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ON QUASI-IDEALS OF SEMIRINGS

CHRISTOPH DÖNGES
Institut für Mathematik
Technische Universität Clausthal
Erzstraße 1
DW-3392 Clausthal-Zellerfeld
Germany

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ABSTRACT. Several statements on quasi-ideals of semirings are given in this paper, where these semirings may have an absorbing element O or not. In Section 2 we characterize regular semirings and regular elements of semi-rings using quasi-ideals (cf. Thms. 2.1, 2.2 and 2.7). In Section 3 we deal with (O-)minimal and canonical quasi-ideals. In particular, if the considered semiring S is semiprime or quasi-reflexive, we present criterions which allow to decide easily whether an (O-)minimal quasi-ideal of S is canonical (cf. Thms. 3.4 and 3.8). If S is an arbitrary semiring, we prove that for (O-)minimal left and right ideals L and R of S the product $\langle RL\rangle \subseteq L\cap R$ is either $\{O\}$ or a canonical quasi-ideal of S (Thm. 3.9). Moreover, for each canonical quasi-ideal Q of a semiring S and each element $a \in S$, Qa is either $\{O\}$ or again a canonical quasi-ideal of S (Thm. 3.11), and the product $\langle Q_1Q_2\rangle$ of canonical quasi-ideals Q_1, Q_2 of S is either $\{O\}$ or again a canonical quasi-ideal of S (Thm. 3.12). Corresponding results to those given here for semirings are mostly known as well for rings as for semigroups, but often proved by different methods. All proofs of our paper, however, apply simultaneously to semirings, rings and semigroups (cf. Convention 1.1), and we also formulate our results in a unified way for these three cases. The only exceptions are statements on semirings and semigroups without an absorbing element O, which cannot have corresponding statements on rings since each ring has its zero as an absorbing element.

KEY WORDS AND PHRASES. Quasi-ideals, regular elements, regular semirings, (O-)minimal quasi-ideals, canonical quasi-ideals.
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1. PRELIMINARIES.

An algebra $S = (S, +, \cdot)$ is called a semiring , iff $(S, +)$ and $(S, \cdot)$ are arbitrary semigroups connected by ring-like distributivity. If there is an element $o \in S$ $[e \in S]$ satisfying $o+a = a+o = a$ $[ea = ae = a]$ for all $a \in S$, it is called the zero [the identity] of S. If there is an element $O \in S$ satisfying $Oa = aO = O$ for all $a \in S$, it is called the absorbing element of S. Note that there are semirings with a zero o satisfying $o = e$ or $oo \neq o$, whereas $O + O = O$ trivially holds for each semiring with an absorbing element O. A semiring with elements o and O which coincide is said to have an absorbing zero $o = O$; among others, each ring is an example for such a semiring. Let $\mathbb{N}$ be the set of positive integers and $\emptyset \neq X, Y \subseteq S$. We introduce the notion

$$(1.1) \qquad \langle X\rangle = \{\sum_{i=1}^{n} x_i \mid x_i \in X, n \in \mathbb{N}\}$$

for the subsemigroup of $(S,+)$ generated by X. Deviating from the usual notion in ring theory, we define XY for semirings (and hence for rings) $(S,+,\cdot)$ in the same way as for semigroups $(S,\cdot)$ by

$$(1.2) \qquad XY = \{xy \mid x \in X, y \in Y\}.$$

The product of subsets mostly used in ring theory can then be expressed by $\langle XY\rangle$ according to

$$(1.3) \qquad \langle XY\rangle = \{\sum_{i=1}^{n} x_i y_i \mid x_i \in X, y_i \in Y, n \in \mathbb{N}\}.$$

Clearly, we write e.g. xY and $\langle xY\rangle$ instead of $\{x\}Y$ and $\langle\{x\}Y\rangle$. We have to use both notions (1.2) and (1.3) extensively, including the following equations concerning (1.3) which are easily checked for all $\emptyset \neq X, Y, Z \subseteq S$:

$$(1.4) \qquad \langle X \cup Y\rangle = \langle\langle X\rangle \cup \langle Y\rangle\rangle$$

$$(1.5) \qquad \langle XY\rangle = \langle\langle X\rangle Y\rangle = \langle X\langle Y\rangle\rangle$$

$$(1.6) \qquad \langle XYZ\rangle = \langle\langle XY\rangle Z\rangle = \langle X\langle YZ\rangle\rangle = \langle\langle X\rangle\langle Y\rangle\langle Z\rangle\rangle \ .$$

Whereas in general only $XY \subseteq \langle XY\rangle$ is true, clearly $xY = \langle xY\rangle$ and $Yx = \langle Yx\rangle$ hold for all $x \in S$ and all $Y \subseteq S$ satisfying $Y = \langle Y\rangle$.

Our main interest is with quasi-ideals of semirings, and we also need left, right and two-sided ideals. Since these concepts differ for semirings and rings, we recall them in a way pointing out this difference: A subset $\emptyset \neq L \subseteq S$ of a (semi)ring $(S,+,\cdot)$ is called a left ideal of $(S,+,\cdot)$ iff L is a sub(semi)group of $(S,+)$ satisfying $SL \subseteq L$. The latter states that L is a left ideal of $(S,\cdot)$, and for an additive sub(semi)group L of a (semi)ring S, clearly, $SL \subseteq L$ and $\langle SL\rangle \subseteq L$ are equivalent. Right ideals and two-sided ideals of $(S,+,\cdot)$ are defined analogously. Due to [3], §12, a subset $\emptyset \neq Q \subseteq S$ of a (semi)ring $(S,+,\cdot)$ is called a quasi-ideal of $(S,+,\cdot)$, iff Q is a sub(semi)group of $(S,+)$ satisfying $\langle SQ\rangle \cap \langle QS\rangle \subseteq Q$. In this case the condition $SQ \cap QS \subseteq Q$, stating that Q is a quasi-ideal of the semigroup $(S,\cdot)$, leads to the concept of a weak quasi-ideal of the (semi)ring $(S,+,\cdot)$, and such a weak quasi-ideal need not be a quasi-ideal (cf. [8], §2).

In general, if $(S,+,\cdot)$ is a ring, one may consider left, right, two-sided and quasi-ideals in the ring-theoretical as well as in the semiring-theoretical meaning. Since each subgroup is all the more a subsemigroup, each ring-theoretical ideal is a semiring-theoretical one. The converse holds e.g. for rings with an identity, but not in general.

As already mentioned in the abstract, most of our results on semirings correspond to already known statements on semigroups and on rings. Moreover, even in cases where those statements have been proved for semigroups and for rings by completely different ideas, our proof for the corresponding statement on semirings will apply also to semigroups and rings. So we want to formulate our considerations in such a way, that they can be read for semirings as well as for semigroups and rings (the latter of course for rings considered as rings and not as a special kind of semirings). For this reason, we use the following

CONVENTION 1.1. a) If $S = (S, +, \cdot)$ is a ring, the terms left, right, two-sided or quasi-ideal of S are used in the ring-theoretical meaning.
b) The notions $\langle X \rangle$, XY and $\langle XY \rangle$ have always the same meaning according to (1.1), (1.2) and (1.3), regardless whether $S = (S, +, \cdot)$ is a semiring or a ring. (This will cause no problems, since in the ring-case $\langle X \rangle$ is a subgroup of $(S, +)$, provided that $-X \subseteq X$ holds.)
c) In the case that $S = (S, \cdot)$ is a semigroup, one has to pay no attention to the brackets $\langle \ \rangle$.

In this context we remark, that each of our results on semirings provides automatically the corresponding one on semigroups. The reason is that any semigroup $(S, \cdot)$ determines a semiring $(S, +, \cdot)$ with respect to the left absorbing addition on S, defined by $a + b = a$ for all $a, b \in S$, such that each left, right, two-sided or quasi-ideal of $(S, \cdot)$ is an ideal of the same kind of $(S, +, \cdot)$, and conversely.
The elementary properties given in the following two Lemmata are well known (cf. e.g. [3], §1-2 for rings and semigroups and [8], §1 for semirings).

LEMMA 1.2. Let S be a semiring, a ring or a semigroup.
a) Each one- or two-sided ideal of S is a quasi-ideal of S.
b) The intersection of any system of quasi-ideals of S is either empty or a quasi-ideal of S.
c) If L is a left and R a right ideal of S, then $RL \subseteq \langle RL \rangle \subseteq L \cap R$ holds and the intersection $Q = L \cap R$ is a quasi-ideal of S.
d) For each $\emptyset \neq X \subseteq S$, $\langle SX \rangle$ is a left ideal, $\langle XS \rangle$ a right ideal, $\langle SXS \rangle$ an ideal and $\langle SX \rangle \cap \langle XS \rangle$ a quasi-ideal of S.
e) Each quasi-ideal of a semiring, a ring or a semigroup S is a subsemiring, a subring or a subsemigroup of S respectively.

LEMMA 1.3. Let S be a semiring, a ring or a semigroup, L a left and R a right ideal of S and consider elements $e = e^2$ and $f = f^2$ of S. Then we have

$$eL = L \cap eS, \quad Re = Se \cap R \quad \text{and} \quad Sf \cap eS = eSf,$$

and all three subsets are quasi-ideals of S.

For each $\emptyset \neq X \subseteq S$ we denote by $(X)_l$, $(X)_r$, $(X)_t$ and $(X)_q$ the left, right, two-sided and the quasi-ideal of S generated by X. We call them principle if they can be generated by one element x and write then e.g. $(x)_q$ instead of $(\{x\})_q$. The following lemma can be checked in a straightforward manner.

LEMMA 1.4. Let S be a semiring, a ring or a semigroup.
a) For each $\emptyset \neq X \subseteq S$, additionally satisfying $-X \subseteq X$ in the case that S is a ring, we have
$(X)_l = \langle X \cup SX \rangle$, $(X)_r = \langle X \cup XS \rangle$,
$(X)_t = \langle X \cup SX \cup XS \cup SXS \rangle$ and $(X)_q = \langle X \cup (\langle SX \rangle \cap \langle XS \rangle) \rangle$.
b) For the principle left, right and two-sided ideals generated by $s \in S$ we have $\langle S(s)_l \rangle = Ss$, $\langle (s)_r S \rangle = sS$ and $\langle S(s)_t S \rangle = \langle SsS \rangle$.

Further, a quasi-ideal Q of S is said to have the intersection property, iff $Q = L \cap R$ holds for suitable left and right ideals L and R of S which clearly implies $Q = (Q)_l \cap (Q)_r$. Whereas each quasi-ideal of a semigroup $(S, \cdot)$ has the intersection property ([3], Prop. 2.6), an example due to Clifford (published in [3], Expl. 2.1) shows that there are rings which contain quasi-ideals without the intersection property . Examples given by Weinert ([8], Prop. 5.2) show that there are even O-minimal (cf. Section 3) quasi-ideals of a semiring $(S,+,\cdot)$ with absorbing element O, which do not have the intersection property. Finally, a bi-ideal B of the (semi)ring $(S,+,\cdot)$ is defined as a sub(semi)ring of $(S,+,\cdot)$ satisfying $BSB \subseteq B$ and hence $\langle BSB\rangle \subseteq B$. Note that $BSB \subseteq B$ states that the subsemigroup B of $(S,\cdot)$ is a bi-ideal of the semigroup $(S,\cdot)$. The following lemma is again easy to check:

LEMMA 1.5. Let S be a semiring, a ring or a semigroup and T a two-sided ideal of S. Then each quasi-ideal Q of T is a bi-ideal of S. Especially each quasi-ideal Q of S is also a bi-ideal of S; hence Q satisfies $\langle QSQ\rangle \subseteq Q$.

2. QUASI-IDEALS AND REGULARITY

Let S be a semiring, a ring or a semigroup. Then an element $s \in S$ is called regular in S iff $s \in sSs$ holds, and S is called regular iff each element of S is regular in S. Note that an element s of a semiring, a ring or a semigroup S is regular iff one of the following statements holds:

(2.1) There is an element $x \in S$ satisfying $xs = e = e^2, \quad s = se$.

(2.1′) There is an element $y \in S$ satisfying $sy = f = f^2, \quad s = fs$.

(2.2) There is an element $e = e^2 \in S$ satisfying $(s)_l = Se = Ss$.

(2.2′) There is an element $f = f^2 \in S$ satisfying $(s)_r = fS = sS$.

Restricted to the special cases that S is a ring or a semigroup, most of the results of this section can be found in [3], §9.

THEOREM 2.1. Let S be a semiring, a ring or a semigroup. Then the following conditions are equivalent:

(1) S is regular.

(2) Each left ideal L and each right ideal R of S satisfy $\langle RL\rangle = L \cap R$ (which in fact implies $RL = \langle RL\rangle = L \cap R$).

(3) Each left ideal L and each right ideal R of S satisfy

a) $\langle L^2\rangle = L$,

b) $\langle R^2\rangle = R$ and

c) $\langle RL\rangle$ is a quasi-ideal of S.

(4) The set $\mathcal{Q}$ of all quasi-ideals of S is a regular semigroup with respect to the "product" $\langle Q_1Q_2\rangle$.

(5) Each quasi-ideal Q of S satisfies $Q = \langle QSQ\rangle$.

Moreover, the statements 3a) and 3b) imply that each quasi-ideal Q of S has the intersection property since it satisfies $Q = \langle SQ\rangle \cap \langle QS\rangle$.

PROOF. At first we prove the last statement. We apply 3a) to the left ideal $(Q)_l = \langle Q \cup SQ\rangle$ of S generated by a quasi-ideal Q of S and obtain $Q \subseteq \langle Q \cup SQ\rangle = \langle\langle Q \cup SQ\rangle^2\rangle \subseteq \langle SQ\rangle$, where the last inclusion is obvious. Similarly we get $Q \subseteq \langle Q \cup QS\rangle = \langle\langle Q \cup QS\rangle^2\rangle \subseteq \langle QS\rangle$ and therefore $Q \subseteq \langle SQ\rangle \cap \langle QS\rangle \subseteq Q$.

(1) $\Rightarrow$ (2): The inclusion $RL \subseteq \langle RL\rangle \subseteq L \cap R$ holds by Lemma 1.2 c). On the other hand, for each $d \in L \cap R$ there exists an $x \in S$ such that $d = dxd$ since S is regular. Now $d \in R$ and $xd \in SL \subseteq L$ imply $d \in RL$.

(2) $\Rightarrow$ (3): For a), let L be a left ideal of S and $(L)_r = \langle L \cup LS\rangle$ the right (in fact two-sided) ideal of S generated by L. Then (2) implies

$$L = L \cap (L)_r = \langle (L)_r L \rangle = \langle \langle L \cup LS \rangle L \rangle \subseteq \langle LL \cup LSL \rangle = \langle LL \rangle \subseteq L.$$

The statement b) can be proved dually and $\langle RL \rangle = L \cap R$ is a quasi-ideal by Lemma 1.2 c).

(3) $\Rightarrow$ (4): At first the multiplication (1.3) is associative by (1.6). Let Q_1 and Q_2 be quasi-ideals of S. Then $L = \langle SQ_1Q_2 \rangle$ and $R = \langle Q_1Q_2S \rangle$ are left and right ideals of S, hence 3a) and 3b) imply $S = \langle S^2 \rangle$ and

$$\langle SQ_1Q_2 \rangle = \langle \langle SQ_1Q_2 \rangle \langle SSQ_1Q_2 \rangle \rangle = \langle S \langle Q_1Q_2S \rangle \langle SQ_1Q_2 \rangle \rangle \tag{2.3}$$

$$\langle Q_1Q_2S \rangle = \langle \langle Q_1Q_2SS \rangle \langle Q_1Q_2S \rangle \rangle = \langle \langle Q_1Q_2S \rangle \langle SQ_1Q_2 \rangle S \rangle. \tag{2.4}$$

Moreover, $\langle RL \rangle$ is a quasi-ideal of S by 3c) and satisfies $\langle SRL \rangle \cap \langle RLS \rangle = \langle RL \rangle$, due to the last statement of the theorem. So we obtain

$$\begin{aligned}\langle SQ_1Q_2 \rangle \cap \langle Q_1Q_2S \rangle &= \langle S \langle Q_1Q_2S \rangle \langle SQ_1Q_2 \rangle \rangle \cap \langle \langle Q_1Q_2S \rangle \langle SQ_1Q_2 \rangle S \rangle \\ &= \langle SRL \rangle \cap \langle RLS \rangle = \langle RL \rangle = \langle \langle Q_1Q_2S \rangle \langle SQ_1Q_2 \rangle \rangle \\ &\subseteq \langle Q_1Q_2SQ_2 \rangle \subseteq \langle Q_1Q_2 \rangle,\end{aligned}$$

the last step by Lemma 1.5. Hence $\langle Q_1Q_2 \rangle$ is again a quasi-ideal of S, i.e. $\mathcal{Q}$ is a semigroup and it remains to show that it is regular. Each quasi-ideal $Q \in \mathcal{Q}$ satisfies $Q = \langle SQ \rangle \cap \langle QS \rangle$, again by the last statement of the theorem, and in a similar way as above we can conclude

$$\begin{aligned}Q &= \langle SQ \rangle \cap \langle QS \rangle = \langle \langle SQ \rangle \langle SSQ \rangle \rangle \cap \langle \langle QSS \rangle \langle QS \rangle \rangle \\ &= \langle S \langle QS \rangle \langle SQ \rangle \rangle \cap \langle \langle QS \rangle \langle SQ \rangle S \rangle = \langle \langle QS \rangle \langle SQ \rangle \rangle = \langle QSQ \rangle.\end{aligned}$$

Hence Q is regular in $\mathcal{Q}$.

(4) $\Rightarrow$ (5): For each quasi-ideal Q of S there exists a quasi-ideal X of S such that $Q = \langle QXQ \rangle \subseteq \langle QSQ \rangle \subseteq \langle SQ \rangle \cap \langle QS \rangle \subseteq Q$, hence $Q = \langle QSQ \rangle$.

(5) $\Rightarrow$ (1): For each element $s \in S$ the intersection $(s)_l \cap (s)_r$ is a quasi-ideal of S by Lemma 1.2 c). Using (5) and Lemma 1.4 b) we conclude

$$\begin{aligned}s \in (s)_l \cap (s)_r &= \langle ((s)_l \cap (s)_r) S ((s)_l \cap (s)_r) \rangle \\ &\subseteq \langle (s)_r S (s)_l \rangle = \langle sS(s)_l \rangle = \langle sSs \rangle = sSs.\end{aligned}$$

Hence each $s \in S$ is regular in S.

For regular elements of a semiring, a ring or a semigroup S there is the following analogue of Theorem 2.1:

THEOREM 2.2. The following statements about an element s of a semiring, a ring or a semigroup S are equivalent:

(1) The element s is regular in S.

(2) The principle left ideal $(s)_l$ and the principle right ideal $(s)_r$ of S satisfy

$$\langle (s)_r (s)_l \rangle = (s)_l \cap (s)_r.$$

(3) The principle left ideal $(s)_l$ and the principle right ideal $(s)_r$ of S satisfy

a) $\langle (s)_l^2 \rangle = (s)_l$,

b) $\langle (s)_r^2 \rangle = (s)_r$ and

c) $\langle (s)_r (s)_l \rangle$ is a quasi-ideal of S.

(4) The principle quasi-ideal $(s)_q$ of S satisfies $(s)_q = \langle (s)_q S (s)_q \rangle$.

PROOF. (1) $\Rightarrow$ (3): For a) let $s \in S$ be regular in S. By (2.2) and (2.2') we have $(s)_l = Se$ for some $e = e^2 \in S$ and $(s)_r = fS$ for some $f = f^2 \in S$. From $(s)_l = Se \subseteq \langle SeSe \rangle = \langle (s)_l^2 \rangle$ we obtain $\langle (s)_l^2 \rangle = (s)_l$ since $(s)_l^2 \subseteq (s)_l$ is clear. The statement b) can be proved dually, and $\langle (s)_r(s)_l \rangle = \langle fSSe \rangle$ is a quasi-ideal of S by Lemma 1.3 since $\langle fSS \rangle$ is a right ideal of S.
(3) $\Rightarrow$ (2): From 3a) one obtains $(s)_l = \langle (s)_l^2 \rangle \subseteq \langle S(s)_l \rangle = Ss$ by Lemma 1.4 b), that is $(s)_l = Ss$. Similarly 3b) yields $(s)_r = sS$. Using again 3a) and 3b) one obtains $(s)_l \cap (s)_r = Ss \cap sS = \langle (Ss)^3 \rangle \cap \langle (sS)^3 \rangle$. Now $\langle (s)_r(s)_l \rangle = \langle sSSs \rangle$ is by 3c) a quasi-ideal of S. From this it follows $\langle SsSSs \rangle \cap \langle sSSsS \rangle \subseteq \langle sSSs \rangle = \langle (s)_r(s)_l \rangle$ and therefore

$$(s)_l \cap (s)_r = \langle (Ss)^3 \rangle \cap \langle (sS)^3 \rangle \subseteq \langle SsSSs \rangle \cap \langle sSSsS \rangle \subseteq \langle (s)_r(s)_l \rangle.$$

This proves (2), since the other inclusion is clear.
(2) $\Rightarrow$ (4): Obviously we have $\langle (s)_q S(s)_q \rangle \subseteq \langle S(s)_q \rangle \cap \langle (s)_q S \rangle \subseteq (s)_q$. From (2) and Lemma 1.4 b) it follows $s \in (s)_l \cap (s)_r = \langle (s)_r(s)_l \rangle \subseteq \langle S(s)_l \rangle = Ss$, hence $(s)_l = Ss$ and similarly $(s)_r = sS$. So we obtain $(s)_q \subseteq (s)_l \cap (s)_r = \langle (s)_r(s)_l \rangle = \langle sSSs \rangle \subseteq \langle (s)_q S(s)_q \rangle$.
(4) $\Rightarrow$ (1): We have $s \in (s)_q = \langle (s)_q S(s)_q \rangle \subseteq \langle (s)_r S(s)_l \rangle = \langle sS(s)_l \rangle = sSs$, the last steps again by Lemma 1.4 b), so s is regular in S.

By the aid of Thm. 2.1, we obtain further properties of quasi-ideals in regular semirings, rings and semigroups. In this context we need the following

LEMMA 2.3. Each two-sided ideal T of a regular semiring, ring or semigroup S is a regular subsemiring, subring or subsemigroup of S.

PROOF. Each element $s \in T \subseteq S$ is regular in S, so there is an $x \in S$ such that $s = sxs = sx(sxs) = s(xsx)s$. Since xsx is an element of T, s is regular in T, too.

THEOREM 2.4. Let S be a regular semiring, ring or semigroup. Then the following statements are true:

a) Each quasi-ideal Q of S satisfies

$$Q = L \cap R = \langle RL \rangle \quad \text{with } L = (Q)_l = \langle SQ \rangle \text{ and } R = (Q)_r = \langle QS \rangle.$$

b) Each quasi-ideal Q of S satisfies $\langle Q^2 \rangle = \langle Q^3 \rangle$.

c) Each bi-ideal B of S is a quasi-ideal of S.

d) Each bi-ideal B of a two-sided ideal T of S is a quasi-ideal of S.

With respect to b) we note, that the regularity of S does not imply $Q = \langle Q^2 \rangle$. Moreover, the statement c) does not imply that S is regular.

PROOF. a) By Theorem 2.1 each quasi-ideal Q of S has the intersection property $Q = \langle SQ \rangle \cap \langle QS \rangle = (Q)_l \cap (Q)_r = L \cap R$ and condition (2) of Theorem 2.1 implies $Q = L \cap R = \langle RL \rangle$.
b) For each quasi-ideal Q of S it follows by Theorem 2.1 (4), that $\langle Q^2 \rangle$ is also a quasi-ideal of S and that there is a quasi-ideal X of S such that $\langle Q^2 \rangle = \langle Q^2 X Q^2 \rangle \subseteq \langle Q^2 S Q^2 \rangle = \langle Q(QSQ)Q \rangle \subseteq \langle Q^3 \rangle$, which yields $\langle Q^2 \rangle = \langle Q^3 \rangle$ since $\langle Q^3 \rangle \subseteq \langle Q^2 \rangle$ is clear.
c) Since $\langle SB \rangle$ is a left ideal and $\langle BS \rangle$ is a right ideal of S, Theorem 2.1 (2) implies $\langle SB \rangle \cap \langle BS \rangle = \langle BSSB \rangle \subseteq \langle BSB \rangle \subseteq B$, so B is a quasi-ideal of S.
d) By Lemma 2.3 the two-sided ideal T of S is a regular sub(semi)ring or subsemigroup of S. So by part c) the bi-ideal B of T is a quasi-ideal of T, hence by Lemma 1.5 a bi-ideal of S and again by part c) a quasi-ideal of S.
The last statements are proved by the following examples:

EXAMPLE 2.5. Let $S = M_{2,2}(K)$ be the set of the (2,2)-matrices over $K = \mathbb{Z}/(2)$. With the usual multiplication (and addition) we consider S as a semigroup, as a semiring and as a ring. Then, in all three cases, S is regular and

$$Q = \left\{ O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

is a quasi-ideal of S (in fact a canonical and hence an O-minimal one, cf. Section 3). Obviously, one has $Q \neq \langle Q^2 \rangle = \{O\}$.

EXAMPLE 2.6. In the semiring $S = \mathbb{N}$ with the usual (commutative) operations each subsemiring of S is an ideal and hence a quasi-ideal of S. Thus the same holds for each bi-ideal of S. However, only the element 1 is regular in S. The corresponding statements hold for the semigroup $S = \mathbb{N}$ with respect to multiplication and for the ring $S = \mathbb{Z}$ of integers.

Whereas the regularity of S does not imply that $\langle Q^2 \rangle = Q$ holds for each quasi-ideal Q of S as just stated, the converse implication is true. Hence the next theorem gives equivalent characterizations for special classes of regular semirings, rings or semigroups:

THEOREM 2.7. The following conditions on a semiring, ring or semigroup S are equivalent:
(1) Each left ideal L and each right ideal R of S satisfy

$$\langle RL \rangle = L \cap R \subseteq \langle LR \rangle.$$

(2) The set $\mathcal{Q}$ of all quasi-ideals of S is an idempotent semigroup with respect to the "product" $\langle Q_1 Q_2 \rangle$.
(3) Each quasi-ideal Q of S satisfies $Q = \langle Q^2 \rangle$.

PROOF. To prove (1) $\Rightarrow$ (2) we state at first, that the equality of condition (1) is just the condition (2) of Thm. 2.1, which yields that S is regular and that the set $\mathcal{Q}$ of all quasi-ideals of S is a regular semigroup with respect to the product $\langle Q_1 Q_2 \rangle$. So we only have to show that it is in fact idempotent. We have $Q = \langle QSQ \rangle$ for each quasi-ideal Q of S by condition (5) of Thm. 2.1 and of course $\langle S^2 \rangle = S$ for the left ideal S of S by condition (3) of Thm. 2.1. Combining this we can conclude $Q = \langle QSQ \rangle = \langle (QSQ)(SS)(QSQ) \rangle = \langle QS\langle QSSQ \rangle SQ \rangle \subseteq \langle QS\langle SQQS \rangle SQ \rangle = \langle (QSQ)(QSQ) \rangle = \langle Q^2 \rangle$, where the inclusion follows from (1) since $\langle SQ \rangle$ and $\langle QS \rangle$ are left and right ideals of S, respectively. This yields $Q = \langle Q^2 \rangle$ since the other inclusion is clear, hence we proved (2). The implication (2) $\Rightarrow$ (3) is only a restriction. To prove (3) $\Rightarrow$ (1) we use that by Lemma 1.2 c) for each left ideal L and each right ideal R of S the inclusion $\langle RL \rangle \subseteq L \cap R$ holds and the intersection $L \cap R$ is a quasi-ideal of S. So (3) implies $L \cap R = \langle (L \cap R)^2 \rangle$ which yields $L \cap R \subseteq \langle RL \rangle$ as well as $L \cap R \subseteq \langle LR \rangle$.

3. (0-)MINIMAL AND CANONICAL QUASI-IDEALS

Whereas a ring always has an absorbing element O (namely its zero), the considered semigroups and semirings may have such an element or not. In particular, the absorbing element O of a semiring S need not be a zero of S, but from $O + O = O$ it follows that $\{O\}$ is a two-sided ideal of S (and hence a quasi-ideal of S, cf. Lemma 1.2 a)) also in this case.
For a semiring, a ring or a semigroup S and each $\emptyset \neq X \subseteq S$ we introduce

$$X' = \begin{cases} X \backslash \{O\} & \text{if } S \text{ has an absorbing element } O \\ X & \text{otherwise} \end{cases}$$

Let Q be a quasi-ideal of a semiring, a ring or a semigroup S. Then Q is called a minimal quasi-ideal of S, iff $S' = S$ holds and Q does not contain a quasi-ideal Q_1 of S properly. Moreover, Q is called $O-$ minimal , iff $S' \subset S$ and $\{O\} \subset Q$ are satisfied and Q does not contain a quasi-ideal $Q_1 \neq \{O\}$ of S properly. For convenience, we use (O-)minimal when dealing with both cases simultaneously. Analogously one defines (O-)minimal left-, right- and two-sided ideals of S. Note that according to this definition a ring S can only have O-minimal quasi-ideals and no minimal ones, which deviates again from the usual ring-theoretical terminology.

A quasi-ideal $Q \neq \{O\}$ of a semiring, a ring or a semigroup S is called canonical iff

(3.1) $Q = L \cap R$ holds with suitable left and right ideals L and R of S and

(3.2) L and R can be chosen (O-)minimal.

We note that (3.1) implies $Q = (Q)_l \cap (Q)_r$ and (3.2) means, that $(Q)_l$ and $(Q)_r$ are (O-)minimal (which conversely implies that Q is canonical).

In the following two theorems we collect known results concerning the relation between the properties that a quasi-ideal Q is (O-)minimal or canonical. At first we assume that Q is canonical and ask if it is also (O-)minimal:

THEOREM 3.1. Let S be a semiring, a ring or a semigroup. Then, for each (O-)minimal left ideal L and (O-)minimal right ideal R the intersection $Q = L \cap R$ is an (O-)minimal quasi-ideal of S or satisfies $Q = \{O\}$.

Consequently, each canonical quasi-ideal Q of S is an (O-)minimal one.

This has been proved for semirings in [8], Prop. 4.3, for semigroups without an absorbing element in [3], Thm. 5.1 and for semigroups with such an element O and for rings in [3], Thm. 6.1.

Secondly, we assume that a quasi-ideal Q is (O-)minimal and ask whether it is also canonical or satisfies at least (3.1). Here we have to distinguish several cases:

THEOREM 3.2. a) Let S be a semiring or a semigroup without an absorbing element. Then each minimal quasi-ideal is also a canonical quasi-ideal of S.

b) Let S be a semigroup with absorbing element O and Q an O-minimal quasi-ideal of S. Then Q has at least the intersection property (3.1).

c) Let S be a semiring or a semigroup with absorbing element O or a ring and Q an O-minimal quasi-ideal of S. Then either Q^2 equals $\{O\}$ or $(Q', \cdot)$ is a subgroup of $(S, \cdot)$. In the latter case, Q satisfies (3.1) since $Sx \cap xS = Q$ holds for all $x \in Q'$. In particular $Q = Se \cap eS = eSe$ holds with the identity e of $(Q', \cdot)$. However, Q need not be canonical, since there are examples such that even the left ideal Se and the right ideal eS of S generated by $e \in Q$ are not O-minimal.

We note that statement a) is proved for semirings in [8], Satz 4.7 and for semigroups in [3], Thm. 5.1. Statement b) is part of Prop. 2.6 in [3]. The positive statements of c) can be obtained for semirings by Satz 4.6 and Satz 4.5 of [8] and for semigroups and rings by Thm. 6.3, Cor.6.4 and Thm. 6.5 of [3]. Finally, Example 2.2 in [1] deals with the semigroup $S = \{O, e, a, b\}$ for which all products are O excepted $e^2 = e$, $ea = a$ and $be = b$. It contains the O-minimal quasi-ideal $Q = eSe = \{O, e\}$, but neither the left ideal Se nor the right ideal eS are O-minimal. We show that examples of this kind yield corresponding ones for rings and semirings:

EXAMPLE 3.3. Let S be a semigroup with above properties, satisfying in particular that the O-minimal quasi-ideal Q consists of the two elements $e^2 = e$ and O. Let A be the algebra over a field F with O as its zero and the elements of $S' = S \backslash \{O\}$ as a basis. Then $eAe = Fe$ is obviously an O-minimal quasi-ideal of A, but neither the left ideal $Ae \supseteq Se$ nor the right ideal $eA \supseteq eS$ are

O-minimal. If one replaces F by a commutative semifield H with an absorbing zero O (e.g. by the semifield of non-negative rational numbers), the same holds for the resulting semiring A, the semialgebra A over H with S' as a basis, cf. [10], Def. 4.1.

In view of Theorem 3.2 one may ask for conditions on S which are necessary and sufficient such that each O-minimal quasi-ideal of S is also a canonical one. To our knowledge, no conditions of this kind are known. In the following we obtain complete answers in the cases that S is semiprime or quasi-reflexive (cf. Thm. 3.4 and Thm. 3.8). Both concepts are well known for a semigroup S with an absorbing element O and for a ring S. For a semiring S with an absorbing element O we define them in the same way, although O need not be the zero of S.

Let S be a semiring or semigroup with absorbing element O or a ring. Then S is called semiprime iff one of the following three conditions (which are easily checked to be equivalent) is satisfied:

(i) if T is a two-sided ideal of S such that $\langle T^2\rangle \subseteq \{O\}$, then $T \subseteq \{O\}$;

(ii) if L is a left ideal of S such that $\langle L^2\rangle \subseteq \{O\}$, then $L \subseteq \{O\}$;

(iii) if R is a right ideal of S such that $\langle R^2\rangle \subseteq \{O\}$, then $R \subseteq \{O\}$.

We recall in this context that $\{O\} = (O)_t$ is a two-sided ideal of S in each of the three cases we consider, which yields that the brackets $\langle\ \rangle$ in (i), (ii) and (iii) are superfluous.

In the case of semiprime rings and semiprime semigroups, the following theorem is essentially Thm. 7.2 in [3].

THEOREM 3.4. Let S be a semiring or semigroup with absorbing element O or a ring such that S is semiprime. Then each O-minimal quasi-ideal Q of S is also a canonical quasi-ideal of S.

PROOF. At first we state, that $\langle SQ\rangle \neq \{O\}$ holds since otherwise $Q \neq \{O\}$ would be a left ideal of S and satisfy $Q^2 = \{O\}$, contradicting that S is semiprime.
The quasi-ideal $\{O\} \subseteq \langle SQ\rangle \cap \langle QS\rangle \subseteq Q$ satisfies either

$$\langle SQ\rangle \cap \langle QS\rangle = \{O\} \quad \text{or} \quad \langle SQ\rangle \cap \langle QS\rangle = Q$$

since Q is O-minimal. To show $\langle SQ\rangle \cap \langle QS\rangle = Q$ we go by contradiction and assume $\langle SQ\rangle \cap \langle QS\rangle = \{O\}$. Then $\langle QSQ\rangle \subseteq \langle SQ\rangle \cap \langle QS\rangle = \{O\}$ implies $\langle\langle SQ\rangle^2\rangle = \langle SQSQ\rangle = \{O\}$, which together with $\langle SQ\rangle \neq \{O\}$ contradicts that S is semiprime. So it remains to show that the one-sided ideals $\langle SQ\rangle$ and $\langle QS\rangle$ of S are O-minimal. Assume $\{O\} \subset L \subseteq \langle SQ\rangle$ for a left ideal L of S. Then the quasi-ideal $\langle SL\rangle \cap \langle QS\rangle$ satisfies either

$$\langle SL\rangle \cap \langle QS\rangle = \{O\} \quad \text{or} \quad \langle SL\rangle \cap \langle QS\rangle = Q,$$

since $\{O\} \subseteq \langle SL\rangle \cap \langle QS\rangle \subseteq \langle SQ\rangle \cap \langle QS\rangle \subseteq Q$ holds and Q is assumed to be O-minimal. Now $\langle QL\rangle \subseteq \langle SL\rangle \cap \langle QS\rangle = \{O\}$ would imply $\langle L^2\rangle \subseteq \langle\langle SQ\rangle L\rangle = \langle SQL\rangle = \{O\}$, but S is assumed to be semiprime. So we obtain $\langle SL\rangle \cap \langle QS\rangle = Q$ which implies $Q \subseteq \langle SL\rangle \subseteq L$ and in turn $\langle SQ\rangle \subseteq \langle SL\rangle \subseteq L$, hence $L = \langle SQ\rangle$, so $\langle SQ\rangle$ is O-minimal. Dually, the right ideal $\langle QS\rangle$ of S is also O-minimal.

Let S be a semiring or semigroup with absorbing element O or a ring. Then S is called quasi-reflexive iff all two-sided ideals A,B of S satisfy the implication $\langle AB\rangle \subseteq \{O\} \Rightarrow \langle BA\rangle \subseteq \{O\}$ or equivalently $AB \subseteq \{O\} \Rightarrow BA \subseteq \{O\}$. Obviously each semiprime semiring, semigroup or ring is also quasi-reflexive, but not conversely.

For rings and semigroups the following lemma is a special case of Prop. 2.1 and Cor. 2.3 in [7].

LEMMA 3.5. Let S be a semiring or a semigroup with absorbing element O or a ring. Then the following conditions are equivalent:

a) All two-sided ideals A, B of S satisfy $AB \subseteq \{O\} \Rightarrow BA \subseteq \{O\}$.

b) All left ideals A, B of S satisfy $AB \subseteq \{O\} \Rightarrow BA \subseteq \{O\}$.

c) All right ideals A, B of S satisfy $AB \subseteq \{O\} \Rightarrow BA \subseteq \{O\}$.

PROOF. The implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) are obvious. For the proof of (a) $\Rightarrow$ (b), assume $AB \subseteq \{O\}$ for any left ideals A, B of S. Then $\langle A \cup AS\rangle$ and $\langle B \cup BS\rangle$ are two-sided ideals of S satisfying

$$\langle A \cup AS\rangle\langle B \cup BS\rangle \subseteq \langle AB \cup ABS \cup ASB \cup ASBS\rangle \subseteq \{O\}$$

and hence by (a) also $\langle B \cup BS\rangle\langle A \cup AS\rangle \subseteq \{O\}$, which clearly implies $BA \subseteq \{O\}$. The implication (a) $\Rightarrow$ (c) is the dual of (a) $\Rightarrow$ (b).

LEMMA 3.6. Let L be an (O-)minimal left ideal of a semiring, a ring or a semigroup S and $O \neq e = e^2 \in L$. Then $((eL)', \cdot)$ is a subgroup of $(S, \cdot)$.

PROOF. Clearly, $(eL, \cdot)$ is a subsemigroup of $(S, \cdot)$ with e as left identity. In fact even $((eL)', \cdot)$ is a subsemigroup of $(S, \cdot)$, since the assumption $ab = O$ for any $a, b \in (eL)'$ (together with $(a)_l = \langle a \cup Sa\rangle = L$ by the (O-)minimality of L) would imply $Lb = \langle a \cup Sa\rangle b = \{O\}$ in contradiction to $O \neq b = eb \in eL$. To show that $((eL)', \cdot)$ is a group consider any element $el \in (eL)'$. Then $O \neq el = eel \in Lel$ and $\langle S(Lel)\rangle = \langle (SL)el\rangle \subseteq \langle Lel\rangle = Lel$ shows that $\{O\} \neq Lel$ is a left ideal of S which is clearly contained in L. This yields $Lel = L$ since L is (O-)minimal. Now we conclude $eLel = eL$ and so there exists an element $x = ek \in eL$ such that $xel = ekel = ee = e$ which shows that $((eL)', \cdot)$ is a group since $x = O$ is of course impossible.

For quasi-reflexive rings and semigroups the following proposition and theorem correspond to Props. 4.1 and 4.3 in [7]. They contain Thm. 7.4 of [3] for semiprime rings and semigroups as special cases.

THEOREM 3.7. Let S be a semiring or semigroup with absorbing element O or a ring such that S is quasi-reflexive. Then for each element $e = e^2 \neq O$ of S the following statements are equivalent:

a) Se is an O-minimal left ideal of S.

a$'$) eS is an O-minimal right ideal of S.

b) eSe is an O-minimal quasi-ideal of S.

c) $((eSe)', \cdot)$ is a subgroup of $(S, \cdot)$.

PROOF. The equivalence (b) $\Leftrightarrow$ (c) is known for arbitrary semirings ([8], Satz 4.5), arbitrary rings ([3], Prop. 6.11) and arbitrary semigroups ([3], Prop. 5.6). From Lemma 3.6 and its dual we obtain (a) $\Rightarrow$ (c) and (a$'$) $\Rightarrow$ (c). For (b) $\Rightarrow$ (a) let $\{O\} \neq L \subseteq Se$ be a left ideal of S. Then $Le = L$ holds. Furthermore, eL is a quasi-ideal of S by Lemma 1.3 and we show $eL \neq \{O\}$ by contradiction. Indeed, $eL = \{O\}$ would imply $SeL = \{O\}$ and hence $LSe = \{O\}$ since S is assumed to be quasi-reflexive. The latter would imply $Le = \{O\}$, contradicting $Le = L \neq \{O\}$. So we have $\{O\} \neq eL = eLe \subseteq eSe$, hence $eL = eSe$ since eSe is assumed to be (O-)minimal. Of course $O \neq e$ is cancellable in the group $((eSe)', \cdot)$, so we conclude $L = Se$, hence the left ideal Se is (O-)minimal. Similarly one shows (b) $\Rightarrow$ (a').

THEOREM 3.8. Let S be a semiring or semigroup with absorbing element O or a ring such that S is quasi-reflexive. Then each O-minimal quasi-ideal Q of S satisfies either $Q^2 = \{O\}$ or it is a canonical quasi-ideal of S.

PROOF. Let Q be an O-minimal quasi-ideal of S satisfying $Q^2 \neq \{O\}$. Then by Theorem 3.2 c) there exists an element $O \neq e = e^2 \in S$ satisfying $Q = Se \cap eS = eSe$. Since S is quasi-reflexive and $Q = eSe$ is assumed to be O-minimal, by Theorem 3.7 the one-sided ideals Se and eS of S are also O-minimal, hence Q is canonical.

Finally, we give some statements on canonical quasi-ideals for arbitrary semirings, semigroups and rings. In the last both cases the following theorem and lemma correspond to Thm. 6.7 a), Thm. 6.7 b) and Prop. 6.9 in [3].

THEOREM 3.9. Let S be a semiring, a ring or a semigroup and let L and R be $(O$-$)$minimal left and right ideals of S. Then $\langle RL\rangle$ is either $\{O\}$ or a canonical quasi-ideal of S satisfying $\langle RL\rangle = L \cap R$.

PROOF. We assume $\langle RL\rangle \neq \{O\}$, which clearly holds if S has no absorbing element, and the same applies to statements like "$\langle SRL\rangle \neq \{O\}$" or "$O \neq x$" appearing in the following. By Lemma 1.2 c) and Theorem 3.1, we have $\langle RL\rangle \subseteq L \cap R = Q$ where Q is a canonical quasi-ideal, and hence an $(O$-$)$minimal quasi-ideal of S. So we have to show that $\langle RL\rangle$ is a quasi-ideal of S, that is $\langle SRL\rangle \cap \langle RLS\rangle \subseteq \langle RL\rangle$. This is trivial if $\langle SRL\rangle = \{O\}$ or $\langle RLS\rangle = \{O\}$, so let us assume $\langle SRL\rangle \neq \{O\}$ and $\langle RLS\rangle \neq \{O\}$. By the $(O$-$)$minimality of L and R we get $\langle SRL\rangle = L$ and $\langle RLS\rangle = R$. Clearly, $\langle SRL\rangle \neq \{O\}$ implies the existence of $O \neq x \in RL \subseteq L \cap R$ with $Sx \neq \{O\}$. Since L is $(O$-$)$minimal, we get $Sx = \langle Sx\rangle = L$ and $SxS = LS$. Now $R = \langle RLS\rangle \neq \{O\}$ implies $\{O\} \neq LS = SxS$, thus $\{O\} \neq xS = \langle xS\rangle \subseteq R$. Since R is $(O$-$)$minimal, this yields $xS = R$. Hence we can conclude $O \neq x \in RL = xSSx \subseteq xSx$, thus x is regular in S. So we can apply Thm. 2.2 to $(x)_l = L$ and $(x)_r = R$ and we obtain $\langle RL\rangle = \langle (x)_r(x)_l\rangle = (x)_l \cap (x)_r = Sx \cap xS = L \cap R$, which proves that $\langle RL\rangle$ is a quasi-ideal of S also in this case.

LEMMA 3.10. Let S be a semiring, a ring or a semigroup and let L be an $(O$-$)$minimal left ideal of S and $a \in S$. Then La is an $(O$-$)$minimal left ideal of S or $La = \{O\}$ holds.

PROOF. For each left ideal L of S and $a \in S$ clearly $La = \langle La\rangle$ is a left ideal of S. Now assume $\{O\} \subseteq A \subseteq La$ for a left ideal A of S and define B by $\{b \in L \mid ba \in A\}$. Then $A = Ba$ holds and B is obviously a left ideal of S. From $\{O\} \subseteq B \subseteq L$ we obtain either $B = \{O\}$ or $B = L$, i.e. $A = \{O\}$ or $A = La$. Hence La is an $(O$-$)$minimal left ideal of S.

Now we prove two theorems which contain Thms. 2.2 and 2.8 in [4] for rings and Thms. 2.3 and 2.9 in [5] for semigroups.

THEOREM 3.11. Let S be a semiring, a ring or a semigroup and Q a canoni-cal quasi-ideal of S. Hence there is a (unique) $(O$-$)$minimal left ideal L and a (unique) $(O$-$)$minimal right ideal R of S satisfying $Q = L \cap R$. Then for each $a \in S$ either $Qa = \langle Qa\rangle$ is a canonical quasi-ideal of S satisfying $Qa = La \cap R$ or $Qa = \{O\}$ holds.

PROOF. We assume $Qa \neq \{O\}$. Then obviously $Qa \subseteq La \cap Ra \subseteq La \cap R = K$ holds. Since La is an $(O$-$)$minimal left ideal of S by Lemma 3.10, K is a canonical quasi-ideal of S. We are now going to show $La \cap R = K \subseteq Qa$. Consider some $k \neq O$ of K. Then $k = ba$ for some $b \in L$ yields $R = (k)_r \subseteq (b)_r$. Since Q is contained in the (O-)minimal left ideal L, each $q \neq O$ of Q satisfies $Sq = L$ or $Sq = \{O\}$. In the last case, $L = (q)_l \subseteq Q$ implies $L = Q$, which makes $K = La \cap R \subseteq Qa$ trivial. Otherwise, there is at least one $q \neq O$ of Q such that $Sq = L$ holds, which yields $sq = b$ for the b considered above and some $s \in S$. This and $q \in Q \subseteq R$ imply $R \subseteq (b)_r = (sq)_r \subseteq sR$. By the dual of Lemma 3.10, sR is an (O-)minimal right ideal of S which yields $R = sR$ and hence $b \in R$. So we have obtained $b \in L \cap R = Q$ and thus $k = ba \in Qa$, i.e. again $K = La \cap R \subseteq Qa$. From $Qa \subseteq K$ as stated above we obtain $Qa = La \cap R$.

THEOREM 3.12. Let S be a semiring, a ring or a semigroup and let Q_1, Q_2 be canonical quasi-ideals of S. Then $\langle Q_1Q_2\rangle$ is a canonical quasi-ideal of S or $\langle Q_1Q_2\rangle = \{O\}$ holds.

PROOF. Since Q_1, Q_2 are canonical quasi-ideals of S, there are (O-)minimal left ideals L_1, L_2 and (O-)minimal right ideals R_1, R_2 of S such that $Q_1 = L_1 \cap R_1$ and $Q_2 = L_2 \cap R_2$. Let us assume $\langle Q_1Q_2\rangle \neq \{O\}$. Then

$$\langle Q_1Q_2\rangle = \langle (L_1 \cap R_1)(L_2 \cap R_2)\rangle \subseteq \langle R_1L_2\rangle \subseteq L_2 \cap R_1.$$

Because of $\langle Q_1Q_2\rangle \neq \{O\}$ there is a $O \neq q_2 \in Q_2$ with $Q_1q_2 \neq \{O\}$. Then Theorem 3.11 implies $Q_1q_2 = L_1q_2 \cap R_1$. From $\{O\} \neq Q_1q_2 \subseteq L_1q_2 \subseteq L_1L_2 \subseteq L_2$ we get $L_1q_2 = L_2$ since L_2 is (O-)minimal, so we have $Q_1q_2 = L_2 \cap R_1$. Now $L_2 \cap R_1 = Q_1q_2 \subseteq Q_1Q_2 \subseteq \langle Q_1Q_2\rangle \subseteq L_2 \cap R_1$ implies $\langle Q_1Q_2\rangle = L_2 \cap R_1$, hence $\langle Q_1Q_2\rangle$ is a canonical quasi-ideal of S.

The last corollary is an immediate consequence of (1.6) and Theorem 3.12.

COROLLARY 3.13. Let S be a semiring, a ring or a semigroup which contains at least one canonical quasi-ideal and let $\mathcal{V}$ denote the set of all canonical quasi-ideals of S. Then $\mathcal{V}$ or $\mathcal{V} \cup \{O\}$ is a semigroup with respect to the "product" $\langle Q_1Q_2\rangle$, which has $\{O\}$ as absorbing element in the latter case.

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