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A METHOD FOR SUMMABILITY OF LAGRANGE INTERPOLATION

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ABSTRACT. The author uses in this paper a technique from numerical integration (see [9]) to get a discretely defined operator, which is a modification of the Lagrange operator. Therefore we improve with the linear summation method $L_n^\star$ a result presented in [12] and we also point out a solution for a problem of P.L.Butzer in the proceedings of the Budapest conference in 1980.

KEY WORDS AND PHRASES. Approximation by positive linear operators, discrete linear operators, Lagrange Interpolation, pointwise estimates, second order modulus of continuity, Lipschitz - type maximalfunction, Butzer's problem.

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1 INTRODUCTION.

Let Π_n be the set of algebraic polynomials of degree $\leq n$ and let M be the triangular matrix of points

$$M = \left\{ \begin{array}{lllll} z_{0,0} & & & & \\ z_{0,1} & z_{1,1} & & & \\ \vdots & \vdots & & & \\ z_{0,n} & z_{1,n} & \cdots & z_{n,n} & \\ \vdots & \vdots & & \vdots & \ddots \end{array} \right.$$

where

$$1 \geq z_{0,n} > z_{1,n} > \cdots > z_{n,n} \geq -1 .$$

For any continuous function $f \in C(I)$ with $I := [-1,1]$ we define the n - th Lagrange interpolation polynomial $L_n f$ with respect to the set of points $z_{k,n} = \cos \frac{2k+1}{2(n+1)}\pi$, $k = 0, 1, \cdots, n$, $n \in \mathbb{N}_0$, to be that polynomial of degree $\leq n$, which interpolates the values $(z_{k,n}, f(z_{k,n}))$.
Therefore the $(n+1)$ points $z_{k,n}$ are the zeros of the $(n+1)$ - th Chebyshev polynomial $T_{n+1}(x) = \cos((n+1)\arccos x)$. These Lagrange operators $L_n : C(I) \rightarrow \Pi_n$, $n \in \mathbb{N}_0$, are given by

$$L_n(z_{0,n}, \cdots, z_{n,n}; f, x) = \sum_{k=0}^{n} f(z_{k,n}) \frac{T_{n+1}(x)}{(x - z_{k,n}){T_{n+1}}'(z_{k,n})} .$$

If we define $\omega_0 = \frac{1}{\pi}$ and $\omega_j = \frac{2}{\pi}$, $j \geq 1$, one can easily see that

$$\frac{T_{n+1}(x)}{(x - z_{k,n})T_{n+1}'(z_{k,n})} = \frac{\pi}{n+1}\sum_{j=0}^{n}\omega_j T_j(x)T_j(z_{k,n}).$$

We use for $f, g : I \to \mathbb{R}$ the inner product

$$[f, g]_n = \frac{\pi}{n+1}\sum_{k=0}^{n} f(z_{k,n})g(z_{k,n}),$$

and so one finds the representation

$$L_n(z_{0,n}, \cdots, z_{n,n}; f, x) = \sum_{j=0}^{n}\omega_j[f, T_j]_n T_j(x).$$

It is known about the sequence $L = (L_n)$ that it does not converge pointwise to the identity on the space $C(I)$.
This negative behaviour was the impulse for some authors to find and to investigate approximation methods in a form of a fine modification, which have better properties.

For instance in this paper we consider linear approximation operators $\hat{L}_n : C(I) \to \Pi_n$, $n \in \mathbb{N}$, given by

$$(\hat{L}_n f)(x) = \sum_{j=0}^{n}\hat{\beta}_{j,n}\omega_j[f, T_j]_n T_j(x). \tag{1.1}$$

An open problem is to find the "convergence - multipliers" $\hat{\beta}_{j,n}$ such that, with respect to the uniform norm, we have for all $f \in C(I)$

$$\lim_{n\to\infty} \| f - \hat{L}_n f \| = 0.$$

Let us use the well - known translation operator $\tau_x : X \to X$, $x \in I$, defined by (see [2], [3])

$$(\tau_x f)(t) = \frac{1}{2}\left[f(xt + \sqrt{1-x^2}\sqrt{1-t^2}) + f(xt - \sqrt{1-x^2}\sqrt{1-t^2})\right], \quad t, x \in I,$$

and denote

$$\hat{b}_n(x) = \sum_{j=0}^{n}\hat{\beta}_{j,n}\omega_j T_j(x). \tag{1.2}$$

Taking into account that

$$(\tau_x T_j)(t) = T_j(x)T_j(t)$$

from (1.1) we see that

$$(\hat{L}_n f)(x) = [f, \tau_x\hat{b}_n]_n \tag{1.3}$$

and we can say that $\hat{L}_n$ is generated by $\hat{b}_n$ or that $(\hat{b}_n)$ is the generating sequence for $(\hat{L}_n)$.
However it is clear that τ_x is a positive operator and moreover that for $n \geq 2$

$$(\hat{L}_n T_k)(x) = \hat{\beta}_{k,n}T_k(x), \quad k \in \mathbb{N}_0. \tag{1.4}$$

Further $\hat{\beta}_{k,n}$, $k = 0, 1, \cdots, n$, are the Chebyshev coefficients of $\hat{b}_n$. This implies that when $\hat{b}_n \geq 0$ on I and

$$\int_{-1}^{1}\hat{b}_n(t)\frac{dt}{\sqrt{1-t^2}} = 1,$$

then $\hat{L}_n : C(I) \to \Pi_n$ is a positive linear operator with $\hat{L}_n e_0 = e_0$, $e_k(t) \doteq t^k$. If we apply $\hat{L}_n$ on the inequalities

$$-1 \le T_k(t) \le 1, \qquad t \in I$$

$$4(1 - T_1(t)) - (1 - T_2(t)) = 2(t-1)^2 \ge 0$$

and then select $x = 1$ (see (1.4)), one finds

$$-1 \le \hat{\beta}_{k,n} \le 1$$

and

$$0 \le 1 - \hat{\beta}_{2,n} \le 4(1 - \hat{\beta}_{1,n}). \tag{1.5}$$

Now

$$\begin{aligned}(\hat{L}_n e_1)(x) &= e_1(x) - x(1 - \hat{\beta}_{1,n}) \\ (\hat{L}_n e_2)(x) &= \hat{\beta}_{2,n} e_2(x) + \frac{1 - \hat{\beta}_{2,n}}{2}.\end{aligned}$$

In conclusion, from (1.5) we see that $\lim_{n\to\infty} \hat{\beta}_{1,n} = 1$ is a sufficient condition for the Korovkin conditions

$$\lim_{n\to\infty} \| e_k - \hat{L}_n e_k \| = 0, \qquad k = 1, 2.$$

In this present paper we will consider the modified summability methods of L_n given by

$$(L_n^\star f)(x) := L_n^\star(z_{0,n}, \cdots, z_{n,n}; f, x) = \sum_{j=0}^{n} \beta_{j,n} \omega_j [f, T_j]_n T_j(x), \tag{1.6}$$

with the generating polynomial (see (4.3) in [9])

$$b_n(x) = \kappa_n \frac{\left(1 - x \cos\frac{2\pi}{n+2}\right)(1 - T_{n+2}(x))}{(1-x)\left(x - \cos\frac{2\pi}{n+2}\right)^2} = \sum_{j=0}^{n} \beta_{j,n} \omega_j T_j(x)$$

where $\kappa_n = \frac{1}{\pi(n+2)} \sin^2 \frac{\pi}{n+2}$ and

$$\beta_{j,n} = \frac{n-j+2}{n+2} \cos^2 \frac{j\pi}{n+2} + \frac{\cos\frac{\pi}{n+2}}{(n+2)\sin\frac{\pi}{n+2}} \cos\frac{j\pi}{n+2} \sin\frac{j\pi}{n+2}. \tag{1.7}$$

Without claim of completeness there are in some papers [5],[6],[12] other examples of similar sequences like presented in (1.6).

For this **discrete linear positive approximation method** in (1.6) we have an instance of algebraic polynomials, for which we can prove (see (2.2)) an estimate of A.F. Timan [13] for all $x \in I$

$$| f(x) - (L_n^\star f)(x) | \le C\omega(f; \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}), \quad n \in \mathbb{N}, \tag{1.8}$$

where $\omega(f;\delta_n) := sup\{|f(t+h) - f(t)|;\ |h| \le \delta_n,\ t, t+h \in I\}$ and the constant C is independent of f and n.

2 MAIN RESULTS

It is the aim of this present note to show that the local order of approximation by means of $(L_n^\star)$ is comparable with those furnished by the best approximation polynomials. At the end of this explanations we get a conclusion to prove a problem of P.L. Butzer (see remark 2.4).

Now let us state our central theorem:

THEOREM 2.1 *Let $L^\star = (L_n^\star)$, $n \in \mathbb{N}$, be defined as in (1.6) and*

$$\Delta_n(x) := \frac{\sqrt{1-x^2}}{n} + \frac{|x|}{n^2}, \qquad x \in I. \tag{2.1}$$

Then for $f \in C(I)$, $x \in I$,

$$|\, f(x) - (L_n^\star f)(x)\,| \le 40\,\omega\,(f; \Delta_n(x)) \tag{2.2}$$

and further

$$|\, f(x) - (L_n^\star f)(x)\,| \le (3 + 2\pi^2)\left\{\omega_2(f; n^{-1}) + \frac{|x|}{n}\omega(f; n^{-1})\right\}, \tag{2.3}$$

where $\omega_2(f;\delta) := \sup\{|f(x-\delta) - 2f(x) + f(x+\delta)|;\ x, x\pm\delta \in I,\ 0 \le \delta \le h\}$ is the second modulus of smoothness of f.

PROOF: Our starting point is to construct together with a sequence $A = (A_n)$ of approximation operators — by means of a Θ - transformation — a new sequence of operators $B = (B_n) = \Theta(A)$.
So, to prove this above estimates, we employ the transformation technique from [9].
Therefore we denote by X the normed linear spaces $C(I)$ or $L_\omega^p(I)$, $1 \le p < \infty$, supplied with norms $\|f\|_{C(I)} = \|f\| := \max_{t\in I}|f(t)|$ for $f \in C(I)$, respectively $\|f\|_p = \left[\int_{-1}^{1} |f(t)|^p\,\omega(t)dt\right]^{\frac{1}{p}}$, where f is an element of the Lebesque space $L_\omega^p(I)$ with the weight $\omega(t) = \frac{1}{\sqrt{1-t^2}}$.
Let us use the well - known convolution product $\star : L_\omega^1(I) \times L_\omega^1(I) \to L_\omega^1(I)$

$$(f \star g)(x) = \int_{-1}^{1} f(t)\,(\tau_x g)\,(t)\,\omega(t)dt\,.$$

According to the Mehler – Hermite quadrature formula, i.e.

$$\int_{-1}^{1} h(t)\omega(t)dt \approx \frac{\pi}{n+1}\sum_{k=0}^{n} h(z_{k,n}),$$

which is exact for algebraic polynomials h of degree $\le 2n+1$, we have

$$[f,g]_n = \int_{-1}^{1} f(t)g(t)\omega(t)dt, \quad fg \in \Pi_{2n+1}\,.$$

The polynomial $L_n^\star f$ can be now expressed as follows

$$\begin{aligned}(L_n^\star f)(x) &= \sum_{j=0}^{n} \beta_{j,n}\omega_j[f,T_j]_n T_j(x) = \frac{\pi}{n+1}\sum_{k=0}^{n} f(z_{k,n})\sum_{j=0}^{n}\beta_{j,n}\omega_j T_j(x)T_j(z_{k,n}) \\ &= \frac{\pi}{n+1}\sum_{k=0}^{n} f(z_{k,n})\,(\tau_x b_n)\,(z_{k,n}).\end{aligned} \tag{2.4}$$

If

$$(B_n f)(x) = \int_{-1}^{1} f(t)\,(\tau_x b_n)\,(t)\omega(t)dt = (f \star b_n)(x), \qquad f \in X, \tag{2.5}$$

where the polynomial sequence $b_n \in \mathcal{P}^+ := \left\{ b = (b_n)\,|\, b_n(x) \ge 0,\ x \in I;\ \int_{-1}^{1} b_n(t)\omega(t)dt = 1 \right\}$ is defined by

$$b_n(x) = \sum_{j=0}^{n} \beta_{j,n}\omega_j T_j(x),$$

then for $f \in \Pi_{n+1}$ we have

$$L_n^\star f = B_n f.$$

Since the translation operator τ_x is a positive linear operator, we can see that these properties transfer to the discrete operator $L_n^\star$ - that means, $L_n^\star$ is linear and positive.

Now let us describe for our purpose the so called Θ - transformation [9]:
We start with a polynomial sequence $a = (a_n) \in \mathcal{P}^1 := \{ a = (a_n) \,|\, a \in \mathcal{P}^+ \text{ and } \forall n \in \mathbb{N}_0$ $\exists\, z_0 := z_0(n) \in I : a_n(z_0) = a_n(1, z_0) = 0\}$, and the mapping $l : \mathcal{P}^1 \to \mathcal{P}^+$,

$$l(a) = b = (b_n),$$

where

$$b_n(x) = \frac{1}{c_n}\frac{a_{n+1}(x,\bar{z}_0)}{1-x}, \qquad c_n = \int_{-1}^{1} \frac{a_{n+1}(t,\bar{z}_0)}{1-t}\,\omega(t)\,dt, \qquad \bar{z}_0 = z_0(n+1).$$

Then the operator sequence $B = (B_n)$, $B_n f = f \star b_n$ is called the Θ - transformation of the sequence $A = (A_n)$, defined by $A_n f = f \star a_n$, and we write

$$B = \Theta(A).$$

If we consider the case $a = (a_n)$, $a_n(x) = \frac{1-T_{n+1}(x)}{\pi(n+1)(1-x)}$, with $z_0 = z_0(n) = \cos\frac{2\pi}{n+1}$, then our (B_n) from (2.5) is the Θ - transformation of Fejér operators

$$A = (F_n),$$

$$(F_n f)(x) = \int_{-1}^{1} f(t) \sum_{j=0}^{n} (1 - \frac{j}{n+1})\omega_j T_j(x) T_j(t)\omega(t)dt, \qquad f \in X.$$

Taking into account that $L_n^\star$ is a discrete form of $B_n = \Theta(F_n)$, we conclude, by means of theorem 4.2 from [9] that

$$|\, f(x) - (L_n^\star f)(x)\,| \le 4\,\omega(f; \epsilon_n(x))$$

with

$$\epsilon_n(x) = \sqrt{1-x^2}\sin\frac{\pi}{n+2} + |x|\sin^2\frac{\pi}{n+2} \le \pi^2\Delta_n(x).$$

At the same time

$$|\, f(x) - (L_n^\star f)(x)\,| \le (3 + 2\pi^2)\left(\omega_2(f; n^{-1}) + \frac{|x|}{n}\omega(f; n^{-1})\right), \qquad \text{(see [9], theorem 4.3)},$$

which completes our proof.

REMARK 2.2 *We get for the convergence multipliers ([9], (3.3))*

$$\beta_{j,n} = \frac{1}{(n+2)c_n}\left(\frac{1 - z_0 - T_{j+1}(z_0) + T_j(z_0)}{(1-z_0)^2} + \frac{(n+1-j)(1+T_j(z_0))}{1-z_0}\right) \tag{2.6}$$

$$= \frac{2}{(n+2)(1-z_0)c_n}\left(\frac{\sin(j+1)\frac{\pi}{n+2}\cos\frac{j\pi}{n+2}}{\sin\frac{\pi}{n+2}} + (n+1-j)\cos^2\frac{j\pi}{n+2}\right)$$

$$= \frac{2}{(1-z_0)c_n}\left(\frac{n-j+2}{n+2}\cos^2\frac{j\pi}{n+2} + \frac{\cos\frac{\pi}{n+2}}{(n+2)\sin\frac{\pi}{n+2}}\sin\frac{j\pi}{n+2}\cos\frac{j\pi}{n+2}\right),$$

where together with $\beta_{0,n} = \frac{2}{(1-z_0)c_n}$

$$c_n = \frac{2}{1-z_0} = \frac{1}{1-\beta_{1,n}} = \frac{1}{\sin^2\frac{\pi}{n+2}}$$

and we find the form in (1.7).

The local behaviour of f is measured by suitable maximal functions taking into consideration the fact that a single local change of f can influence the polynomial operator $L_n^\star$ as a whole. Therefore the local error $f - L_n^\star f$ is estimated by the pointwise Lipschitz type maximal function

$$f_\alpha^{\sim}(x) := \sup_{t\neq x, t\in I} \frac{|f(x)-f(t)|}{|t-x|^\alpha}$$

for $f \in L_1^1(I)$ bounded and $x \in I$, $\alpha \in (0,1]$, introduced by B. Lenze [7]. For similar results with other operators see [8],[11],[10].

THEOREM 2.3 *Let $n \in N$, $\alpha \in (0,1]$ and $f \in L_1^1(I)$ be bounded. Then for all $x \in I$*

$$|f(x) - (L_n^\star f)(x)| \leq \pi^2 f_\alpha^{\sim}(x)(\Delta_n(x))^\alpha \tag{2.7}$$

PROOF: From the definition of $f_\alpha^{\sim}(x)$ and with Hölder inequality for $p = \frac{1}{\alpha} \geq 1$

$$\begin{aligned} |f(x) - (L_n^\star f)(x)| &\leq L_n^\star(|f(x)-f(t)|;x) \leq f_\alpha^{\sim}(x)\, L_n^\star(|x-t|^\alpha;x) \\ &\leq f_\alpha^{\sim}(x)\,(L_n^\star(|x-t|^{p\alpha};x))^{\frac{1}{p}} = f_\alpha^{\sim}(x)\,(L_n^\star(|x-t|;x))^\alpha . \end{aligned}$$

With $1-\beta_{1,n} = \sin^2\frac{\pi}{n+2}$ and

$$\frac{1-\beta_{2,n}}{2} = \frac{2}{n+2}\sin^2\frac{\pi}{n+2}\left(2(n+1)\cos^2\frac{\pi}{n+2}+1\right) \leq \frac{2\pi^2}{(n+2)^2},$$

we find the following enclosure

$$\begin{aligned} |x|\sin^2\frac{\pi}{n+2} \leq L_n^\star(|t-x|;x) &\leq \frac{\sqrt{2(1-x^2)}\pi}{n+2} + |x|\sin^2\frac{\pi}{n+2} \\ &\leq \pi^2\Delta_n(x) \qquad ([9], (2.6)), \end{aligned}$$

which proves this theorem.

REMARK 2.4 P.L.Butzer presented on a conference in Budapest [1] the following problem: *Can one construct a triangular matrix of distinct nodes $\{x_{k,n}\}_{k=0}^n$, $n \in \mathbb{N}_0$, $x_{k,n} \in I$, and a triangular matrix of positive fundamental functions $\{\varphi_{k,n}\}_{k=0}^n$, defined on I, such that the linear summator operators*

$$(L_nf)(x) := \sum_{k=0}^n f(x_{k,n})\varphi_{k,n}(x), \qquad f \in C(I),$$

are algebraic polynomials of degree n, and satisfy

$$\|L_nf - f\|_{C(I)} = \mathcal{O}(n^{-\alpha}), \tag{2.8}$$

provided $f \in Lip_2(\alpha, C)$, $0 < \alpha \leq 2$.

We get a solution of this problem, if we use for $\delta \in (0,1]$ and $f \in Lip_2(\alpha, C)$, $0 < \alpha \leq 2$, the estimates for $\omega_2(f;\delta) \leq C\delta^\alpha$, $C := const.$ and

$$\delta\omega(f;\delta) \leq \begin{cases} 2\,\|f\|\delta^\alpha & , \quad \alpha \in (0,1] \\ \\ \|f'\|\delta^\alpha & , \quad \alpha \in (1,2] \end{cases}$$

We find so with a positive constant $M = M(f)$

$$\omega_2(f;\delta) + |x|\delta\omega(f;\delta) \leq M\delta^\alpha, \quad \alpha \in (0,2],\ \delta \in (0,1],\ x \in I.$$

If we choose $\delta = \frac{1}{n}$, from (2.3) we get

$$|\,f(x) - (L_n^\star f)(x)\,| \leq \frac{M}{n^\alpha}, \quad x \in I.$$

Finally the linear summator operators $(L_n^\star)$ have the co - domain in Π_n and satisfy (2.8) provided $f \in Lip_2(\alpha, C)$, $0 < \alpha \leq 2$, i.e. a solution for the problem proposed by P.L.Butzer.

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