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DECAY OF SOLUTIONS OF A NONLINEAR HYPERBOLIC SYSTEM IN NONCYLINDRICAL DOMAIN

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ABSTRACT. In this paper we study the existence of solutions of the following nonlinear hyperbolic system

$$\left|\begin{array}{l} u'' + A(t)u + b(x)G(u) = f \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(0) = u^o \quad u'(0) = u^1 \end{array}\right.$$

where Q is a noncylindrical domain of $\mathbf{R}^{n+1}$ with lateral boundary Σ, $u = (u_1, u_2)$ a vector defined on Q, $\{A(t), 0 \leq t < +\infty\}$ is a family of operators in $\mathcal{L}(H_o^1(\Omega), H^{-1}(\Omega))$, where $A(t)u = (A(t)u_1, A(t)u_2)$ and $G: \mathbf{R}^2 \to \mathbf{R}^2$ a continuous function such that $x.G(x) \geq 0$, for $x \in \mathbf{R}^2$.

Moreover, we obtain that the solutions of the above system with dissipative term u' have exponential decay.

KEY WORDS AND PHRASES. Weak solutions, exponential decay, noncylindrical domain.

1. INTRODUCTION.

Let Q be a noncylindrical domain of $\mathbf{R}^n \times [0, +\infty[$ with lateral boundary Σ, $G: \mathbf{R}^2 \to \mathbf{R}^2$ a continuous function and $u: Q \to \mathbf{R}^2$, $u(x,t) = (u_1(x,t), u_2(x,t))$. In Q we consider the following mixed hyperbolic problem:

$$u'' + A(t)u + b(x)G(u) = f \quad \text{in} \quad Q \tag{1.1}$$

$$u = 0 \qquad \text{on} \qquad \Sigma \tag{1.2}$$

$$u(x,0) = u^o(x), \quad u'(x,0) = u^1(x) \tag{1.3}$$

where $\rho > -1$ is a real number, $\{A(t), 0 \leq t < +\infty\}$ is a family in $\mathcal{L}(H_o^1(\Omega), H^{-1}(\Omega))$. In this case the vector $(A(t)u_1, A(t)u_2)$, for $u \in (H_o^1(\Omega))^2$, is designated by $A(t)u$.

The linear and nonlinear wave equations in noncylindrical domains have been treated by a number of authors. Among them we can mention Lions [6] who introduced the so-called penalty method to solve the problem of existence of solutions. Using this method, Medeiros [8] proved the existence of weak solutions of the mixed problem for the equation

$$u'' - \Delta u + \beta(u) = f \quad \text{in} \quad Q. \tag{1.4}$$

For a wide class of $\beta(u)$ such that $\beta(u)u \geq 0$, Cooper-Bardos [4] studied the existence and uniqueness of weak solutions of (1.4) for the case $\beta(u) = |u|^\alpha u$ ($\alpha \geq 0$) and Σ globally "time-like" and Cooper-Medeiros [3] included the above results in a general model

$$u'' - \Delta u + f(u) = 0$$

where f is continuous and $sf(s) \geq 0$ and Σ globally "time-like".

Cooper [2] considered the local decay property of solutions of linear wave equations (in some exterior domain) assuming that the boundary is "time-like" at each point. Inoue [5] succeeded in proving the existence of classical solutions of (1.4) for the case $n = 3$ and $\beta(u) = u^3$ when the body is "time-like" at each point. Clark [1] proved the existence of weak solutions of the mixed problem for the equation

$$k_2(x)u'' + k_1(x)u' + A(t)u + |u|^\rho u = f \quad \text{in} \quad Q.$$

Nakao-Narazaki [11] studied the decay of weak solutions for a wave equation with nonlinear dissipative terms in noncylindrical domains. On the other hand, Milla Miranda and Medeiros obtained weak solutions for problems (1.1)-(1.3) for the case $A(t) = -\Delta$ and $b(x) = 1$ (Medeiros-Milla Miranda [9]) and $b(x) = -1$ (Milla Miranda-Medeiros [10]) in a cylindrical domain.

In this paper we study the existence of weak solutions of problem (1.1)-(1.3) and the decay of weak solutions for the system (1.1) perturbed by the dissipative term u'. Under the hypothesis that the domain is monotone increasing we prove that these solutions decay exponentially as $t \to +\infty$.

2. PRELIMINARIES.

By $\mathcal{D}(\Omega)$ we denote the space of infinitely differentiable functions with compact support contained in Ω. The inner product and norm in $(L^2(\Omega))^2$ and $(H_o^1(\Omega))^2$ will be represented by $(\cdot, \cdot)$, $|\cdot|$, $((\cdot,\cdot))$, $||\cdot||$ respectively and defined by:

$$(u,v) = \sum_{j=1}^{2}(u_j, v_j)_{L^2(\Omega)}, \quad |u|^2 = (u,u),$$

$$((u,v)) = \sum_{j=1}^{2}((u_j, v_j)), \quad ||u||^2 = ((u,u))$$

where $u = (u_1, u_2)$, $v = (v_1, v_2)$.

For $w = (w_1, w_2) \in (L^p(\Omega))^2$, we have

$$||w||^2_{L^p(\Omega))^2} = ||w_1||^2_{L^p(\Omega)} + ||w_2||^2_{L^p(\Omega)}, \quad \text{for } 1 \leq p \leq \infty.$$

We denote by u', u'', $D_i u$, $0 \leq i \leq n$, the vectors

$$u' = \left(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}\right), \quad u'' = \left(\frac{\partial^2 u_1}{\partial t^2}, \frac{\partial^2 u_2}{\partial t^2}\right), \quad D_i u = \left(\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_i}\right).$$

If X is a Banach space we denote by $L^p(0,T;X)$, $1 \leq p < +\infty$, the Banach space of vector valued functions $u:]0,T[\to X$ which are measurable and $||u(t)||_X \in L^p(0,T)$ with the norm

$$||u||_{L^p(0,T;X)} = \left[\int_0^T ||u(t)||_X^p \right]^{1/p},$$

and by $L^\infty(0,T;X)$ the Banach space of vector valued functions $u:]0,T[\to X$ which are measurable and $||u(t)||_X \in L^\infty(0,T)$ with the norm

$$||u||_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} ||u(t)||_X.$$

Let Ω be a bounded, connected and open subset of $\mathbf{R}^n$ with smooth boundary Γ, $Q \subset \Omega \times]0,+\infty[$ an open noncylindrical domain. We will use the following notations: $\Omega_s = Q \cap \{t = s\}$ for $s > 0$, $\Omega_o = \operatorname{int}(\overline{Q} \cap \{t = 0\})$, $\Gamma_s = \partial\Omega_s$, $\Sigma = \bigcup_{0\le s<\infty} \Gamma_s$ and $\partial Q = \overline{\Omega}_o \cup \Sigma$ is the boundary of Q. Of course, $\Omega_o \neq \phi$.

Our assumptions on Q are:

(H1) Ω_t is monotone increasing, that is, $\Omega_t^* \subset \Omega_s^*$ if $t < s$, where Ω_t^* is the projection of Ω_t in the hyperplane $t = 0$.

(H2) For each $t \in]0,+\infty[$, Ω_t has the following property of regularity: if $u \in H_o^1(\Omega)$ and $u = 0$ a.e. in $\Omega\backslash\Omega_t^*$, the restriction of u to Ω_t^* belongs to $H_o^1(\Omega_t^*)$.

For simplicity we will identify Ω_t^* with Ω_t. We define $L^q(0,\infty;(L^p(\Omega_t))^2)$ as the space of functions $w \in L^q(0,\infty;(L^p(\Omega))^2)$ such that $w = 0$ a.e. in $\Omega\times]0,+\infty[\backslash Q$. When $1 \le q < \infty$ we consider the norm

$$||w||_{L^q(0,\infty;(L^p(\Omega_t))^2)} = \left[\int_0^\infty ||w(t)||^q_{L^p(\Omega_t))^2}\, dt \right]^{1/q},$$

which agrees with $||w||_{L^q(0,\infty;(L^p(\Omega))^2)}$. For the case $q = \infty$ we consider

$$||w||_{L^\infty(0,\infty;(L^p(\Omega_t))^2)} = \operatorname*{ess\,sup}_{0<t<\infty} ||w(t)||_{(L^p(\Omega_t))^2}.$$

We observe that $L^q(0,\infty;(L^p(\Omega_t))^2)$ is a closed subspace of $L^q(0,\infty;(L^p(\Omega))^2)$ for $1 \le q \le \infty$. In the same way we define $L^q(0,\infty;(H_o^1(\Omega_t))^2)$ as the space of functions $w \in L^q(0,\infty;(H_o^1(\Omega))^2)$ such that $w = 0$ a.e. in $\Omega\times]0,+\infty[\backslash Q$ with the norm:

$$||w||_{L^q(0,+\infty;(H_o^1(\Omega_t))^2)} = \left[\int_0^{+\infty} ||w(t)||^q_{(H_o^1(\Omega_t))^2}\, dt \right]^{1/q}, \quad 1 \le q < \infty,$$

and

$$||w||_{L^\infty(0,\infty;(H_o^1(\Omega_t))^2)} = \operatorname*{ess\,sup}_{0<t<\infty} ||w(t)||_{(H_o^1(\Omega_t))^2}.$$

It follows by (H2) that these norms agree with the norms in $L^q(0,\infty;(H_o^1(\Omega))^2)$ for $1 \le q \le \infty$. We also have that $L^q(0,\infty;(H_o^1(\Omega_t))^2)$ is a closed subspace of $L^q(0,\infty;(H_o^1(\Omega))^2)$.

Let us consider the following family of operators in $\mathcal{L}(H_o^1(\Omega), H^{-1}(\Omega))$

$$A(t) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left[a_{ij} \frac{\partial}{\partial x_i} \right],$$

where $a_{ij} = a_{ji}$ and $a_{ij}, \frac{\partial}{\partial t} a_{ij} \in L^\infty(0,+\infty;L^\infty(\Omega))$ $(i,j = 1,\ldots,n)$. Here $\frac{\partial}{\partial t} a_{ij}$ denotes the derivative in distributional sense of a_{ij} with relation to t. We suppose:

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \alpha(|\xi_1|^2 + \cdots + |\xi_n|^2) \tag{2.1}$$

for all $(t,\xi) \in [0,+\infty[\times \mathbf{R}^n$ and a.e. in Ω, with $\alpha > 0$ a constant.

For $u, v \in (H_o^1(\Omega))^2$ we denote by $a(t,u,v)$ the family of linear forms defined as:

$$a(t,u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,t)D_i u.D_j v\, dx$$

associated to the operator $A(t)$ defined in $(H_o^1(\Omega))^2$ by $A(t)u = (A(t)u_1, A(t)u_2)$, where $u = (u_1,u_2)$.

From the hypothesis about a_{ij}, we obtain that $a(t,u,v)$ is symmetrical and of (2.1)

$$a(t,u,u) \geq \alpha ||u||^2, \quad \text{for} \quad u \in (H_o^1(\Omega))^2,\ t \in [0,+\infty[\tag{2.2}$$

Still, if we define $h(t) = a(t,u,v)$ for u, v fixed in $(H_o^1(\Omega))^2$, we have that $h, h' \in L^1_{\text{loc}}(0,+\infty)$ where

$$h'(t) = \sum_{i,j=1}^{n} \int_\Omega \frac{\partial}{\partial t} a_{ij}(x,t)D_i u.D_j v\, dx$$

which we denote as $a'(t,u,v)$. Let us suppose that

$$a'(t,u,u) \leq 0, \quad \text{for} \quad u \in (H_o^1(\Omega))^2. \tag{2.3}$$

We consider the continuous function $G\colon \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$G(s,t) = \left(|t|^{\rho+2}|s|^\rho s, |s|^{\rho+2}|t|^\rho t\right).$$

We easily verify that

$$x.G(x) \geq 0, \quad \text{for} \quad x \in \mathbf{R}^2. \tag{2.4}$$

Let $b(x)$ be a function such that $b \in L^\infty(\Omega)$ and to facilitate the computation we assume that

$$|b(x)| \leq 1 \quad \text{a.e. in} \quad \Omega. \tag{2.5}$$

For $u, v \in (H_o^1(\Omega_t))^2$ we use

$$a(t,u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x,t)D_i\tilde{u}.D_j\tilde{v}\, dx$$

where $\tilde{u}$, $\tilde{v}$ are extensions of u, v by zero outside of Ω_t.

Finally in this section we give a lemma due to Nakao [11], which will be needed for the proof of decay property of solutions.

LEMMA 2.1: Let $\phi(t)$ be a nonnegative decreasing function on $\mathbf{R}^+$, satisfying

$$\phi(t+1) - d_2\phi(t) \leq d_3(\phi(t) - \phi(t+1)) \tag{2.6}$$

with some constants $0 < d_2 < 1$, $d_3 > 0$. Then we have

$$\phi(t) \leq c_o\, \phi(0) e^{-\delta t}, \quad \text{for} \quad t \in \mathbf{R}^+$$

where c_o, δ are positive constants.

3. MAIN RESULTS.

THEOREM 3.1: Let $a(t,u,v)$ and $b(x)$ be as in (2.2), (2.3), (2.4) and $f \in L^1(0,+\infty;(L^2(\Omega_t))^2)$, $u^o \in (H_o^1(\Omega_o))^2$, $u^1 \in (L^2(\Omega_o))^2$ satisfy:

$$||u^o||_{(H_o^1(\Omega_o))^2} < \left[\frac{\alpha}{C_o'^{2(\rho+2)}}\right]^{\frac{1}{2(\rho+1)}} \tag{3.1}$$

$$\theta < \left[\frac{1}{2}\left(\frac{\alpha}{C_o'^2}\right)^{\frac{\rho+2}{\rho+1}}\left(\frac{\rho+1}{\rho+2}\right)\right]^{1/2} \tag{3.2}$$

where

$$\theta = \left[|u^1|^2_{(L^2(\Omega_o))^2} + a(0,u^o,u^o) + \frac{1}{\rho+2}(b(x)G(u^o),u^o)_{(L^2(\Omega))^2}\right]^{1/2} + \int_0^{+\infty} |f(t)|_{(L^2(\Omega_t))^2}\, dt$$

$\rho > -1$, if $n = 1,2$; $-1 < \rho < \frac{4-n}{n-2}$ if $n \geq 3$ and C_o' is the constant of the continuous embedding of $H_o^1(\Omega)$ in $L^{2(\rho+2)}(\Omega)$. Then, under the assumptions (H1) and (H2), there exists a function u satisfying

$$u \in L^\infty(0,+\infty;(H_o^1(\Omega_t))^2) \tag{3.3}$$

$$u' \in L^\infty(0,+\infty;(L^2(\Omega_t))^2) \tag{3.4}$$

$$u'' \in L^1(0,+\infty;(H^{-1}(\Omega_o))^2) \tag{3.5}$$

$$u'' + A(t)u + b(x)G(u) = f \quad \text{in} \quad (\mathcal{D}'(Q))^2 \tag{3.6}$$

$$u(0) = u^o \tag{3.7}$$

$$u'(0) = u^1 \tag{3.8}$$

REMARK: Theorem 3.1 (replacing ∞ by T) is also valid of we do not consider (2.3) and replace (3.2) by

$$\theta_1 < \left[\frac{1}{2}\left(\frac{\alpha}{C_o'^2}\right)^{\frac{\rho+2}{\rho+1}}\left(\frac{\rho+1}{\rho+2}\right)\right]^{1/2} \exp\left[\frac{-nN}{2\alpha}\left(\frac{\rho+2}{\rho+1}\right)T\right]$$

where $N = \max\limits_{1\leq i,j\leq n} \operatorname*{ess\,sup}\limits_{\Omega\times]0,T[} \left|\frac{\partial}{\partial t} a_{ij}(x,t)\right|$ and

$$\theta_1 = \frac{1}{2}\left[|u^1|^2_{(L^2(\Omega_o))^2} + a(0,u^o,u^o) + \frac{1}{\rho+2}(b(x)G(u^o),u^o)_{(L^2(\Omega_o))^2}\right]^{1/2} +$$
$$+ \frac{1}{\sqrt{2}}\int_0^{+\infty} |f(t)|_{(L^2(\Omega_t))^2}\, dt.$$

THEOREM 3.2: Let ρ, α and C_o be as in Theorem 3.1 and $u^o \in (H_o^1(\Omega_o))^2$, $u^1 \in (L^2(\Omega_o))^2$, $a(t,u,v)$, $b(x)$ as in (2.2), (2.3), (2.4) such that

$$||u^o||_{(H_o^1(\Omega_o))^2} < \left[\frac{\alpha}{2C_o^{2(\rho+2)}}\right]^{\frac{1}{2(\rho+1)}} \tag{3.9}$$

$$\theta < \left(\frac{\alpha}{2C_o^2}\right)^{\frac{(\rho+2)}{\rho+1}} \left(\frac{2\rho+3}{2\rho+4}\right) \tag{3.10}$$

where

$$\theta = |u^1|^2_{(L^2(\Omega_o))^2} + a(0,u^o,u^o) + \frac{1}{\rho+2}\int_{\Omega_o} b(x)G(u^o(x)).u^o(x)dx.$$

Then, under assumptions (H1), (H2), there exists a function u satisfying (3.3), (3.4), (3.5), (3.7), (3.8) and

$$u'' + A(t)u + b(x)G(u) + u' = 0 \quad \text{in} \quad (\mathcal{D}'(Q))^2 \tag{3.11}$$

$$E(t) \leq c\, e^{-\beta t} \quad \text{for} \quad t \in [0,+\infty[\tag{3.12}$$

where $c > 0$, $\beta > 0$ are constants independent of u, and

$$E(t) = \frac{1}{2}\Bigg[|u'(t)|^2_{(L^2(\Omega_t))^2} + \sum_{i,j=1}^{n}\int_{\Omega_t} a_{ij}(x,t)D_i u(x,t).D_j u(x,t)dx +$$

$$+ \frac{1}{\rho+2}\int_{\Omega_t} b(x)G(u(x,t)).u(x,t)dx\Bigg].$$

4. PROOF OF THE RESULTS.

PROOF OF THEOREM 3.1. We observe by (3.3), (3.4) and (3.5) that the initial conditions make sense. From (3.1) we have $\theta \geq 0$. To prove the theorem we consider $\tilde{u}^o$, $\tilde{u}^1$ extensions of u^o, u^1 by zero outside of Ω_o and

$$M(x,t) = \begin{cases} 1 & \text{in} \quad \Omega \times [0,+\infty[\backslash(Q \cup \Omega_o \times \{0\}) \\ 0 & \text{in} \quad Q \cup \Omega_o \times \{0\} \end{cases}$$

It is clear that $\tilde{u}^o \in (H_o^1(\Omega))^2$, $\tilde{u}^1 \in (L^2(\Omega))^2$ and that they satisfy (3.1) and (3.2) with the norms in the respective spaces.

Let $(w_\nu)_{\nu\geq 1}$ be a basis of $(H_o^1(\Omega))^2$ and $V_m = [w_1,\dots,w_m]$ the subspace generated by the m first vectors of the basis (w_ν). For each $\varepsilon > 0$, we determine the penalized approximate solutions $u_{\varepsilon m}$: $[0,t_{\varepsilon m}[\rightarrow V_m$ as solutions of the following system

$$(u''_{\varepsilon m}(t),z) + a(t,u_{\varepsilon m}(t),z) + (b(x)G(u_{\varepsilon m}(t)),z) + \frac{1}{\varepsilon}(M(t)u_{\varepsilon m}(t),z) =$$

$$= (f(t),z), \quad \text{for } z \in V_m \tag{4.1}$$

$$u_{\varepsilon m}(0) = u_{om} \rightarrow \tilde{u}^o \quad \text{strongly in} \quad (H_o^1(\Omega))^2, u_{om} \in V_m \tag{4.2}$$

$$u'_{\varepsilon m}(0) = u_{1m} \rightarrow \tilde{u}^1 \quad \text{strongly in} \quad (L^2(\Omega))^2, u_{1m} \in V_m \tag{4.3}$$

Let be $\phi_\mu(x,t) = \mu\int_t^{t+\frac{1}{\mu}} M(x,s)ds$. We can prove that $\phi_\mu \in C([0,+\infty), L^\infty(\Omega))$, ϕ_μ is differentiable with respect to t and

$$\frac{\partial}{\partial t}\phi_\mu(x,t) \leq 0 \quad \text{for} \quad t \in [0,+\infty[, \quad \text{a.e. in } \Omega,$$

being the derivative in distributional sense of ϕ_μ with respect to t agree with $\frac{\partial}{\partial t}\phi_\mu$. Moreover, we have:

$$\frac{\partial}{\partial t}\phi_\mu \in L^\infty(0,+\infty; L^\infty(\Omega))$$

$$|\phi_\mu(x,t)| \leq 1, \text{ for } (x,t) \in \Omega \times [0,+\infty[\text{ and } \mu > 0$$

$$\phi_\mu(x,t) \to M(x,t), \text{ for } t \in]0,+\infty[\text{ a.e. in } \Omega, \text{ when } \mu \to \infty$$

$$\int_0^t (\phi_\mu(s)w(s), w'(s))ds = \frac{1}{2}(\phi_\mu(t)w(t), w(t)) - \frac{1}{2}(\phi_\mu(0)w(0), w(0)) - \frac{1}{2}\int_0^t (\phi'_\mu(s)w(s), w(s))ds,$$

for $w \in L^\infty(0,+\infty;(H_o^1(\Omega))^2)$ such that $w' \in L^\infty(0,+\infty;(L^2(\Omega))^2)$.

When we take $w = u_{\varepsilon m}$ in the last equality and then make $\mu \to \infty$, we obtain by the above results for ϕ_μ that:

$$\int_0^t (M(s)u_{\varepsilon m}(s), u'_{\varepsilon m}(s))ds \geq \frac{1}{2}|M(t)u_{\varepsilon m}(t)|^2 - \frac{1}{2}|M(0)u_{om}|^2. \tag{4.4}$$

It follows by (4.2) and $H_o^1(\Omega) \hookrightarrow L^{2(\rho+2)}(\Omega)$ that there exists a subsequence of (u_{om}), still denoted by the same symbol, such that

$$(b(x)G(u_{om}), u_{om}) \to (b(x)G(\tilde{u}^o), \tilde{u}^o) \tag{4.5}$$

From (4.2) we also obtain

$$\frac{1}{\varepsilon}M(0)u_{om} \to 0 \quad \text{strongly in} \quad (L^2(\Omega))^2 \quad \text{when} \quad m \to +\infty \tag{4.6}$$

Since $u_{\varepsilon m} \in C^1([0,t_{\varepsilon m}[;V_m)$ and (3.1) is valid for $\tilde{u}^o$ it follows that there exists $T_{o\varepsilon m}$ such that $0 < T_{o\varepsilon m} < t_{\varepsilon m}$ and

$$||u_{\varepsilon m}(t)|| < \left[\frac{\alpha}{C_o^{2(\rho+2)}}\right]^{1/2(\rho+1)} \quad \text{for} \quad t \in [0, T_{o\varepsilon m}[, \ m \geq m_o. \tag{4.7}$$

So, from (4.2)-(4.7) by using similar arguments as in Tartar [12], we have

$$||u_{\varepsilon m}(t)|| \leq \gamma < C_1, \quad |u'_{\varepsilon m}(t)|^2 < C_2, \quad \frac{1}{\varepsilon}|M(t)u_{\varepsilon m}(t)|^2 < C_2, \tag{4.8}$$

for $t \in [0, T_{o\varepsilon m}[$ and $m \geq m_1$, where $C_1 = \left(\frac{\alpha}{C_o^{2(\rho+2)}}\right)^{1/2(\rho+1)}$ and $C_2 \doteq \left(\frac{\alpha}{C_o^2}\right)^{(\rho+2)/(\rho+1)}\left(\frac{\rho+1}{\rho+2}\right)$.

By continuity of $u_{\varepsilon m}$ in $T_{o\varepsilon m}$ and (4.8) we can show that for all $t \in [0, t_{\varepsilon m}[$ and $m \geq m_1$:

$$||u_{\varepsilon m}(t)|| < C_1, \qquad |u'_{\varepsilon m}(t)|^2 < C_2, \qquad \frac{1}{\varepsilon}|M(t)u_{\varepsilon m}(t)|^2 < C_2. \tag{4.9}$$

Therefore we can extend the solutions to $[0,+\infty[$. One observes that the above constants are independent of ε and m, so there exists a subsequence of $(u_{\varepsilon m})$, still denoted by $(u_{\varepsilon m})$ such that

$$u_{\varepsilon m} \to u_\varepsilon \quad \text{weak-star in} \quad L^\infty(0,+\infty;(H_o^1(\Omega))^2) \tag{4.10}$$

$$u'_{\varepsilon m} \to u'_\varepsilon \quad \text{weak-star in} \quad L^\infty(0,+\infty;(L^2(\Omega))^2) \tag{4.11}$$

$$\frac{1}{\varepsilon}Mu_{\varepsilon m} \to \frac{1}{\varepsilon}Mu_\varepsilon \quad \text{weak-star in} \quad L^\infty(0,+\infty;(L^2(\Omega))^2) \tag{4.12}$$

To prove the convergence of nonlinear term of (4.1), first we show that they are bounded in $L^\infty(0,+\infty;(L^r(\Omega))^2)$ and then using (4.10), (4.11), compactness arguments (Lions [7]) and

Lions Lemma 1.3 op. cit., we conclude

$$b(x)G(u_{\varepsilon m}) \to b(x)G(u_\varepsilon) \quad \text{weak-star in} \quad L^\infty(0,\infty;(L^r(\Omega))^2) \tag{4.13}$$

From the convergences (4.10)-(4.13) and passing to the limit in (4.1) when $m \to +\infty$ it follows that

$$u_\varepsilon'' + A(t)u_\varepsilon + b(x)G(u_\varepsilon) + \frac{1}{\varepsilon} Mu_\varepsilon = f \quad \text{in} \quad \mathcal{D}'(0,+\infty;(H^{-1}(\Omega))^2) \tag{4.14}$$

One observes that the estimates (4.9) are also valid for u_ε, so there exists a subsequence, still denoted by (u_ε), which satisfy

$$u_\varepsilon \to u \quad \text{weak-star in} \quad L^\infty(0,+\infty;(H_o^1(\Omega))^2), \tag{4.15}$$

$$u_\varepsilon' \to u' \quad \text{weak-star in} \quad L^\infty(0,+\infty;(L^2(\Omega))^2), \tag{4.16}$$

Proceeding as in (4.12) we have

$$\frac{1}{\varepsilon} M(t)u_\varepsilon \to \chi_1 \quad \text{weak-star in} \quad L^\infty(0,+\infty;(L^2(\Omega))^2) \tag{4.17}$$

By (4.17) we see that $Mu = 0$. From this we conclude that $u = 0$ a.e. $\Omega\times]0,+\infty[\backslash Q$, so that $u \in L^\infty(0,+\infty;(H_o^1(\Omega_t))^2)$. Therefore, we obtain $u' = 0$ a.e. in $\Omega\times]0,+\infty[\backslash Q$. So, $u' \in L^\infty(0,+\infty;(L^2(\Omega_t))^2)$.

Multiplying the equation (4.14) by $\tilde{\phi} \in (\mathcal{D}(\Omega \times (0,+\infty)))^2$, where $\tilde{\phi}$ is the extension of $\phi \in (\mathcal{D}(Q))^2$ we obtain by (4.15), (4.16) and by definition of M, letting $\varepsilon \to 0$,

$$u'' + A(t)u + b(x)G(u) = f \quad \text{in} \quad (\mathcal{D}'(Q))^2 \tag{4.18}$$

Let $Q_o = \Omega_o\times]0,+\infty[\subset Q$. It follows by (4.18) that

$$u'' + A(t)u + b(x)G(u) = f \quad \text{in} \quad (\mathcal{D}'(Q_o))^2 \tag{4.19}$$

From u satisfying (3.3), (3.4) and (4.19) we obtain (3.5) and (3.8). From the convergences (4.15), (4.16) we obtain (3.7).

PROOF OF THEOREM 3.2. We only prove (3.12) because the other results follow as in Theorem 3.1. For the proof of (3.12) it suffices to show that the approximate solutions $u_{\varepsilon m}$ (m large enough) satisfy the decay estimate of the theorem with c and β independent of ε and m.

Proceeding as before, we obtain

$$||u_{\varepsilon m}(t)|| < C_3, \quad \text{for} \quad t \geq 0, \qquad m \geq m_1 \tag{4.20}$$

where $C_3 = \left(\frac{\alpha}{2C_o^{2(\rho+2)}}\right)^{1/2(\rho+1)}$

From Banach-Steinhaus's theorem we obtain the same estimates for u.

From the penalized problem associated to (3.11) it follows that

$$E_{\varepsilon m}(t) + \int_0^t |u_{\varepsilon m}'(s)|^2\, ds \geq E_m(0), \qquad m \geq m_1$$

where

$$E_{\varepsilon m}(t) = \frac{1}{2}\Big[|u'_{\varepsilon m}(t)|^2 + a(t, u_{\varepsilon m}(t), u_{\varepsilon m}(t)) + \\ + \frac{1}{\rho+2}\int_\Omega b(x)G(u_{\varepsilon m}(x,t)).u_{\varepsilon m}(x,t)dx + \frac{1}{\varepsilon}|M(t)u_{\varepsilon m}(t)|^2\Big]$$

Applying similar arguments as Theorem 3.1, we conclude that

$$E_{\varepsilon m}(t) \leq \left(\frac{\alpha}{2C_o^2}\right)^{\frac{\rho+2}{\rho+1}}\left[\frac{2\rho+3}{2\rho+4}\right], \quad m \geq m_1, \ t \geq 0 \tag{4.21}$$

and

$$E_{\varepsilon m}(t+1) + \int_t^{t+1} |u'_{\varepsilon m}(s)|^2\, ds \leq E_{\varepsilon m}(t) \tag{4.22}$$

Therefore, from (4.20) and (4.21) we have that $E_{\varepsilon m}(t) \geq 0$ for $t \geq 0$, $m \geq m_1$ and $E_{\varepsilon m}(t)$ is decreasing.

From (4.21), there exist $t_1 \in (t, t+1/4)$, $t_2 \in (t+3/4, t+1)$ such that for $m \geq m_1$,

$$|u_{\varepsilon m}(t_i)| \leq 2F_{\varepsilon m}(t), \quad i = 1, 2 \tag{4.23}$$

where $F^2_{\varepsilon m}(t) = E_{\varepsilon m}(t) - E_{\varepsilon m}(t+1)$.

Letting $z = u_{\varepsilon m}(t)$ in (4.1), we obtain

$$\int_{t_1}^{t_2} \frac{1}{2}\Big[a(s, u_{\varepsilon m}(s), u_{\varepsilon m}(s)) + \frac{1}{\rho+2}\int_\Omega b(x)G(u_{\varepsilon m}(x,s)).u_{\varepsilon m}(x,s)dx + \frac{1}{\varepsilon}|M(s)u_{\varepsilon m}(s)|^2\Big]ds \leq \\ \leq KF^2_{\varepsilon m}(t) + \frac{1}{4\rho+6}E_{\varepsilon m}(t) = \frac{1}{2}H^2_{\varepsilon m}(t) \tag{4.24}$$

From (4.22) and (4.24) we see that there exists a time $t^* \in (t_1, t_2) \subset (t, t+1)$ such that

$$E_{\varepsilon m}(t^*) \leq (2K+1)(E_{\varepsilon m}(t) - E_{\varepsilon m}(t+1)) + \frac{1}{2\rho+3}E_{\varepsilon m}(t) \tag{4.25}$$

Since $E_{\varepsilon m}(t)$ is monotone decreasing and $\rho > -1$ we have by (4.25)

$$E_{\varepsilon m}(t+1) - d_2E_{\varepsilon m}(t) \leq d_3(E_{\varepsilon m}(t) - E_{\varepsilon m}(t+1)),$$

where $0 < d_2 = \frac{1}{2\rho+3} < 1$ and $d_3 = 2K+1 > 0$.

Applying Lemma 2.1 we obtain the desired result.

From the boundedness (4.21) and Arzelá-Ascoli's Theorem it follows that for each $t_o \geq 0$ we have

$$u_{\varepsilon m}(t_o) \to u(t_o) \quad \text{weakly in} \quad (H_o^1(\Omega))^2$$

$$u'_{\varepsilon m}(t_o) \to u'(t_o) \quad \text{weakly in} \quad (L^2(\Omega))^2$$

and by (4.20) and Banach-Steinhaus's theorem we can conclude that

$$E(t) \leq c\,e^{-\beta t} \quad \text{for} \quad t > 0.$$

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