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CERTAIN INTEGRALS INVOLVING LOGARITHMIC AND EXPONENTIAL FUNCTIONS

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ABSTRACT. In this paper we have evaluated the integrals

$$\int_0^\infty x^{n-1} \ln x \exp(-ax - bx^{-1})dx$$

and

$$\int_0^\infty x^{n-2}(ax^2 - b)(\ln x)^2 \exp(-ax - bx^{-1})dx$$

for all $n = 1, 2, 3, \ldots$. Some applications of the results are discussed and an open problem is posed.

KEY WORDS AND PHRASES. Macdonald functions, logarithmic and exponential function, statistical inference theory.

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1. INTRODUCTION.

The integrals

$$\int_0^\infty x^{n-1} \ln x \exp(-ax - bx^{-1})dx, \qquad a > 0,\ b > 0, \tag{1}$$

and

$$\int_0^\infty x^{n-2}(ax^2 - b)(\ln x)^2 \exp(-ax - bx^{-1})dx, \qquad a > 0,\ b > 0, \quad (n = 1, 2, 3, \ldots) \tag{2}$$

arise in statistical inference theory when Frechet and other distributions are applied to solve important problems related to ocean engineering technology, resource management, and weather phenomena such as estimation and forecasting of wind velocity, flood, rainfall, etc. See [3] and [6,7,8].

The closed form solutions of the integrals (1) and (2) are not known for any $n = 1, 2, 3, \ldots$. Some numerical techniques are used to evaluate these integrals [2,8]. We have evaluated these integrals (1) and (2). Some special cases of the results are also discussed.

2. LEMMA.

For $n = 0, 1, 2, 3, \ldots$

$$(\frac{-d}{dx})^n\{\ln(b/x)K_0(2\sqrt{bx})\}$$
$$= \frac{1}{x^n}[(bx)^{n/2}\ln(b/x)K_n(2\sqrt{bx}) + (n!)\sum_{j=1}^{n}\frac{(bx)^{(n-j)/2}K_{n-j}(2\sqrt{bx})}{j(n-j)!}]$$
$$(a > 0,\ x > 0). \tag{3}$$

PROOF. This follows from the Leibnitz rule of differentiation and from the recursion formulas [5, p 970].

3. THEOREM.

$$\int_0^\infty x^{n-1}\ln x\,\exp(-ax - bx^{-1})dx$$
$$= \frac{1}{a^n}\left[(ab)^{n/2}\ln(b/a)K_n(2\sqrt{ab}) + (n!)\sum_{j=1}^{n}\frac{(ab)^{(n-j)/2}K_{n-j}(2\sqrt{ab})}{j(n-j)!}\right]$$
$$(a > 0,\ b > 0,\ n = 0, 1, 2, 3, \ldots). \tag{4}$$

PROOF. It is known that [5, p 313]

$$\int_0^\infty x^{\alpha-1}\exp(-ax - bx^{-1})dx = 2(b/a)^{\alpha/2}K_\alpha(2\sqrt{ab})$$
$$a > 0,\ b > 0. \tag{5}$$

Performing formal differentiation with respect to the parameter α we obtain the equality

$$\int_0^\infty x^{\alpha-1}\ln x\,\exp(-ax - bx^{-1})dx = 2(b/a)^{\alpha/2}\times$$
$$\times\left[\frac{1}{2}\ln(b/a)K_\alpha(2\sqrt{ab}) + \frac{\partial}{\partial\alpha}K_\alpha(2\sqrt{ab})\right]. \tag{6}$$

The process of the formal differentiation is justified [4, pp. 427-448].

Using the integral representation [5, p. 358] for the Macdonald function K_α we get

$$\frac{\partial}{\partial\alpha}[K_\alpha(z)] = \int_0^\infty t\,e^{-z\cosh(t)}\sinh(\alpha t)dt \tag{7}$$

which implies

$$\frac{\partial}{\partial\alpha}[K_\alpha(z)]_{\alpha=0} = 0 \tag{8}$$

Letting $\alpha \to 0$ in (6) and using (8) we get

$$\int_0^\infty \frac{1}{x}\ln x\,\exp(-ax - bx^{-1})dx = \ln(b/a)K_0(2\sqrt{ab}),$$
$$(a > 0,\ b > 0). \tag{9}$$

We can rewrite (9) in the operational form as follows:

$$L\left\{\frac{\ln x}{x}\exp(-bx^{-1}); a\right\} = \ln(b/a)K_0(2\sqrt{ab}), \tag{10}$$

where L is the Laplace transform operator [1].

By using the lemma and property [1]

$$L\{x^n f(x); a\} = (-\frac{d}{da})^n F(a) \qquad (n = 1, 2, 3, \ldots) \tag{11}$$

of the Laplace transformation we get the result from (9) and (11).

COROLLARY 1. See [5, p. 577]

$$\int_0^\infty \exp(-\mu(x/c + c/x)) \frac{ln\, x}{x} dx = 2\,(\ln\, c)\, K_0(2\mu), \qquad (c > 0,\ \mu > 0).$$

PROOF. This follows from (9) when we take $a = \mu/c$ and $b = \mu c$.

COROLLARY 2.

$$\int_{-\infty}^\infty t\ \exp(\beta t - 2a \cosh \beta t) dt = 1/a\beta^2 K_0(2a) \qquad (a > 0,\ \beta > 0).$$

PROOF. This follows from (4) when we take $n = 1$, $b = a$ and use the transformation $x = e^{\beta t}$, $\beta > 0$.

COROLLARY 3. For all $n = 0, 1, 2, 3, \ldots$ and $a > 0$, $b > 0$,

$$\int_0^\infty x^{n-2}(ax^2 - b)(\ln\, x)^2 \exp(-ax - bx^{-1}) dx = \frac{2}{a^n}\left[(ba)^{n/2} \ln(b/a) K_n(2\sqrt{ab}) + n! \sum_{j=1}^n (ab)^{(n-j)/2} \frac{K_{n-j}(2\sqrt{ab})}{j(n-j)!}\right] \tag{12}$$

PROOF. This follows by applying integration by parts to the integral in (4).

In particular for $n = 0$, $a = \mu/c$, $b = \mu c$, $\mu > 0$, $c > 0$ in (12), we get

$$\int_0^\infty \frac{(\ln\, x)^2}{x^2}(x^2 - 1) \exp(-\mu(x/c + c/x)) dx = 4 ln\, c\, K_0(2\mu) \qquad (\mu > 0,\ c > 0).$$

We state here an open problem. The solution to the problem will have far-reaching consequences in statistical inference theory. It should be noted that the solution to the problem is not known even for $n = 2$ and for any value of α.

4. STATEMENT OF THE OPEN PROBLEM.

$$\text{Evaluate } \int_0^\infty (ln\, x)^n x^{\alpha-1} \exp(-ax - bx^{-1}) dx \qquad (a > 0,\ b > 0,\ n \geq 2).$$

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