $B\in E^+$, then there exists a $v\in \mathfrak{g}$ such that

$$u=v^B-v.$$

PROOF. The idea of the proof is same as it is in the Lemma 1.1. Without loss of generality we assume that $B=\begin{pmatrix}k & 0\\ 0 & 1/k\end{pmatrix}$. Then for $u=\begin{pmatrix}a & b\\ c & -a\end{pmatrix}\in \mathfrak{g}$.

$$(df)\,(B)\,(u)=\lim_{t\to 0}\frac{f\left\{\begin{pmatrix}k & 0\\ 0 & 1/k\end{pmatrix}\begin{pmatrix}1+at & bt\\ ct & 1-at\end{pmatrix}+o\,(t)\right\}-f\begin{pmatrix}k & 0\\ 0 & 1/k\end{pmatrix}}{t}$$

$$=\lim_{t\to 0}\frac{f\left\{\begin{pmatrix}k(1+at) & kbt\\ 1/kct & 1/k(1-at)\end{pmatrix}+o\,(t)\right\}-f\begin{pmatrix}k & 0\\ 0 & 1/k\end{pmatrix}}{t}$$

$$=\lim_{t\to 0}\frac{(k+1/k)+at(k-1/k)+o(t)-(k+1/k)}{t}$$

$$=a(k-1/k).$$

Hence if $u\in ker(df)\,(B), a=0$; that is, $u=\begin{pmatrix}0 & b\\ c & 0\end{pmatrix}$. We check that there exists a $v=\begin{pmatrix}a' & b'\\ c' & -a'\end{pmatrix}\in \mathfrak{g}$ such that

$$B^{-1}vB-v=\begin{pmatrix}0 & b\\ c & 0\end{pmatrix}.$$

Since $B=\begin{pmatrix}k & 0\\ 0 & 1/k\end{pmatrix}$, $B^{-1}vB-v=\begin{pmatrix}0 & b'(1/k^2-1)\\ c'(k^2-1) & 0\end{pmatrix}$. We choose $b'=\frac{b}{1/k^2-1}, c'=\frac{c}{k^2-1}$ and a' arbitrarily. This completes the proof of the lemma.

Once again, we observe that $(df)(B)$ is surjective for $B\in E^+$, since $a\neq 0$ and $k^2\neq 1$. Hence at each point of $E^+\,df$ has maximal rank, and hence $E^+=f^{-1}(k+1/k)$ is a submanifold of $SL(2,\mathbf{C})$ of dimension 2. Moreover,

$$T_B(E^+)=ker(df)(B).$$

Hence

$$T_B(E^+)=\{u\in \mathfrak{g}; u=v^B-v \text{ for some } v\in \mathfrak{g}\}.$$

Similarly, we can prove the same results for E^- as well as for E^o. When $k^2\neq -1$, E^+ and E^- are submanifolds of $SL(2,\mathbf{C})$. Since E^+ and E^- project to E in G, E is a submanifold of G. When $k^2=-1, E^o$ is a submanifold of $SL(2,\mathbf{C})$. Hence $E=E^o/\pm I$ is a submanifold of G. Thus we prove the following.

COROLLARY 2. E is a submanifold of G of dimension 2. Moreover, for $g\in E$,

$$T_g(E)=\{u\in \mathfrak{g}; u=v^g-v \text{ for some } v\in \mathfrak{g}\}.$$

We introduce a function F on $G_{2p,n}$ defined by

$$F(S_i,T_i,W_j)=\prod_{i=1}^{p}[S_i,T_i],\prod_{j=1}^{n}W_j$$

This is a complex analytic function from $G_{2p,n}$ into G. The subset

$$R=\{(S_i,T_i,W_j)\in G_{2p,n}; F(S_i,T_i,W_j)=I\}$$

is then a complex analytic subvariety of $G_{2p,n}$; the mapping

$$Hom^*(\Gamma,G)\ni \chi\to(\chi(A_i),\chi(B_i),\chi(C_j))\in G_{2p,n}$$

identifies $Hom^*(\Gamma,G)$ with this subvariety and thus establishes a complex structure on $Hom^*(\Gamma,G)$. $G_{2p,n}$ is a complex analytic manifold of dimension $6p+2n$. We show that the subset of $Hom^*(\Gamma,G)$ consisting of those homomorphisms χ for which $\chi(\Gamma)$ are non-elementary is the set of regular points in R. The case when $n=0$ has been discussed by Gunning in [10]. Following

Gunning we can find $d_\chi F$ at $\chi = (S_i, T_i, W_j) \in G_{2p,n}$. The tangent space of $G_{2p,n}$ at the point χ is denoted by $T_\chi(G_{2p,n})$. Then

$$T_\chi(G_{2p,n}) \cong \mathfrak{g}^{2p} \times \prod_{j=1}^{n} \mathfrak{g}_{w_j},$$

where $\mathfrak{g}_{w_j} = T_{w_j}(P)$ for $j = 1, 2, \cdots, m$ and $\mathfrak{g}_{w_j} = T_{w_j}(E_j)$ for $j = m+1, \cdots, n$.

Let $(X_1, X_2, \cdots, X_p, Y_1, Y_2, \cdots, Y_p, Z_1, Z_2, \cdots, Z_n)$, denoted by (X_i, Y_i, Z_j) for short, be a point in $\mathfrak{g}^{2p} \times \prod_{j=1}^{n} \mathfrak{g}_{w_j}$.

Then by definition,

$$d_\chi F(X_i, Y_i, Z_j) = \lim_{t \to 0} \frac{F\left(S_i e^{tX_i},\ T_i e^{tY_i},\ W_j e^{tZ_j}\right) - f(S_i, T_i, W_j)}{t}$$

In other words, $d_\chi f(X_i, Y_i, Z_j)$ is the coefficient of t in the Taylor expansion of $F(S_i e^{tX_i}, T_i e^{tY_i}, W_j e^{tZ_j})$.

After a long calculation we find that

$$d_\chi F(X_i, Y_i, Z_j) = \sum_{i=1}^{p} AdS_i^{-1}\, T_i^{-1} \prod_{k=i+1}^{p} [S_k, t_k]((I - AdS_i)Y_i - (I - AdT_i)X_i)$$
$$+ \sum_{j=1}^{n} Ad \prod_{k=j+1}^{n} W_k(Z_j)$$

which is essentially same as the expression obtained in Gunning [10] except the second term.

We define an action of Γ on $\mathfrak{g}$ as follows:

For $u \in \mathfrak{g}$ and $\gamma \in \Gamma$, we define

$$u \cdot \gamma = u \cdot \chi(\gamma) = Ad\chi(\gamma)(u).$$

We rewrite the above expression in the following way,

$$d_\chi F(X_i, Y_i, Z_j) = \sum_{i=1}^{p} (X_i \cdot (B_i - I) + Y_i(I - A_i)) \cdot A_i^{-1} B_i^{-1} \prod_{k=i+1}^{p} [A_k, B_k] + \sum_{j=1}^{n} Z_j \cdot \sum_{k=j+1}^{n} C_k.$$

We want to check when $d_\chi F$ is surjective. To do that we follow Ahlfor's method in ([2], §5). We introduce notations $R_0 = I$ and

$$R_i = A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_i B_i A_i^{-1} B_i^{-1}$$
$$R_{p+j} = R_p C_1 C_2 \cdots C_j$$
$$\overline{A}_i = R_{i-1} B_i^{-1} R_i^{-1}$$
$$\overline{B}_i = R_i A_i^{-1} R_{i-1}^{-1}$$
$$\overline{C}_j = R_{p+j} C_j R_{p+j}^{-1} (1 \le i \le p, 1 \le j \le n)$$

Then $\overline{A}_i, \overline{B}_i, \overline{C}_j$ are generators of Γ. Moreover,

$$d_\chi F(X_i, Y_i, Z_j) = \sum_{i=1}^{p} X_i \cdot A_i^{-1} R_{i-1}^{-1}(I - \overline{A}_i) + \sum_{i=1}^{p} Y_i \cdot B_i^{-1} R_i^{-1}(\overline{B}_i - I) + \sum_{j=1}^{n} Z_j \cdot R_{p+j}^{-1}.$$

We suppose that the map

$$d_\chi F : \mathfrak{g}^{2p} \times \sum_{j=1}^{n} \mathfrak{g}_{w_j} \to \mathfrak{g}$$

is not surjective. Then there exists a nonzero linear functional v^* on $\mathfrak{g}$ that vanishes on all the subspaces $\mathfrak{g} \cdot (\overline{A}_i - I)$, $\mathfrak{g} \cdot (\overline{B}_i - I)$ and $\mathfrak{g} \cdot (C_j - I) R_{p+j}^{-1} = \mathfrak{g} \cdot R_{p+j}^{-1} (\overline{C}_j - I) = \mathfrak{g} \cdot (\overline{C}_j - I)$. If v^* annihilates $v \cdot (A - I)$ and $v \cdot (B - I)$ for all $v \in \mathfrak{g}$, it annihilates $v \cdot (AB - I) = v \cdot A(B - I) + v \cdot (A - I)$.

Since $\{\bar{A}_i, \bar{B}_i, \bar{C}_j\}$ is a system of generators of Γ, it follows that v^* annihilates $v\cdot(A-I)$ for all $V \in \mathfrak{g}$ and all $A \in \Gamma$.

We assume first that there is a loxodromic element $\chi(A)$, $A \in \Gamma$. We may take

$$\chi(A)(z) = k^2 z;\ |k^2| \neq 1.$$

For $v = \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \in \mathfrak{g}, v\cdot(A-I) = \begin{pmatrix} 0 & q\left(\frac{1}{k^2}-1\right) \\ r(k^2-1) & 0 \end{pmatrix}$.

Therefore, v^* must be multiple of the linear functional that maps any v on its first entry. It follows that the first entry of $v\cdot(B-I)$ is zero for all $v \in \mathfrak{g}$ and all $B \in \Gamma$. We take $\chi(B)(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$, and apply the above result on $v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then we get $\alpha\beta = \gamma\delta = 0$. This is true only when $\chi(B)$ is a multiple of z or $1/z$.

Next, we assume that there is a parabolic element $\chi(A), A \in \Gamma$. We take

$$\chi(A)(z) = z + 1.$$

Then for $v = \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \in \mathfrak{g}, v\cdot(A-I) = \begin{pmatrix} -r & 2p-r \\ 0 & r \end{pmatrix}$. Therefore, v^* must be a multiple of the linear functional that maps any v on its third entry. It follows that $v\cdot(B-I)$ has zero third entry for all $v \in \mathfrak{g}$, all $B \in \Gamma$. As before, we assume that $\chi(B)(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$, and apply the above result on $v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We get $\gamma = 0, \alpha^2 = 1$. This is true only when $\chi(B)(z) = z + \beta'; \beta' \neq 0$.

Finally, we assume that there is no loxodromic or parabolic element in $\chi(\Gamma)$; that is, all elements of $\chi(\Gamma)$ are elliptic. Hence $\chi(\Gamma)$ is finite.

Combining all these we conclude that $d_\chi F$ is surjective if none of the following statements holds.

(i) $\chi(\Gamma)$ is finite;

(ii) all elements of $\chi(\Gamma)$ are multiples of z or $\frac{1}{z}$;

(iii) all elements of $\chi(\Gamma)$ are of the form $z \to z + \beta, \beta \neq 0$.

Thus we have the following.

PROPOSITION. Let R_0 be the subset of $Hom^*(\Gamma, G)$ consisting of those homomorphisms χ for which $\chi(\Gamma)$ is nonelementary; that is, $\chi(\Gamma)$ is not a finite extension of an Abelian group. Then R_0 is a complex manifold of dimension $6p + 2n - 3$.

REMARK. It follows from condition (iii) that the above proposition also holds when $\chi(\Gamma)$ is some of the elementary groups.

SECTION 2.

DEFINITION. Let a group Γ act discontinuously on a domain $\Omega \subset \hat{\mathbb{C}}$. We denote by $Q_2(\Omega, \Gamma)$ the complex vector space of quadratic differentials for Γ; $Q_2(\Omega, \Gamma)$ consists of functions ϕ, holomorphic on Ω satisfying $(\phi \circ \gamma)\gamma'^2 = \phi$ for all $\gamma \in \Gamma$.

We denote by $B_2(\Omega, \Gamma)$ the subspace of $Q_2(\Omega, \Gamma)$ consisting of bounded quadratic differentials for Γ; $B_2(\Omega, \Gamma)$ consists of $\phi \in Q_2(\Omega, \Gamma)$ for which

$$Sup\{\lambda_\Omega^{-2} |\phi(z)|\} < \infty$$

where λ_Ω is the Poincaré metric on Ω.

DEFINITION. A **deformation** of Γ is a pair (f, χ), where f is a holomorphic local homeomorphism of U into $\hat{\mathbb{C}}$ and χ is a homomorphism of Γ into G, the group of all Moebius transformations, satisfying

$$f \circ \gamma = \chi(\gamma) \circ f \text{ for all } \gamma \in \Gamma.$$

The local homeomorphism f also describes a projective structure on the Riemann surface U/Γ (provided Γ is torsion free). We also call f a projective structure on U/Γ. We call two projective structures f and g equivalent if $g = Aof$ for some Moebius transformation A. There is a one-to-one correspondence between the set of equivalence classes of projective structures on U/Γ and the space of quadratic differentials $Q_2(U,\Gamma)$.

DEFINITION. Let w be a quasiconformal selfmap of U, normalized by the conditions $w(0)=0, w(1)=1$, and $w(\infty)=\infty$. w is compatible with the group Γ if $w \circ \gamma o w^{-1}$ is conformal for every $\gamma \in \Gamma$. Two such quasi-conformal self maps of U, w_1 and w_2 are equivalent if they coincide on the real line.

The **Teichmüller space** $T(\Gamma)$ of Γ is the set of equivalence classes $[w]$ of normalized quasi-conformal self maps of U which are Γ-compatible.

Let $L_\infty(U)$ denote the complex Banach space of bounded measurable functions μ on U. let $L_\infty(U)_1$ be its open unit ball. Let $L_\infty(U,\Gamma)$ be the subspace of $L_\infty(U)$ consisting of μ satisfying

$$\mu(\gamma(z))\overline{\gamma'}(z)/\gamma'(z) = \mu(z) \text{ for all } \gamma \in \Gamma \text{ and } z \text{ in } U.$$

Let $L_\infty(U,\Gamma)_1 = L_\infty(U)_1 \cap L_\infty(U,\Gamma)$. For every q.c. self map w of U, its Beltrami coefficient, $\mu = w_z/w_{\overline{z}} \in L_\infty(U)_1$. Every $\mu \in L_\infty(U)_1$ determines a unique normalized self map w of U satisfying $w_z = \mu w_{\overline{z}}$, Ahlfors [1]. We denote this w by w_μ. It is easy to check that w_μ is Γ-compatible if and only if $\mu \in L_\infty(U,\Gamma)$. $T(\Gamma)$ can be endowed with the quotient topology associated with the surjective map $\mu \to [w_\mu]$. $T(\Gamma)$ with this topology, can be realized as a bounded open set in $B_2(U^*,\Gamma)$. Since it is an open set in $B_2(U^*,\Gamma)$, $T(\Gamma)$ is a complex manifold modeled on $B_2(U^*,\Gamma)$ and has dimension $3P-3+n$ when Γ is of type $(p,n,0)$.

We take $\mu \in L_\infty(U,\Gamma)_1$ and extend it to be zero on the rest of $\hat{\mathbb{C}}$. There exists a unique q.c. self-map w of $\hat{\mathbb{C}}$ fixing $0,1,\infty$ which has Beltrami coefficient μ on U and which is conformal on U^*, Ahlfors [1]. We denote this w by $w^\mu \cdot w^\mu|_{\mathbf{R}}$, hence $w^\mu|U^*$ depends only on $[w_\mu]$, Ahlfors [1]. Therefore, $w^\mu(U)$ depends only on $[w_\mu]$. We denote $w^\mu(U)$ by $D(t)$, where $t=[w_\mu] \in T(\Gamma)$. The boundary of $w^\mu(U)$ is $w^\mu(\hat{\mathbf{R}})$. The group $w^\mu\Gamma(w^\mu)^{-1}$ fixes this boundary which is a Jordan curve. Hence the group is quasi-Fuchsian. We denote $w^\mu\Gamma(w^\mu)^{-1}$ by $\Gamma(t)$. The **Bers' fibre space** $F(t)$ over $T(\Gamma)$ is the set of pairs (t,z) with $t \in T(\Gamma)$, $z \in D(t)$.

For each $t \in T(\Gamma)$, there exists a quasi-Fuchsian group $\Gamma(t)$ and a Jordan domain $D(t) = w^\mu(U)$. To each t, we associate the complex vector space $B_2(D(t),\Gamma(t))$ of bounded quadratic differentials for $\Gamma(t)$. We form $\mathfrak{B}(T(\Gamma)) = \bigcup_{t \in T(\Gamma)} B_2(D(t),\Gamma(t))$ as a fibre space over $T(\Gamma)$. $\mathfrak{B}(T(\Gamma))$ forms a complex vector bundle of rank $3p-3+n$ over $T(\Gamma)$. We denote the points of $\mathfrak{B}(T(\Gamma))$ by $(t,\phi(t))$ where $\phi(t) \in B_2(D(t),\Gamma(t))$.

Each $\phi(t) \in B_2(D(t),\Gamma(t))$ determines a holomorphic local homeomorphism

$$f(z,t)\colon D(t) \to \hat{\mathbb{C}}$$

such that the Schwarzian derivative of f, $Sf = (f''/f')' - 1/2(f''/f')^2$, is ϕ. We notice that (i) $S(f \circ \gamma^t) = Sf$, for $\gamma^t \in \Gamma(t)$, and hence (ii) $f \circ \gamma^t = \hat{\gamma} \circ f$ for some $\hat{\gamma} \in G$. Both (i) and (ii) follow from properties of Schwarzian derivatives. The map $\gamma \to \hat{\gamma}$ determines a homomorphism χ_o from $\Gamma(t)$ into G.

Let $\Theta^\mu\colon \gamma \to \gamma^t$ be the isomorphism of Γ into $\Gamma(t)$ induced by w^μ. We take $\chi = \chi_o \circ \Theta^\mu$. Thus we get a homomorphism χ of Γ into G induced by $f \circ w^\mu$ and we have

$$f \circ w^\mu \circ \gamma = \chi(\gamma) \circ f \circ w^\mu \qquad \text{for all } \gamma \in \Gamma. \tag{2.1}$$

For $A \in G$, f and $A \circ f$ have the same Schwarzian derivative ϕ. Since replacing f by $A \circ f$ has the effect of replacing χ by $A\chi A^{-1}$, we have a well defined map

$$\Phi: \mathfrak{B}(T(\Gamma)) \rightarrow Hom(\Gamma, G)/G.$$

We call Φ the monodromy map. We prove the following:

THEOREM 1. The monodromy map is a holomorphic local homeomorphism.

We want to study the local behavior of Φ. For this purpose we fix the origin $t_0 \in T(\Gamma)$ so that $D(t_0) = U$ and $\Gamma(t_0) = \Gamma$. We consider the vector space W of the functions $\mu: \mathbf{C} \rightarrow \mathbf{C}$ satisfying the following conditions.

$$\begin{aligned} \mu(z) &= (Im\ z)^2\ \overline{\phi(z)},\ z \in U, \text{ for some } \phi \in B_2(U, \Gamma) \\ &= 0, \text{ outside } U. \end{aligned}$$

Let W_1 be the subset of W consisting of μ with $\|\mu\|_\infty < 1$. For each $\mu \in W_1$ there exists a unique quasi-conformal self map $w = w^\mu$ of $\hat{\mathbf{C}}$, fixing $0, 1, \infty$, and such that w has the Beltrami coefficient μ in U. Moreover, $w^\mu(U)$ is a Jordan domain and $w^\mu\Gamma(w^\mu)^{-1}$ is a quasi-Fuchsian group fixing $w^\mu(U)$. There exists a neighborhood W_0 of zero in W_1 which provides a local coordinate at t_0 in such a way that for every t in a sufficiently small neighborhood of t_0, $D(t)$ is the Jordan domain $w^\mu(U)$ and $\Gamma(t)$ is the quasi-Fuschian group $w^\mu\Gamma(w^\mu)^{-1}$ for some $\mu \in W_0$. We choose W_0 so small that a point $z_0 \in w^\mu(U)$ for all $\mu \in W_0$ whenever $z_0 \in U$.

Now for $\mu \in W_0$ and $\phi \in B_2(w^\mu(U), w^\mu\Gamma(w^\mu)^{-1})$, we consider the Schwarzian differential equation

$$Sf = \left(\frac{f''}{f'}\right)' - 1/2\left(\frac{f''}{f'}\right)^2 = \phi. \tag{2.2}$$

Let $g = g_\phi$ be the unique solution of (2.2) satisfying

$$g(z_0) = 0, g'(z_0) = 1, g''(z_0) = 0. \tag{2.3}$$

Any function f satisfying $Sf = \phi$ is given by $f = A \circ g$ for some $A \in G$. Hence for $\mu \in W_0$ and $\phi \in B_2(w^\mu(U), w^\mu\Gamma {w^\mu}^{-1})$, we have from (2.1).

$A \circ g \circ w^\mu(\gamma(z)) = \chi(\gamma) \circ A \circ g \circ w^\mu(z)$ for all $\gamma \in \Gamma,\ z \in U$.

We take $h = A \circ g \circ w^\mu$. Then h is a C^∞-function satisfying $h \circ \gamma = \chi(\gamma) \circ h$ for all $\gamma \in \Gamma$. Since g depends on ϕ and w^μ depends on the Beltrami coefficient μ, h is a function of A, μ and ϕ. Hence so is χ. We denote the map

$$G \times \mathfrak{B}(T(\Gamma)) \ni (A, \mu, \phi) \rightarrow \chi \in Hom(\Gamma, G)$$

by Φ^*. We shall show that Φ^* is holomorphic. To prove this we need some Lemmas which have been proved already in Earle [5]. These Lemmas do not need adjustment for the parabolic or elliptic elements in Γ. Hence we state these lemmas without proofs.

LEMMA 2.1 (Earle [5]). Let A, μ, ϕ be functions of a complex variable τ such that $A(z,\tau) \in G, \mu(z,\tau) \in W_0$ and $\phi(z,\tau)$ is in $B_2(w^\mu(U), w^\mu\Gamma {w^\mu}^{-1})$ for all τ; $|\tau| < \varepsilon$.

We assume that

$$\begin{cases} A(z,\tau) = A_0(z) + \tau\dot{A}(z) + o(\tau) \\ \mu(z,\tau) = \qquad\qquad \tau\dot{\mu}(z) + o(\tau) \\ \phi(z,\tau) = \phi_0(z) + \tau\dot{\phi}(z) + o(\tau), \qquad |\tau| < \varepsilon. \end{cases} \tag{2.4}$$

where $A_0(z) = A(z,0), \phi_0(z) = \phi(z,0)$ and the dot denotes the derivative with respect to τ at $\tau = 0$. We set $\mu_0(z) = \mu(z,0) = 0$.

Then h has a power series expansion

$$h(z,\tau) = h_0(z) + \tau \dot{h}(z) + o(\tau), \text{ for } |\tau| < \varepsilon \tag{2.5}$$

where $h_0(z) = h(z,0)$ and $\dot{h}(z) = \frac{\partial h}{\partial \tau}|_{\tau = 0}$.

LEMMA 2.2 (Earle [5]).

Let $h^* = \frac{\dot{h}}{h_0}$. Then $h^* = 0 \Leftrightarrow \dot{A} = \dot{\mu} = \dot{\phi} = 0$.

With the help of Lemma 2.1 it can be proved that χ depends holomorphically on A, μ and ϕ. To show this we need the following:

LEMMA 2.3 (Earle [5]). Let A, μ, ϕ satisfy (2.4) and let h satisfy (2.5). Then $\chi(\gamma), \gamma \in \Gamma$, has the following power series expansion

$$\chi(\gamma) = \chi_0(\gamma) + \tau \dot{\chi}(\gamma) + o(\tau) \text{ for } |\tau| < \varepsilon \tag{2.6}$$

and for all $\gamma \in \Gamma$ where

$$\dot{\chi}(\gamma)(h_0(z)) = (h_0 \circ \gamma)'(z)(h^*(\gamma)\gamma'(z)^{-1} - h^*(z)), z \in U. \tag{2.7}$$

The Lemma 2.3 has the following

COROLLARY 4. $\dot{\chi}(\gamma) = 0$ for $\gamma \in \Gamma$ if and only if $h^* = 0$ in U.

We need some adjustments to prove the corollary for the presence of parabolic elements. We include the proof.

PROOF. In (2.7) we use $h_0 \circ \gamma = \chi_0(\gamma) \circ h_0$ and we get

$$\frac{\dot{\chi}(\gamma)(h_0(z))}{\chi_0(\gamma)'(h_0(z))} = h_0'(z)(h^*(\gamma(z))\gamma'(z)^{-1} - h^*(z)).$$

Since $\frac{\dot{\chi}(\gamma)(z)}{\chi_0(\gamma)'(z)}$ is a polynomial and $h_0(U)$ is open, $\dot{\chi}(\gamma) = 0$ if $h^* = 0$ in U.

Now we assume that $\dot{\chi}(\gamma) = 0$ for all $\gamma \in \Gamma$. Then $h^*(\gamma(z))\gamma'(z)^{-1} = h^*(z)$, for all $\gamma \in \Gamma, z \in U$. Hence h^* is a $C^\infty(-1)$ differential for Γ. We shall show that h^* is actually holomorphic in U under the assumption, that $\dot{\chi}(\gamma) = 0$ for all $\gamma \in \Gamma$. We intend to apply Stoke's theorem on U/Γ. Since U/Γ has punctures, Stoke's theorem cannot be applied directly. We follow Bers [3] to handle this situation. U/Γ has m punctures. Thus one can construct a fundamental domain D for Γ containing m cusped regions belonging to punctures.

We draw in each cusped region a smooth curve C_s, $s = 1, 2, \cdots, m$ so that (i) C_s joins two points ζ_s and ζ_s' on ∂D which are identified by an element of Γ, and (ii) C_s and $C_{s'}$ do not meet, for $s \neq s'$. In this manner we obtain a relatively compact subset D^* of D which is bounded by part of ∂D and the curves $C_1, C_2, \cdots C_m$.

For any $\phi \in B_2(U,\Gamma), h^*\phi$ is a C^∞-differential for Γ.

Let ϕ be arbitrary. By Stoke's theorem we have

$$\int\int_{D^*} d(h^*\phi dz) = \int_{\partial D^*} h^*\phi dz = \sum_{s=1}^{m} \int_{C_s} h^*\phi\, dz;$$

the integrals along two identified sides on ∂D cancel each other, since $h^*\phi dz$ is Γ-invariant. The integral $\int\int_{D^*} d(h^*\phi dz) \to \int\int_D d(h^*\phi dz)$ whenever $\zeta_s \to a_s$; a_s is the fixed point of the parabolic transformation A_s identifying ζ_s and ζ_s'. Hence we can show that

$$\int\int_D d(h^*\phi dz) = 0$$

by showing that $\lim_{\zeta_s \to a_s} \int_{C_s} h^*\phi dz = 0$, for $s = 1, 2, \cdots, m$.

It suffices to assume that $s = 1, A_s(z) = z + 1$ and $a_s = \infty$. Then the cusped region belonging to ∞ is the region

$$U_c = \{z \in C; 0 \leq Rez < 1, Imz > c\}$$

Hence

$$\int_{c_1} h^* \phi dz = \int_0^1 h^*(x + ib)dx, \tag{2.8}$$

where $\zeta_1 = ib; b > c$, hence $\zeta_1' = 1 + ib$. Since $\phi \in B_2(U, \Gamma), \phi(z+1) = \phi(z)$ which implies that $\phi(z)$ has a Fourier series expansion

$$\phi(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z}, \quad z \in U.$$

Since $\sup_{z \in U} \{(Imz)^2 \mid \phi(z) \mid\} < \infty, a_n = 0$ for $n \leq 0$. Hence $\phi(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$.

Therefore,

$$\mid \phi(x + ib) \mid < Const. e^{-2\pi b}. \tag{2.9}$$

Since $h = fow^{\mu}, \dot{h} = f_0(z)\dot{w} + \dot{f}$ and hence

$$h^* = \dot{w} + f^* \tag{2.10}$$

where $\dot{w}$ *is* given by the following integral (see Ahlfors [1], chap. V)

$$\dot{w}(z) = \frac{z(z-1)}{2\pi i} \int\int_U \frac{\dot{\mu}(\zeta) d\zeta \wedge d\bar{\zeta}}{(\zeta - z)\zeta(\zeta - 1)}$$

It is known (Kra [13], chap. IV) that

$$\dot{w}(z) = 0(\mid z \mid log \mid z \mid) \text{ as } z \to \infty,$$

and hence

$$\mid \dot{w}(x + ib) \mid < const.(x^2 + b^2)^{\frac{1}{2}} log(x^2 + b^2) \text{ as } b \to \infty. \tag{2.11}$$

Finally, we shall find a growth condition on f^*. For this purpose, we study the behavior of f^* in the cusped region U_c.

From (2.1) it follows that

$$f \ o \ w^{\mu} \circ A_1 \circ (w^{\mu})^{-1} = \chi(A_1) \circ f. \tag{2.12}$$

Let $A_\tau = w^{\mu} \circ A_1 \circ (w^{\mu})^{-1}$. Then A_τ is parabolic, since A_1 is parabolic. Since w^{μ} fixes $0, 1$ and ∞, A_τ fixes ∞, and takes 0 to 1.

Hence $A_\tau(z) = z + 1$ for all τ. Moreover, $\chi(A_1)$ is parabolic if A_1 is parabolic by Kra [12]. Let $B_\tau(z) = \frac{1}{z - P_\tau}$, where P_τ is the fixed point of $\chi(A_1)$; and hence $\dot{B} = 0$.

Then

$$B_\tau \circ \chi(A_1) \circ B_\tau^{-1}(z) = z + b_\tau, \ b_\tau \neq 0.$$

We replace f by $B_\tau \circ f$ so that $\chi(A_1)$ is replaced by $B_\tau \circ \chi(A_1) \circ B_\tau^{-1}$, and we get from (2.12)

$$B_\tau \circ f \circ A_\tau = B_\tau \circ \chi(A_1) \circ B_\tau^{-1} \circ B_\tau \circ f. \tag{2.13}$$

We take $F = B_\tau \circ f$ and check that $\frac{\dot{F}}{F_0} = \frac{\dot{f}}{f_0}$, since $\dot{B} = 0$.

From (2.13), we have

$$F \circ A_\tau(z) = B_\tau \circ \chi(A_1) \circ B_\tau^{-1} \circ F(z);$$

$$F(z+1) = F(z) + b_\tau, z \in w^{\mu}(U).$$

Differentiating with respect to z we get $F'(z+1) = F'(z)$.

Therefore, $F'(z)$ is periodic in z and has a Fourier series expansion

$$F'(z, \tau) = \sum_{n=-\infty}^{\infty} a_n(\tau) e^{2\pi i n z}, \quad z \in w^{\mu}(U). \tag{2.14}$$

Now we follow the arguments of Kra [13] keeping in mind that F is a function in two variables z and τ. Thus from (2.14) we get

$$F'(z,\tau) = a_0(\tau) + \sum_{k=1}^{\infty} a_k(\tau)e^{2\pi ikz}, \text{ where } a_0(\tau) = b_\tau \neq 0. \tag{2.15}$$

Moreover,

$$b_\tau = a_0(\tau) = \int_{z_0}^{z_0+1} F'(z,\tau)dz, \text{ and}$$

$$a_k(\tau) = \int_{z_0}^{z_0+1} e^{-2k\pi iz}F'(z,\tau)dz, z_0 \in w^\mu(U).$$

Integrating (2.15) we get

$$F(z,\tau) = b_\tau z + \sum_{k=1}^{\infty} c_k(\tau)e^{2\pi ikz}, \ c_k(\tau) = \frac{a_k(\tau)}{2k\pi i}. \tag{2.16}$$

b_τ and $c_k(\tau)$ are holomorphic in τ, hence they have power series expansions in τ which are uniformly convergent in $\Delta_\varepsilon = \{\tau; |\tau| < \varepsilon\}$. Thus from (2.16), taking derivative with respect to τ at $\tau = 0$, we get

$$\dot{F}(z) = \dot{b}z + \sum_{k=1}^{\infty} \dot{c}_k e^{2\pi ikz}, \ z \in U.$$

We know that

$$B_\tau \circ \chi(A_1) \circ B_\tau^{-1}(z) = z + b_\tau; \text{ that is,}$$

$$B_\tau \circ \chi(A_1)(z) = B_\tau(z) + b_\tau.$$

Differentiating with respect to τ at $\tau = 0$ we get

$$B_0'(\chi_0(A_1))(z)\dot{\chi}(A_1) = \dot{B}(z) + \dot{b} = \dot{b},$$

since $\dot{B} = 0$. Thus $\dot{\chi}(A_1) = 0$ implies that $\dot{b} = 0$, and we have

$$\dot{F}(z) = \sum_{k=1}^{\infty} \dot{c}_k \ e^{2\pi ikz}, \ z \in U.$$

From (2.16), we also get

$$F_0'(z) = F'(z,0) = b_0 + \sum_{k=1}^{\infty} c_k(0) \ e^{2\pi ikz}, \ z \in U.$$

Hence

$$\frac{\dot{F}(z)}{F_0'(z)} = \sum_{k=1}^{\infty} \dot{c}_k e^{2\pi ikz}(b_0 + \sum_{k=1}^{\infty} c_k(0)e^{2\pi ikz})^{-1} = \sum_{k=1}^{\infty} d_k e^{2\pi ikz}.$$

Hence we have

$$f^*(z) = F^*(z) = \sum_{k=1}^{\infty} d_k e^{2\pi ikz}, \ z \in U. \tag{2.17}$$

From (2.17) it follows that

$$|f^*(x+ib)| < const.e^{-2\pi b}. \tag{2.18}$$

We recall that in the integral (2.8)

$$h^*\phi = (f^* + \dot{w})\phi = f^*\phi + \dot{w}\phi.$$

From (2.10), (2.11) and (2.18) we conclude that

$$|h^*(x+ib)\phi(x+ib)| < const.(e^{-4\pi b} + \frac{(x^2+b^2)^{\frac{1}{2}}log(x^2+b^2)}{e^{2\pi b}}) \to 0 \text{ as } b \to \infty$$

and hence

$$\lim_{b\to\infty}\int_{c_1} h^*\phi dz = \lim_{b\to\infty}\int_0^1 h^*(x+ib)\phi(x+ib) = 0.$$

Thus we have

$$\iint_D d(h^*\phi dz) = 0; \text{ that is,}$$

$$\iint_D h^*_{\bar z}\,\phi\; dz\wedge d\bar z = 0.$$

From (2.10) we know that $h^*_{\bar z} = \dot w_{\bar z} = \dot\mu$, hence we have

$$\iint_D \dot\mu\phi dz\wedge d\bar z = 0 \text{ for any } \phi\in B_2(U,\Gamma) \tag{2.19}$$

Since $\dot\mu\in W$ for $\mu\in W_0$, we have that

$$\dot\mu(z) = (Im\ z)^2\bar\phi_0(z), z\in U, \text{ for some } \phi_0\in B_2(U,\Gamma).$$

We now take $\phi=\phi_0$ in (2.19). Then we have

$$\iint_D (Im\ z)^2\,|\,\phi_0(z)\,|^2 dz\wedge d\bar z = 0$$

$$\phi_0 = 0 \Rightarrow \dot\mu = 0 \Rightarrow \dot w = 0 \Rightarrow h^*_{\bar z} = 0.$$

Hence h^* is holomorphic in U. Furthermore, $h^*=f^*$. Thus h^* is a (-1) differential for Γ. Following Kra [13], we define

$$red\ ord_p h^* = \frac{ord_p h^*}{|\,\Gamma_p\,|}, \text{ for } p\in U,$$

($|\,\Gamma_p\,|$ is the order of the stabilizer of P.)
and for each *cusp* a_s of Γ, $red\ ord_{a_s} h^* = r$ if the Fourier series expansion of h^* at ∞ is

$$h^*(z) = \sum_{k=r}^{\infty} a_k\, e^{2\pi ikz},\ a_r\neq 0,\ z\in U.$$

Since h^* is holomorphic in U, $red\ ord_p h^*\geq 0$ if $p\in U$. From (2.17)

$$red\ ord_{a_s} h^*\geq 1 \text{ for } s=1,2,\cdots,m.$$

Thus $\sum_{p\in\bar D_0} red\ ord_p h^* > 0$, where D_0 is a fundamental set in $\bar U$ for Γ. But

$$\sum_{p\in\bar D_0} red\ ord_p h^* = -(2p-2)+\sum_{j=1}^{n}(1-\tfrac{1}{\nu_j})$$

by Kra [13], and it is negative since $2p-2+\sum_{j=1}^{n}(1-\frac{1}{\nu_j})>0$.

This contradiction leads to the conclusion that $h^*=0$. This completes the proof of the corollary.

PROOF OF THE THEOREM. For an arbitrary point $t\in T(\Gamma)$, there exists a map taking t to a given point $t_0\in T(\Gamma)$. This map is a holomorphic homeomorphism by Bers [4]. Hence it is sufficient to prove the theorem in a neighborhood of the origin $t_0\in T(\Gamma)$.

We have noticed earlier that, in a neighborhood of t_0, Φ is induced by Φ^*. Φ^* is holomorphic by the Lemma 2.3. The Lemma 2.2 and the Corollary of the Lemma 2.3 together imply that the differential of Φ^* is injective. It is known that χ preserves the parabolic elements and the multipliers of the elliptic elements in Γ. Moreover, $\chi(\Gamma)$ is nonelementary by Kra [12]. Hence the image χ of Φ^* is a manifold point in $Hom(\Gamma,G)$ by the Theorem 1. Since $G\times\mathfrak{B}(T(\Gamma))$ and $Hom(\Gamma,G)$ have the same dimension $6p+2n-3$, Φ^* is a local homeomorphism. Replacing (I,t,ϕ) by (A,t,ϕ) in $G\times\mathfrak{B}(T(\Gamma))$ has the effect of conjugating χ by A. Hence we conclude that Φ is holomorphic and a local homeomorphism in a neighborhood of t_0. This completes the proof.

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REFERENCES

1. AHLFORS, L.V., Lectures on quasiconformal mappings, Princeton, New Jersey, 1966.
2. AHLFORS, L.V., The structure of a finitely generated Kleinian group, *Acta Math.* **122** (1969), 1-17.
3. BERS, L., Inequalities for finitely generated Kleinian Groups, *J. d'Analyse Math.* **18** (1967), 23-41.
4. BERS, L., Fiber spaces over Teichmüller spaces, *Acta Math.*, **130** (1973), 89-126.
5. EARLE, C.J., On variation of projective structures, *Proc. of Stony Brook Conference* (1978), 87-99.
6. FALTINGS, G., Real projective structures on Riemann surfaces, *Compositio Math.* **48** (1983), 223-269.
7. GALLO, D.M. & PORTER, R.M., Embedding the deformation space of a Fuchsian group of the first kind, *Bol. Soc. Mat. Mex.* **26** (1981), 49-53.
8. GALLO, D.M. & PORTER, R.M., Extended monodromy for bordered Riemann surfaces, *J. d'Analyse Math.* **54** (1990), 1-20.
9. GARDINER, F. & KRA, I., Stability of Kleinian groups, *Indiana Univ. Math. Journal* 21 (1972), 1037-1059.
10. GUNNING, R.C., Lectures on vector bundles over Riemann surfaces, Princeton, New Jersey, 1967.
11. KRA, I., Affine and projective structures, *J. d'Analyse Math.* **22** (1969), 285-298.
12. KRA, I., Deformation of Fuchsian groups, II, *Duke Math. Journal* **38** (1971), 499-508.
13. KRA, I., Automorphic Forms and Kleinian Groups, Benjamin-Reading, Massachusetts, 1972.
14. MASKIT, B., Kleinian Groups, Springer-Verlag, 1987.
15. SENGUPTA, J., A uniqueness theorem of reflectable deformations of a Fuchsian group, *Proc. of AMS, Vol. 104, No. 4*, 1148-1152.