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SOME RESULTS ON π-SOLVABLE AND SUPERSOLVABLE GROUPS

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ABSTRACT. For a finite group G, $\phi_p(G)$, $S_p(G)$, L(G) and $S_{\wp}(G)$ are generalizations of the Frattini subgroup of G. We obtain some results on π-solvable, p-solvable and supersolvable groups with the help of the structures of these subgroups.

KEY WORDS AND PHRASES. p-solvable, π-solvable, supersolvable.
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1. INTRODUCTION.

Many authors have considered various generalizations of the Frattini subgroup of a finite group. Deskins [6] considered the subgroup $\phi_p(G)$, Mukherjee and Bhattacharya [4] the subgroup $S_p(G)$ and Bhatia [3] the subgroup L(G). In [7], we introduced the subgroup $S_{\wp}(G)$ and investigated its influence on solvable group. In this paper, our aim is to prove some results which imply a finite group G to be π-solvable, p-solvable and supersolvable. All groups are assumed to be finite. We use standard notations as found in Gorenstein [8] and denote a maximal subgroup M of G by $M \lessdot G$.

2. PRELIMINARIES.

DEFINITION. Let H and K be two normal subgroups of a group G with $K \subset H$. Then the factor group H/K is called a chief factor of G if there is no nórmal subgroup N of G such that $K \subset N \subset H$, with proper inclusion. Let M be a maximal subgroup of G. Then H is said to be a normal supplement of M in G if MH = G. The normal index of M in G is defined as the order of a chief factor H/K, where H is minimal in the set of all normal supplements of M in G and is denoted by $\eta(G : M)$.

(2.1) (Deskins [6,(2.1)], Beidleman and Spencer [2, Lemma-1])

If M is a maximal subgroup of a group G then $\eta(G : M)$ is uniquely determined.

(2.2) (Beidleman and Spencer [2, Lemma-2])

If N is a normal subgroup of a group G and M is a maximal subgroup of G such that $N \subseteq M$ then $\eta(G/N : M/N) = \eta(G : M)$

(2.3) (Mukherjee [9, Theorem-1])

If M is a maximal subgroup of a group G and $M \triangleleft G$ then $\eta(G:M)=[G:M]$=a prime.

(2.4) (Baer [1, Lemma-3])

If the group G possesses a maximal subgroup with core 1 then the following properties of G are equivalent.

(1) The indices in G of all the maximal subgroups with core 1 are powers of one and the same prime p.

(2) There exists one and only one minimal normal subgroup of G and there exists a common prime divisor of all the indices in G of all the maximal subgroups with core 1.

(3) There exists a non-trivial solvable normal subgroup of G.

DEFINITION. Let G be a group and p be any prime. The four characteristic subgroups of G, which are analogous to the Frattini subgroup $\phi(G)$, are defined as follows :

$$S_p(G) = \bigcap\{M : M \in \Sigma_p(G)\}$$
$$\phi_p(G) = \bigcap\{M : M \in \Upsilon_p(G)\}$$
$$L(G) = \bigcap\{M : M \in \Lambda(G)\}$$
$$S_{\mathcal{P}}(G) = \bigcap\{M : M \in \Sigma_{\mathcal{P}}(G)\}$$

where

$$\Sigma_p(G) = \{M : M \lessdot G,\ [G:M]_p = 1 \text{ and } [G:M] \text{ is composite}\}$$
$$\Upsilon_p(G) = \{M : M \lessdot G,\ [G:M]_p = 1\}$$
$$\Lambda(G) = \{M : M \lessdot G,\ [G:M] \text{ is composite}\}$$
$$\Sigma_{\mathcal{P}}(G) = \{M : M \lessdot G,\ \eta(G:M)_p = 1 \text{ and } \eta(G:M) \text{ is composite}\}$$

In case $\Sigma_{\mathcal{P}}(G)$ is empty then we define $G = S_{\mathcal{P}}(G)$ and the same thing is done for the other three subgroups.

(2.5) If H is a subgroup with finite index n in a group G then $core_G H$ has finite index dividing n!

(2.6) (Dutta and Bhattacharyya [7, Theorem-3.5])

If G is p-solvable then $S_{\mathcal{P}}(G)$ is solvable.

DEFINITION. Let M be a maximal subgroup of a group G. Then M is said to be c-maximal if [G:M] is composite.

3. SOME RESULTS ON p-SOLVABLE AND π-SOLVABLE GROUPS.

THEOREM 3.1. Let p be the largest prime dividing $|G|$ and $\Sigma_p(G) \neq \emptyset$. Then G is p-solvable if and only if $\eta(G:M)_p = [G:M]_p$ for each M in $\Sigma_p(G)$.

PROOF. Let G satisfy the hypothesis of the theorem. Then G is not simple. For, otherwise $|G|_p = \eta(G:M)_p = [G:M]_p = 1$, where M belongs to $\Sigma_p(G)$, which contradicts the fact that p divides $|G|$. Let N be a minimal normal subgroup of G. If p does not divide $|G/N|$ then G/N is a p'-group and hence it is p-solvable. If p divides $|G/N|$ then p is the largest prime dividing $|G/N|$. If $\Sigma_p(G/N) = \emptyset$ then $G/N = S_p(G/N)$. By Theorem-8(i) [10], $S_p(G/N)$ is solvable and hence G/N is p-solvable. We now assume that $\Sigma_p(G/N) \neq \emptyset$. By Lemma-2 [2], we obtain $\eta(G/N:M/N)_p = [G/N : M/N]_p$ for each M/N in $\Sigma_p(G/N)$. So by induction, G/N is p-solvable. We note that $S_p(G) \neq G$, since $\Sigma_p(G) \neq \emptyset$. If $N \subsetneq S_p(G)$ then N is solvable and so it is p-solvable and consequently G is p-solvable. If $N \not\subseteq S_p(G)$ then there exists M in $\Sigma_p(G)$ such that $N \not\subseteq M$ and so G = MN. By hypothesis $|N|_p = \eta(G:M)_p = [G:M]_p = 1$ and so N is p-solvable and hence G is p-solvable.

The converse follows directly from Theorem 1 [2].

THEOREM 3.2. Let p be the largest prime dividing $|G|$. Then G is p-solvable if the following hold.

(i) G has a p-solvable c-maximal subgroup M with $\eta(G:M)_p = [G:M]_p$

(ii) If M_1 and M_2 are c-maximal subgroups of G with $\eta(G:M_1)_p = \eta(G:M_2)_p$ then $[G:M_1]_p = [G:M_2]_p$

REMARK 3.3. The converse of the above theorem is not necessarily true. Let G be a p-group, where p is any prime. Then G is p-solvable, but it has no c-maximal subgroup and so G does not satisfy the hypothesis (i) of the above theorem. If the group G has a c-maximal subgroup then the converse of Theorem 3.2 follows from Theorem 1 [2].

THEOREM 3.4. Let G be a p-solvable group and $\Sigma_{\mathcal{P}}(G) \neq \emptyset$. Then G is π-solvable if and only if $\eta(G:M)_\pi = [G:M]_\pi$ for each M in $\Sigma_{\mathcal{P}}(G)$.

PROOF. Let the condition of the theorem hold. Let G be simple. Then it immediately follows that either G is a p'-group or is of prime order p. If G is of prime order p then it is solvable and hence π-solvable. If G is a p'-group then $|G|_p = 1$. Also $|G|$ is composite. For, otherwise, G is cyclic and hence it is π-solvable. Let $|G|_\pi \neq 1$ and $p_1, p_2, \ldots p_n$ be the set of prime divisors of $|G|$, which belong to π. Let $S(p_i)$ (i = 1,2, ..., n) denote the Sylow p_i-subgroup of G. Then $S(p_i) \neq G$ for i = 1,2,...,n. For, otherwise, G is solvable and hence G is π-solvable. Let M_i be the maximal subgroups of G such that $S(p_i) \subsetneq M_i \subset G$ and so $[G:M_i]_{p_i} = 1$ (i = 1,2,...,n). By hypothesis $|G|_\pi = \eta(G:M_i)_\pi = [G:M_i]_\pi$ (i=1,2,...,n). As each $p_i \in \pi$, it follows that $|G|_\pi = 1$, a contradiction. So $|G|_\pi = 1$ and hence G is π-solvable. We now suppose that G is not simple. Let N be a minimal normal subgroup of G. Then G/N is a p-solvable group. If $\Sigma_{\mathcal{P}}(G/N) = \emptyset$ then $G/N = S_{\mathcal{P}}(G/N)$ and so by (2.6), it follows that G/N is solvable and hence it is π-solvable. We now assume that $\Sigma_{\mathcal{P}}(G/N) \neq \emptyset$. Using Lemma 2 [2], we obtain $\eta(G/N : M/N)_\pi = [G/N : M/N]_\pi$ for each M/N in $\Sigma_{\mathcal{P}}(G/N)$. By induction, G/N is π-solvable. Let N_1 be another minimal normal subgroup of G. Then G/N_1 is π-solvable. Since $G = G/N \cap N_1$ is isomorphic to a subgroup of the π-solvable group $G/N \times G/N_1$, it follows that G is π-solvable. We may now assume that N is the unique minimal normal subgroup of G. We shall now show that N is π-solvable. We note that $S_{\mathcal{P}}(G) \neq G$, since $\Sigma_{\mathcal{P}}(G) \neq \emptyset$. If $N \subsetneq S_{\mathcal{P}}(G)$ then by (2.6) it follows that N is solvable and hence it is π-solvable. If $N \not\subset S_{\mathcal{P}}(G)$ then there exists M_o in $\Sigma_{\mathcal{P}}(G)$ such that $N \not\subset M_o$ and so $G = M_oN$ and $core_G(M_o) = \langle 1 \rangle$. Let M be any maximal subgroup of G with core 1. Then $N \not\subset M$ and so G = MN. Clearly M belongs to $\Sigma_{\mathcal{P}}(G)$. By hypothesis $|N|_\pi = \eta(G:M)_\pi = [G:M]_\pi$. If $|N|_\pi = 1$ then N is π-solvable. If $|N|_\pi \neq 1$ then there exists a common prime divisor of all the indices in G of all the maximal subgroups with core 1. So by (2.4), N is solvable and hence it is π-solvable. Thus G/N and N are both π-solvable. So G is π-solvable.

The converse follows directly from Theorem 2 [9].

THEOREM 3.5. Let G be a group with $\lambda(G) \neq \emptyset$. Then G is π-solvable if and only if $\eta(G:M)_\pi = [G:M]_\pi$ for each M in $\lambda(G)$, where $\lambda(G) = \{M : M \lessdot G, \eta(G:M) \text{ is composite}\}$.

THEOREM 3.6. Let G be a group with $|\lambda(G)| \geq 2$. Then G is π-solvable if and only if $\eta(G:M_1)_\pi = \eta(G:M_2)_\pi$ implies $[G:M_1]_\pi = [G:M_2]_\pi = \eta(G:M_1)_\pi$ for any M_1, M_2 in $\lambda(G)$.

PROOF. Let the condition of the theorem hold. If $|G|_\pi = 1$ then G is a π'-group and hence it is π-solvable. So we assume that $|G|_\pi \neq 1$. Let G be simple and $p_1, p_2, \ldots, p_n$ be the set of prime divisors of $|G|$, which belong to π. Then as in the proof of Theorem 3.4, we can show that there exist maximal subgroups M_i of G such that $[G:M_i]_{p_i} = 1$ (i=1,2,...n).

By hypothesis, $|G|_\pi = [G:M_1]_\pi = [G:M_2]_\pi = \cdots = [G:M_n]_\pi$. As each $p_i \in \pi$, it follows that $|G|_\pi = 1$, a contradiction. So G can not be simple. Let N be a minimal normal subgroup of G. If $\lambda(G/N)$ is empty then $\Lambda(G/N)$ is also empty and so by definition, $L(G/N) = G/N$ and consequently by the supersolvability of the group $L(G/N)$, it follows that G/N is π-solvable. If $\lambda(G/N)$ consists of only one element M/N, say, then either $\Lambda(G/N)$ is empty or $\Lambda(G/N) = \{M/N\}$. If $\Lambda(G/N)$ is empty then as above G/N is supersolvable. If $\Lambda(G/N) = \{M/N\}$ then $M/N = L(G/N)$ and consequently M/N is normal in G/N. So by Theorem 1 [9], $\eta(G/N:M/N)=[G/N:M/N]$ = a prime, a contradiction, since $M/N \in \Lambda(G/N)$. We now assume that $|\lambda(G/N)| \geq 2$. It can be shown that G/N satisfies the hypothesis of the theorem. So by induction, G/N is π-solvable. As before, we can assume that N is the unique minimal normal subgroup of G. Also we see that $L(G) \neq G$. If $N \subsetneq L(G)$ then N is solvable and hence it is π-solvable. If $N \not\subseteq L(G)$ then there exists M_0 in $\Lambda(G)$ such that $N \not\subseteq M_0$ and so $G = M_0N$ and $\mathrm{core}_G(M_0) = \langle 1 \rangle$. Let M be any maximal subgroup of G with core 1. Then $N \not\subseteq M$ and so $G = MN$. Consequently $\eta(G:M)=|N|=\eta(G:M_0)$, whence it follows that M belongs to $\lambda(G)$. By hypothesis $[G:M]_\pi = |N|_\pi$. If $|N|_\pi = 1$ then N is π-solvable. If $|N|_\pi \neq 1$ then using (2.4), we have N is solvable and hence it is π-solvable. Thus G/N and N are both π-solvable and consequently G is π-solvable.

The converse follows directly from Theorem 5 [9].

THEOREM 3.7. Let G be a p-solvable group and $|\Sigma_{\mathcal{P}}(G)| \geq 2$. Then G is π-solvable if and only if $\eta(G:M_1)_\pi = \eta(G:M_2)_\pi$ implies $[G:M_1]_\pi = [G:M_2]_\pi = \eta(G:M_1)_\pi$ for any M_1, M_2 in $\Sigma_{\mathcal{P}}(G)$.

THEOREM 3.8. Let G be a p-solvable group and $|\Sigma_{\mathcal{P}}(G)| \geq 2$. Then G is π-solvable if and only if the following hold.

(i) G has a π-solvable maximal subgroup M with $\eta(G:M)_\pi = [G:M]_\pi$.

(ii) $\eta(G:M_1)_\pi = \eta(G:M_2)_\pi$ implies $[G:M_1]_\pi = [G:M_2]_\pi$ for any M_1, M_2 in $\Sigma_{\mathcal{P}}(G)$.

THEOREM 3.9. Let G be a group with $|\lambda(G)| \geq 2$. Then G is π-solvable if and only if the following hold.

(i) G has a π-solvable maximal subgroup M with $\eta(G:M)_\pi = [G:M]_\pi$.

(ii) $\eta(G:M_1)_\pi = \eta(G:M_2)_\pi$ implies $[G:M_1]_\pi = [G:M_2]_\pi$ for any M_1, M_2 in $\lambda(G)$.

PROPOSITION 3.10. Let G be a p-solvable group and $|\Sigma_{\mathcal{P}}(G)| \geq 2$. Then G is π-solvable if $\eta(G:M_1)_\pi = \eta(G:M_2)_\pi = 1$ for all M_1, M_2 in $\Sigma_{\mathcal{P}}(G)$ with equal normal index.

PROPOSITION 3.11. Let G be a group with $\Lambda(G) \neq \emptyset$. Then G is π-solvable if $\eta(G:M)_\pi = 1$ for each M in $\Lambda(G)$.

PROPOSITION 3.12. Let G be a p-solvable group or p be the largest prime dividing $|G|$ and $\Sigma_p(G) \neq \emptyset$. Then G is π-solvable if $\eta(G:M)_\pi = 1$ for each M in $\Sigma_p(G)$.

PROPOSITION 3.13. Let G be a group with $|\lambda(G)| \geq 2$. Then G is π-solvable if $\eta(G:M_1)_\pi = \eta(G:M_2)_\pi = 1$ for all M_1, M_2 belonging to $\lambda(G)$ with equal normal index.

PROPOSITION 3.14. If a group G has a π-solvable maximal subgroup M with $\eta(G:M)_\pi = 1$ then G is π-solvable.

PROOF. Let G satisfy the hypothesis of the proposition. Then G is not simple. For, otherwise, $|G|_\pi = \eta(G:M)_\pi = 1$ and so G is π-solvable. Let N be a minimal normal subgroup of G. If $N \subsetneq M$ then N is π-solvable and also, by induction, G/N is π-solvable and hence G is π-solvable. If $N \not\subseteq M$ then $G=MN$ and since $G/N \cong M/M\cap N$, G/N is π-solvable. Also by hypothesis $|N|_\pi = \eta(G:M)_\pi = 1$ and so N is π-solvable. Hence G is π-solvable.

4. SOME RESULTS ON SUPERSOLVABLE GROUPS.

THEOREM 4.1. Let G be a p-solvable group and suppose that for each c-maximal subgroup M of G, $[G:M]_p = 1$ or p. Then G is supersolvable if and only if $\eta(G:M)$ is square-free for each M in $\Sigma_{\mathcal{P}}(G)$.

PROOF. Let G satisfy the hypothesis of the theorem. We claim that $\Sigma_{\mathcal{P}}(G)$ is empty. If possible, let there exist M in $\Sigma_{\mathcal{P}}(G)$. Then G is not simple. For otherwise, $|G| = \eta(G:M)$ is square-free and so G is supersolvable. Let $\eta(G:M)=|H/K|$, where H/K is a chief factor of G and H is minimal in the set of normal supplements of M in G. By hypothesis $|H/K|$ is square-free and hence H/K is supersolvable. Thus H/K is a solvable minimal normal subgroup of G/K. So H/K is an elementary abelian q-group for some prime q. Consequently $\eta(G:M)=|H/K|=q$, a prime, which is a contradiction. So $\Sigma_{\mathcal{P}}(G)$ is empty. By definition $G=S_{\mathcal{P}}(G)$ and hence G is solvable we shall now show that $\Lambda(G)$ is empty. If possible, let there exist M in $\Lambda(G)$. Then since $\eta(G:M)=[G:M]$, [2, Corollary of Theorem 1], it follows that $\eta(G:M)$ is composite and hence p divides [G:M]. Now the solvability of G implies that [G:M] is the power of the prime p. By hypothesis, $[G:M]=[G:M]_p=p$, a prime, which is a contradiction. Hence $\Lambda(G)$ is empty and consequently G=L(G). Hence G is supersolvable.

Conversely if G is supersolvable then $\eta(G:M)=[G:M]$=a prime for each maximal subgroup M of G and hence the assertion immediately follows.

PROPOSITION 4.2. Let p,q be two distinct primes. Suppose that G is either p-solvable or q-solvable. Then G is supersolvable if and only if $\eta(G:M)$ is square-free for every M in $\Sigma_{\mathcal{P}}(G)$ or $\Sigma_{\mathcal{Q}}(G)$.

PROPOSITION 4.3. If G contains a supersolvable maximal subgroup M such that $core_G(M)=\langle 1\rangle$ and $\eta(G:M)$ is square-free then G is supersolvable.

PROOF. Let G be simple. By hypothesis, $|G|=\eta(G:M)$ is square-free. So G is supersolvable. We now assume that G is not simple. Let N be a minimal normal subgroup of G. Since $core_G(M)=\langle 1\rangle$, it follows that $N\not\subset M$ and so G=MN. By hypothesis $|N|=\eta(G:M)$ is square-free and so N is supersolvable. Since $G/N \cong M/M\cap N$, it follows that G/N is supersolvable. Thus G/N and N are both solvable. Hence G is solvable. Now since N is a minimal normal subgroup of the solvable group, it follows that N is an elementary abelian p-group for some prime p. Hence $|N|=p$ and consequently N is cyclic. Therefore G is supersolvable.

PROPOSITION 4.4. If G contains a supersolvable maximal subgroup M such that $\eta(G:M)$ is square-free and the Fitting subgroup. F(G), is not contained in M then G is supersolvable.

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