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SOLUTIONS TO LYAPUNOV STABILITY PROBLEMS: NONLINEAR SYSTEMS WITH CONTINUOUS MOTIONS

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Abstract. The necessary and sufficient conditions for accurate construction of a Lyapunov function and the necessary and sufficient conditions for a set to be the asymptotic stability domain are algorithmically solved for a nonlinear dynamical system with continuous motions. The conditions are established by utilizing properties of o-uniquely bounded sets, which are explained in the paper. They allow arbitrary selection of an o-uniquely bounded set to generate a Lyapunov function.

Simple examples illustrate the theory and its applications.

Key Words and Phrases: Stability, Lyapunov Method, Lyapunov Functions, Nonlinear Systems, Dynamical Systems.

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1. INTRODUCTION

In his fundamental dissertation [1] Lyapunov referred to papers by Poincaré [2], [3] as those inspiring him to establish a method that has become fundamental for qualitative and stability analysis of motions of a very general class of nonlinear systems.

The promising methodological effectiveness of the Lyapunov method has not been fully achieved due to the need to construct a system Lyapunov function. Significant results on a Lyapunov function generation were initiated by Zubov [14]. The literature on the Lyapunov method is too vast [9]-[11],[13],[14] to be referred to herein.

The problem of the necessary and sufficient conditions for constructing a Lyapunov function and the problem of the necessary and sufficient conditions for a set to be the asymptotic stability domain have not yet been solved. Solutions to these problems will be established by using properties of o-uniquely bounded sets. Their features will be explained briefly by referring to [7],[8], where they were discovered and studied.

2. NOTATION

$A, R^n \supseteq A$	- an open connected neighborhood of $x = 0$,
$B_\delta = \{x: \|x\| < \delta\}, R^n \supseteq B_\delta$,	- an open hyperball,
$\overline{B}_\delta = \{x: \|x\| \le \delta\}, R^n \supseteq \overline{B}_\delta$,	- the closure of B_δ,
$\partial B_\delta = \{x: \|x\| = \delta\}, R^n \supset \partial B_\delta$,	- the boundary of both B_δ and $\overline{B}_\delta$,
$C(S)$	- the set of all functions of x continuous on S,

$D_a, D_s, D, R^n \supseteq D_{(\cdot)}$,	- the domain of attraction, of stability, of asymptotic stability, respectively, of $x = 0$,
$D^+v(x) = \limsup\{\langle v[x(\theta;x)] - v(x)\rangle/\theta: \theta \to 0^+\}$	- the Dini derivative of v along the system motion (Yoshizawa [13]),
$E(S;f)$	- a family of functions determined by Definition 5,
$f: R^n \to R^n$	- a given nonlinear vector function,
$I_0, R_+ \supseteq I_0$,	- the largest subinterval of R_+ over which a motion $x(t;x_0)$ exists,
$n \in \{1,2,\ldots\}$	- the dimension of the system,
$N, R^n \supseteq N$,	- an open connected neighborhood of $x = 0$,
$\overset{\circ}{N}$	- the interior of N (in fact $\overset{\circ}{N} = N$),
R	- the set of real numbers,
R_+	$= [0, +\infty[= \{\alpha: \alpha \in R, 0 \leq \alpha < +\infty\}$,
$S, R^n \supseteq S$,	- an open neighborhood of $x = 0$,
$U, R^n \supset U$,	- an o-uniquely bounded set,
$u: R^n \to R$	- the generating function of the o-uniquely bounded set U,
$U_\zeta = \{x: u(x) < \zeta\}$	- a set generated by the function u and a positive number ζ,
$v: R^n \to R$	- a tentative Lyapunov function of the system,
$x: R_+ x R^n \to R^n$	- the system motion (solution), $x(t;x_0) = x(t)$, $x(0;x_0) = x_0$,
$\lVert \cdot \rVert: R^n \to R_+$	- Euclidean norm on R^n,
$\varnothing$	- the empty set.

3. SYSTEM DESCRIPTION

Systems to be analyzed are described by the following equation

$$\frac{dx}{dt} = f(x). \tag{3.1}$$

They are assumed to possess either of the following two features:

Weak Smoothness Property:

(i) There is an open neighborhood S of $x = 0$, $R^n \supseteq S$, such that for every $x_0 \in S$

(a) the system (1) has the unique solution $x(t;x_0)$ through x_0 at $t = 0$, and

(b) the motion $x(t;x_0)$ is defined and continuous in $(t,x_0) \in I_0 \times S$.

(ii) For every $x_0 \in (R^n - S)$ every motion $x(t;x_0)$ of the system (1) is continuous in $t \in I_0$.

Strong Smoothness Property:

(i) The system (1) has Weak Smoothness Property.

(ii) If the boundary ∂S of S is non-empty then every motion of the system (1) passing through $x_0 \in \partial S$ at $t = 0$ obeys $\inf[\lVert x(t;x_0)\rVert : t \in I_0] > 0$ for every $x_0 \in \partial S$.

4. DEFINITIONS

4.1 ON THE DEFINITIONS OF STABILITY DOMAINS

For the definitions of the attraction domain D_a see [4]-[6],[9],[11],[14]. The stability domain D_s and the asymptotic stability domain D of $x = 0$ are defined in [5],[6]. We shall refer to those definitions in the sequel.

For the system (1) with Weak Smoothness Property, the stability domains are mutually related as follows:

LEMMA 1. If the state $x = 0$ of the system (1) possessing Weak Smoothness Property has both the domain of attraction D_a, $S \supseteq D_a$, and the domain of stability D_s, then they and the asymptotic stability domain D are interrelated by

$$D_s \supseteq D_a, \quad D = D_a.$$

PROOF. Let $x = 0$ have D_a, $S \supseteq D_a$, and D_s. Then it has also D because $D = D_a \cap D_s$ and both D_a and D_s are neighborhoods of $x = 0$ [5],[6]. Let $x_0 \in D_a$. Then $\mathbf{x}(t;x_0) \to 0$ as $t \to +\infty$. This and continuity of $\mathbf{x}(t;x_0)$ in $t \in I_0$ (Weak Smoothness Property) imply $\max[\|\mathbf{x}(t;x_0)\| : t \in R_+] = \alpha < +\infty$. Let $\varepsilon = 2\alpha$. Hence, $\|\mathbf{x}(t;x_0)\| < \varepsilon$, $\forall t \in R_+$, which yields [5],[6] $x_0 \in D_s$ so that $D_s \supseteq D_a$ and $D = D_a = D_a \cap D_s$ [5],[6].

4.2 ON THE DEFINITION OF A POSITIVE DEFINITE FUNCTION

The notion of a positive definite function is used in a broader Lyapunov sense [1].

DEFINITION 1. A function $v: R^n \to R$ is a positive definite if and only if there is an open connected neighborhood A of $x = 0$, $R^n \supseteq A$, such that

1) $v(x)$ is uniquely determined by $x \in A$ and v is continuous on A: $v(x) \in C(A)$,

2) $v(0) = 0$, and

3) $v(x) > 0$ for every $(x \neq 0) \in A$.

4.3 DEFINITIONS AND PROPERTIES OF O-UNIQUELY BOUNDED SETS

O-uniquely bounded sets were introduced, defined and studied in [7],[8].

DEFINITION 2. A set U, $R^n \supset U$, is o-uniquely bounded if and only if it is bounded and for every $(x \neq 0) \in R^n$ there is exactly one positive number λ, $\lambda = \lambda(x;U)$, such that $(\lambda x) \in \partial U$.

DEFINITION 3. A function $u: R^n \to R$ is radially increasing on an open neighborhood N of $x = 0$ if and only if for every $(x \neq 0) \in N$ and any μ_i, $i = 1,2$, obeying both $0 \le \mu_1 < \mu_2$ and $\mu_i x \in N$ it satisfies $u(\mu_1 x) < u(\mu_2 x)$.

PROPERTY U. Let N be an open neighborhood of $x = 0$ and U, $N \supset \overline{U}$, be a given bounded set. There is a function $u: R^n \to R$ that obeys the following:

(a) u is continuous on N: $u(x) \in C(N)$,

(b) if $N = R^n$ then $u(x) \to +\infty$ as $\|x\| \to +\infty$,

(c) $u(0) = 0$,

(d) $u(x) > 0$ for all $(x \neq 0) \in N$,

(e) there is positive number ξ, $\xi = \xi(U)$, such that both 1. and 2. hold:

1. $u(x) \le \xi$ for $x \in N$ if and only if $x \in \overline{U}$,

2. $u(x) = \xi$ for $x \in N$ if and only if $x \in \partial U$,

(f) $u(\lambda_i x) = \xi$, $i = 1,2$, holds for any $(x \neq 0) \in N$ if and only if $\lambda_1 = \lambda_2 = \lambda(x;U) \in]0,+\infty]$,

(g) u is radially increasing on N.

Definition 2 implies the next result due to Definition 2, Corollary 1 and Proposition 4 in [8].

LEMMA 2. For a bounded subset U of an open neighborhood N of $x = 0$ to be o-uniquely bounded it is both necessary and sufficient that it possesses Property U.

DEFINITION 4. (i) A function u is the generating function on N of an o-uniquely bounded set U if and only if they have Property U.

(ii) The function u is the generating function of the uniquely bounded set U if and only if they obey (i) for $N = R^n$.

Lemma 2 and Definition 4 imply the following corollary [8].

COROLLARY 1. If a function u is the generating function on N of an o-uniquely bounded set U then for any $\zeta > 0$ for which $N \supseteq N_\zeta$ the subset U_ζ of N is a connected open neighborhood of $x = 0$ that is also an o-uniquely bounded set with the generating function u on N.

5. SOLUTIONS VIA O-UNIQUELY BOUNDED SETS

We shall make use of the family $E(S;f)$ defined as follows.

DEFINITION 5. A function $u: R^n \to R$ belongs to the family $E(S;f)$ if and only if

1) u is continuous on S; $u(x) \in C(S)$, and
2) the following equations along the motions of the system (3.1),

$$D^+v(x) = -u(x), \tag{5.1a}$$

$$v(0) = 0, \tag{5.1b}$$

have a solution v that is well defined in R and continuous for every $x \in \overline{B}_\mu$ for some $\mu \in]0, +\infty[, \mu = \mu(u,f)$.

THEOREM 1. In order for the state $x = 0$ of the system (1) with Strong Smoothness Property to have the domain D of asymptotic stability and for a set N, $R^n \supseteq N$, to be the domain of its asymptotic stability, $N = D$, it is both necessary and sufficient that

1) the set N is an open connected neighborhood of $x = 0$ and $S \supseteq N$,
2) $f(x) = 0$ for $x \in N$ if and only if $x = 0$, and
3) for arbitrarily selected o-uniquely bounded set U, $S \supset \overline{U}$, with the generating function u on S obeying $u \in E(S;f)$, the equations (5.1) have a unique solution function v on N with the following properties:
   (i) v is positive definite on N, and
   (ii) if the boundary ∂N of N is non-empty then $v(x) \to +\infty$ as $x \to \partial N, x \in N$.

PROOF. *Necessity.* Let $x = 0$ of the system (3.1) with Strong Smoothness Property have the asymptotic stability domain D. Definitions of D_a and D [5],[6] show that it has also the attraction domain D_a, $D_a \supseteq D$. It is a neighborhood of $x = 0$ due to Definition of D_a, and S is a neighborhood of $x = 0$ in view of the smoothness property. Hence, $D_a \cap S \neq \varnothing$. Let us prove $S \supseteq D_a$. If $\partial S = \varnothing$, then $S = R^n$ and $S \supseteq D_a$ due to $R^n \supseteq D_a$. If $\partial S \neq \varnothing$, then we shall consider both $x_0 \in \partial S$ and $x_0^* \in (R^n - \overline{S})$. If $x_0 \in \partial S$, then $x_0 \notin D_a$ due to (ii) of Strong Smoothness Property. Therefore, $\partial S \cap D = \varnothing$. If $x_0^* \in (R^n - \overline{S})$, then for $\mathbf{x}(t: x_0^*) \to 0$ as $t \to +\infty$ it is necessary that there is $t^* \in R_+$ such that $\mathbf{x}(t^*; x_0^*) \in \partial S$, because D and S are neighborhoods of $x = 0, x_0^* \notin \overline{S}$ and the motion $\mathbf{x}(t; x_0)$ is continuous in $t \in R_+$ due to (ii) of Weak Smoothness Property ensured by (i) of Strong Smoothness Property. However, $\mathbf{x}(t^*; x_0^*) \in \partial S$ implies that $\mathbf{x}(t; x_0)$ does not converge to $x = 0$ because of (ii) of Strong Smoothness Property. This yields $x_0^* \notin D$ and $(R^n - \overline{S}) \cap D = \varnothing$. By connecting the above results, that is $D_a \cap S \neq \varnothing, D_a \cap \partial S = \varnothing$ and $D_a \cap (R^n - \overline{S}) = \varnothing$, we conclude that $S \supseteq D_a$. Therefore, $D = D_a$ (Lemma 1) and $S \supseteq D$. Let $N = D$ so that $S \supseteq N$. Hence, N is open connected neighborhood of $x = 0$ due to (i-b) of Weak Smoothness Property, $N = D = D_a$, and invariance of D_a with respect to system motions (Theorem 1.5.14 by Bhatia and Szegö [4], Theorem 33.3 by Hahn [9]). This proves necessity of the condition 1). From $N = D = D_a$, $D_s \supseteq D_a$, and Definitions of D_a and D it results that $x = 0$ is the unique equilibrium state of the system (1) in N, which implies $f(x) = 0$ for $x \in N = D$ if and only if $x = 0$ (Proposition 7 in [6]) and proves necessity of the condition 2).

From $N = D$ it follows that the interval I_0 of existence of $\mathbf{x}(t;x_0)$ equals R_+, $I_0 = R_+$, for every $x_0 \in N$, due to Definitions of D_a, D_s and D [5],[6]. Let U be arbitrarily selected open o-uniquely bounded set such that $N \supset \overline{U}$ and its generating function u on S obeys $u \in E(S;f)$. Such a set U exists because S is open neighborhood of $x = 0$ (Lemma 2). Definition 3, Property U, and Lemma 2 show that the function u is also positive definite on S. Since $S \supseteq N = D$ then the function u is the positive definite generating function on N, too. The property of $u \in E(S;f)$ ensures existence of $\mu > 0$ such that there exists a solution function v to the equations (5.1), which is well defined in R and continuous for every $x \in \overline{B}_\mu$, that is that

$$|v(x)| < +\infty \quad \text{for every} \quad x \in \overline{B}_\mu \quad \text{and} \quad v(x) \in C(\overline{B}_\mu) . \tag{5.2}$$

Let $\zeta \in]0, +\infty[$ be such that

$$\overline{B}_\mu \cap U \supseteq \overline{U}_\zeta . \tag{5.3}$$

The existence of such ζ is assured by Corollary 1. Let $\tau \in [0, +\infty[$, $\tau = \tau(x_0;f;u;\zeta)$, be such that for any $x_0 \in N$ the following condition holds,

$$\mathbf{x}(t;x_0) \in U_\zeta \quad \text{for every} \quad t \in [\tau, +\infty[. \tag{5.4}$$

Such τ exists in view of Definitions of D_a and D, $D_a = D$, $N = D$ and $x_0 \in N$. Notice that $x_0 \in N$ implies also

$$\mathbf{x}(+\infty;x_0) = 0 . \tag{5.5}$$

After integrating (5.1a) from $t \in R_+$ to $+\infty$ we derive

$$v[\mathbf{x}(+\infty;x_0)] - v[\mathbf{x}(t;x_0)] = - \int_t^{+\infty} u[\mathbf{x}(\sigma;x_0)]d\sigma \quad \text{for every} \quad (t,x_0) \in R_+ x N . \tag{5.6}$$

Since $u \in E(S;f)$ then the following holds,

$$v(0) = 0 . \tag{5.7}$$

Now, (5.5)-(5.7) yield

$$v[\mathbf{x}(t;x_0)] = \int_t^{+\infty} u[\mathbf{x}(\sigma;x_0)]d\sigma \quad \text{for every} \quad (t,x_0) \in R_+ x N . \tag{5.8}$$

This can be written in the following form,

$$v[\mathbf{x}(t;x_0)] = \int_t^{+\infty} u[\mathbf{x}(\sigma;x_0)]d\sigma \quad \text{for every} \quad (t,x_0) \in R_+ x N . \tag{5.9}$$

Positive invariance of D with respect to system motions, $N = D$, continuity of the motions $\mathbf{x}$ due to the smoothness property, continuity of u on N, the definition of τ (5.4) and (5.2), and compactness of $[\tau, t]$ for any $t \in R_+$ prove

$$\left| \int_t^{\tau} u[\mathbf{x}(\sigma;x_0)]d\sigma \right| < +\infty \quad \text{for every} \quad (t,x_0) \in R_+ x N . \tag{5.10}$$

From (5.2)-(5.4) we obtain

$$\left| \int_\tau^{+\infty} u[\mathbf{x}(\sigma;x_0)]d\sigma \right| < +\infty \quad \text{for every} \quad x_0 \in N . \tag{5.11}$$

(5.9)-(5.11) together prove boundedness of $v[\mathbf{x}(t;x_0)]$ expressed as

$$|v[\mathbf{x}(t;x_0)]| < +\infty \quad \text{for every} \quad (t,x_0) \in R_+ x N . \tag{5.12}$$

Hence, by setting $t = 0$ and $x_0 = x$ in (5.12) we derive

$$|v(x)| < +\infty \quad \text{for every} \quad x \in N . \tag{5.13}$$

Continuity of the motion $\mathbf{x}$ in $x_0 \in N$, continuity of u in $x \in N$, and of v in $x \in \overline{B}_\mu$, $\overline{B}_\mu \supseteq \overline{U}_\xi$, positive invariance of $N = D$ with respect to system motions, (5.4), (5.9) and (5.12) prove continuity of v in $x \in N$

$$v(x) \in C(N). \tag{5.14}$$

Positive invariance of N with respect to system motions, positive definiteness of u on N and (5.8) imply

$$v(x) > 0 \quad \text{for all} \quad (x \neq 0) \in N. \tag{5.15}$$

Now, (5.7), (5.14) and (5.15) prove necessity of the positive definiteness of v on N.

To prove uniqueness of the solution v to (5.1.*ab*) we shall suppose that there are two solutions v_1 and v_2 to (5.1). Hence,

$$v_1(x_0) - v_2(x_0) = \int_0^{+\infty} \{u[\mathbf{x}_1(\sigma;x_0)] - u[\mathbf{x}_2(\sigma;x_0)]\}d\sigma \quad \text{for every} \quad x_0 \in N. \tag{5.16}$$

Since $u(x)$ is uniquely determined by $x \in N$, due to (a) of Property U and Definition 4, and the motion of the system is unique through x_0, $\mathbf{x}_1(\sigma;x_0) = \mathbf{x}_2(\sigma;x_0)$ and $u[\mathbf{x}_1(\sigma;x_0)] = u[\mathbf{x}_2(\sigma;x_0)]$ so that $v_1(x_0) - v_2(x_0) = 0$ for every $x_0 \in N$. This proves uniqueness of the solution v to (5.1) and completes the proof of 3(i).

Let ∂N be non-empty, $x_1, x_2, \ldots, x_k, \ldots$ be a sequence converging to x', $x_k \to x'$ as $k \to +\infty$, where $x_k \in N$, for all $k = 1,2,\ldots$, and $x' \in \partial N$. Let $\xi \in]0,+\infty[$ be arbitrarily chosen so that $U_\xi = \{x: u(x) < \xi\}$, $U \supseteq U_\xi$. Such ξ exists because the set U is o-uniquely bounded and the function u is its generating function on N (Definitions 2 and 3, Property U, Lemma 2 and Definition 4). The set U_ξ is a connected open neighborhood of $x = 0$ (Corollary 1). Let T_k, $T_k = T(x_k, \xi) \in [0,+\infty[$, be the first instant obeying the following

$$\mathbf{x}(t;x_k) \in \overline{U}_\xi \quad \text{for all} \quad t \in [T_k, +\infty[. \tag{5.17}$$

The existence of such T_k is guaranteed by $x_k \in N$ and $N = D$ (Definitions of D_a and D [5], [6]). Continuity of the motions $\mathbf{x}$ in $(t,x_0) \in R_+ x N$ due to Strong Smoothness Property and $\overset{\circ}{N} = N = D$ (Theorem 33.1 by Hahn [9]) and $S \supseteq D$ imply $T_k \to +\infty$ as $k \to +\infty$ (Theorem 33.2 by Hahn [9]). Let m be a natural number such that $x_k \in (N - \overline{U}_\xi)$ for all $k = m, m+1, \ldots$. Such m exists because N is open, $N \supset \overline{U}_\xi$ and $x_k \to \partial N$ as $k \to +\infty$. Let α' be defined by (18),

$$\alpha' = \min[u(x): x \in (N - U_\xi)]. \tag{5.18}$$

The o-unique boundedness of the set U_ξ, the fact that the function u is its generating function on N (Corollary 1), $N \supset \overline{U}$, and $U \supseteq U_\xi$ guarantee (Property U and Lemma 2) that α' defined by (5.18) satisfies

$$\alpha' = \xi \in]0,+\infty[. \tag{5.19}$$

From (5.9) we get, after replacing τ by T_k,

$$v[\mathbf{x}(t;x_k)] = \int_t^{Tk} u[\mathbf{x}(\sigma;x_k)]d\sigma + \int_{Tk}^{+\infty} u[\mathbf{x}(\sigma;x_k)]d\sigma \quad \text{for every} \quad (t,x_k) \in R_+ x N,$$

$$\text{and for} \quad k = m, m+1, \ldots . \tag{5.20}$$

Setting $t = 0$ in (5.20) and using (5.18) and (5.19) we derive

$$v(x_k) \geq \int_0^{Tk} \xi d\sigma + \int_{Tk}^{+\infty} u[\mathbf{x}(\sigma;x_k)]d\sigma \quad \text{for} \quad x_k \in N \quad \text{and all} \quad k = m, m+1, \ldots . \tag{5.21}$$

Positive invariance of $N = D$ with respect to system motions, positive definiteness of u on N, and (5.21) imply

$$v(x_k) \geq \xi T_k \quad \text{for} \quad x_k \in N \quad \text{and all} \quad k = m, m+1, \ldots . \tag{5.22}$$

Since $T_k \to +\infty$ as $k \to +\infty$, the last inequality, the definitions of T_k, $T_k = T(x_k, \xi)$, and of x_k, and $\alpha > 0$ imply

$$v(x_k) \to +\infty \quad \text{as} \quad x_k \to \partial N \quad \text{due to} \quad k \to +\infty, \quad x_k \in N,$$

which proves necessity of the condition (3-ii).

Sufficiency. Let all the conditions of Theorem 1 hold. Then, $S \supseteq N$. Two possible cases will be considered separately: a) N is a bounded set, b) N is an unbounded set.

a) Let N be a bounded set. Then, under the conditions of the theorem to be proved all the conditions of Theorem 1 by Vanelli and Vidyasagar [12] are satisfied, which proves $N = D_a$. Since $D_a = D$ (in view of Weak Smoothness Property implied by Strong Smoothness Property and Lemma 1), $N = D$.

b) Let N be an unbounded set. Under the conditions of the theorem to be proved the zero state $x = 0$ of the system (1) is asymptotically stable (cf. Yoshizawa [13]). Hence, it has the domain of asymptotic stability D. Since both N and D are open connected neighborhoods of $x = 0$,

$$N \cap D \neq \varnothing . \tag{5.23}$$

Since $S \supseteq N$, S is also unbounded. If ∂S is empty, then $S = R^n$, which implies $S \supseteq D$. If ∂S is non-empty, then $\partial S \cap D = \varnothing$ due to (ii) of Strong Smoothness Property and Definitions of D_a, D_s and D [5],[6]. This result implies $S \supseteq D$ because both D and S are neighborhoods of $x = 0$ and D is also connected. Altogether, in both cases $S \supseteq D$. We shall treat separately the cases of non-empty ∂D and of empty ∂D. The definition of the function v, $S \supseteq D$, and the proof of the necessity part prove continuity of v on D and $v(x) \to +\infty$ as $x \to \partial D$, which together with continuity of v also on N, $S \supseteq N$ and $v(x) \to +\infty$ as $x \to \partial N$ [the condition 3(ii)] imply both

$$\partial D \cap N = \varnothing \quad \text{and} \quad D \cap \partial N = \varnothing .$$

These equations and (5.23) prove both $\partial D = \partial N$ and $D = N$ due to the fact that both D and N are open connected neighborhoods of $x = 0$. Let now ∂D be empty. Then $D = R^n$. Hence, v is positive definite on R^n (see the proof of the necessity part). Thus, it is continuous on R^n, which implies $v(x) < +\infty$ for every $x \in R^n$. Therefore, $\partial N \cap R^n = \varnothing$ due to the conditions 3(ii), which yields $N = R^n$ so that $N = D$. Finally, $N = D$ in all the cases, which completes the proof. ▮

The conditions slightly change if the system (3.1) possesses Weak Smoothness Property rather than Strong Smoothness Property.

THEOREM 2. For the state $x = 0$ of the system (1) possessing Weak Smoothness Property to have the domain D of asymptotic stability and that a subset N of S, $S \supseteq N$, equals $D : N = D$, it is both necessary and sufficient that

1) the set N is an open connected neighborhood of $x = 0$,
2) $f(x) = 0$ for $x \in N$ if and only if $x = 0$, and
3) for arbitrarily selected o-uniquely bounded set U, $S \supset \overline{U}$, with the generating function u on R^n obeying $u \in E(S; f)$, the equations (5.1) have a unique solution function v on N with the following properties:
   (i) v is positive definite on N, and
   (ii) if the boundary ∂N of N is non-empty then $v(x) \to +\infty$ as $x \to \partial N$, $x \in N$.

PROOF. *Necessity.* Let the system (3.1) possess Weak Smoothness Property. Let $x = 0$ have the asymptotic stability domain D, $S \supseteq D$, and let N, $S \supseteq N$, be equal to D. Let an o-uniquely bounded set U, $S \supset U$, with the generating function u obeying $u \in E(S;f)$, be arbitrarily selected. From this point on we have to repeat the proof of the necessity part of Theorem 1 to show that the conditions 1)-3) of Theorem 2 hold. In that way we complete the proof of the necessity part.

Sufficiency. Let the system (3.1) possess Weak Smoothness Property and the conditions 1)-3) be valid. Then $x = 0$ of the system (3.1) is asymptotically stable [1]. Therefore, $x = 0$ has the domain of asymptotic stability (Definitions of D_a, D_s and D [5],[6]). Let $x_0 \in (R^n - \overline{N})$. Since $\mathrm{x}(t; x_0)$ is continuous in $t \in I_0$, then it can enter N iff it passes through ∂N. But $v(x) \to +\infty$ as $x \to \partial N$, $x \in N$ [the condition 3(ii)]. This and $D^+v(x) < 0$ for $x \in (R^n - N)$ in view of positive definiteness of u on R^n and (5.1a), show that $\mathrm{x}(t; x_0)$ cannot reach ∂N. Hence, $\mathrm{x}(t; x_0) \in (R^n - \overline{N})$ for all $t \in I_0$. Therefore, $\overline{N} \supset D$. Furthermore, (5.1a) and positive definiteness of u on R^n imply (see the proof of the necessity part of Theorem 1) $v(x) \to +\infty$ as $x \to \partial D$, $x \in D$, which together with the condition 3(i) proves $\partial D \cap N = \varnothing$. This result, $\overline{N} \supset D$, and the fact that D and N are non-empty open connected neighborhoods of $x = 0$ imply $D = N$ and complete the proof. ∎

The properties of the generating function u of an o-uniquely bounded set U are essential for the accurate one-shot determination of the asymptotic stability domain. However, such properties are not needed for asymptotic stability of $x = 0$ only. This is clarified by the next result.

THEOREM 3. For the state $x = 0$ of the system (3.1) possessing Weak Smoothness Property to be asymptotically stable it is both necessary and sufficient that for any positive definite function $p \in E(S;f)$ there exists a unique solution function v to (5.24) with (5.24*a*) determined along system motions,

$$D^+v(x) = -p(x), \tag{5.24a}$$

$$v(0) = 0, \tag{5.24b}$$

which is also positive definite.

PROOF. *Necessity.* Let the system (3.1) possess the Weak Smoothness Property. Let $x = 0$ be asymptotically stable. Then it has D_a, D_s and D, and $D_a \cap S \neq \varnothing$, $D_s \cap S \neq \varnothing$ and $D \cap S \neq \varnothing$, because D_a, D_s, D and S are neighborhoods of $x = 0$. Let $p \in E(S;f)$ be an arbitrarily selected positive definite function (Definition 1). Such properties of p and its membership to $E(S;f)$ guarantee existence of a solution v to the equations (5.24), which is well defined in R and continuous (see the proof of the necessity part of Theorem 1) on the set A determined in Definition 1. The set $L = A \cap D$, $D \supseteq L$, is also an open connected neighborhood of $x = 0$ (see the proof of Theorem 1 for such a property of D). Let ε satisfying $L \supseteq B_\varepsilon$ be arbitrarily selected. Then $D \supseteq B_\varepsilon$. Let $\rho \in]0, \varepsilon[$ obeying $D_s(\varepsilon) \supseteq B_\rho$ be also arbitrarily selected, where $D_s(\varepsilon)$ is defined [5],[6] as the neighborhood of $x = 0$ such that $\| \mathbf{x}(t;x_0) \| < \varepsilon$ for all $t \in R_+$ holds iff $x_0 \in D_s(\varepsilon)$. By following the proofs of (5.13) and (5.14), we prove that v, defined by (5.24), has the following properties since $A \supseteq L \supseteq B_\varepsilon \supseteq D_s(\varepsilon) \supseteq B_\rho$,

$$|v(x)| < +\infty \quad \text{for every} \quad x \in B_\rho, \tag{5.25a}$$

$$v(x) \in C(B_\rho). \tag{5.25b}$$

Notice that $D_s(\varepsilon) \supseteq B_\rho$ and the definitions of $D_s(\varepsilon)$ and D guarantee [5],[6] $\mathbf{x}(t;x_0) \in B_\varepsilon$ for every $(t, x_0) \in R_+ x B_\rho$. This result, $A \cap D \supseteq B_\varepsilon$, positive definiteness of the function p on A, $\mathbf{x}(+\infty;x_0) = 0$ for every $x_0 \in B_\rho$ (because $D \supseteq B_\rho$) and (5.24*a*), integrated from $t = 0$ to $t = +\infty$, together with (5.24*b*) prove (5.26),

$$v(x_0) > 0 \quad \text{for every} \quad (x_0 \neq (0)) \in B_\rho. \tag{5.26}$$

Now, (5.24*b*) through (5.26) prove positive definitness of the solution v to (5.24) on B_ρ. Its uniqueness is proved in the same way as in the proof of Theorem 1, which completes the proof of the necessity part.

Sufficiency. Sufficiency of the conditions of Theorem 3 for asymptotic stability of $x = 0$ of the system (3.1) with Weak Smoothness Property is well known [13]. This completes the proof of Theorem 3. ∎

6. EXAMPLES

Example 1. Let $n = 1$,

$$\frac{dx}{dt} = -x + h(x), \quad h(x) = \begin{cases} x\,|x| & \text{for} \quad |x| \in [0,1], \\ x(|x|)^{1/2} & \text{for} \quad |x| \in [1, +\infty[\end{cases} \tag{6.1}$$

The system possesses Strong Smoothness Property because $f(x) = -x + h(x)$ is Lipschitzian on R^1. The equilibrium states are $x_{e1} = -1$, $x_{e2} = 0$ and $x_{e3} = +1$. They suggest $S =]-1, +1[$ and $U = \{x : x \in R^1, |x| < \alpha\} =]-\alpha, +\alpha[$, for $\alpha \in]0, 1[$. The generating function u on N, $u(x) = |x|$, of the o-uniquely bounded set U and (5.1ab) yield

$$D^{+}v(x) = -|x|, \quad x \in S.$$

The solution v to this equation is

$$v(x) = -\ln(1-|x|), \quad x \in S. \tag{6.2}$$

The function v(27) and the set $N = S =]-1,+1[$ satisfy all the requirements of Theorem 1, that is that,

1) $N =]-1,+1[$ is an open connected neighborhood of $x = 0$ and $N = S$,
2) $f(x) = -x + h(x) = 0$ for $x \in N$ iff $x = 0$,
3) (i) $v(x) = 0$ for $x \in N$ iff $x = 0$, $v(x) \in C(N)$, and $v(x) > 0$ for every $(x \neq 0) \in N$, which prove positive definiteness of v on N,
   (ii) $v(x) \to +\infty$ as $x \to \partial N = \{-1,+1\}, x \in N$.

Hence $N =]-1,+1]$ is the domain D of asymptotic stability of $x = 0$,

$$D =]-1,+1[.$$

Notice that $|f(x)| = |x|\,|1-|x||, x \in N$, is not a generating function on N of any o-uniquely bounded set because it is not radially increasing on N.

Example 2. Let the function h be defined as in Example 1 and

$$\frac{dx}{dt} = x - h(x). \tag{6.3}$$

It is clear that the system possesses Strong Smoothness Property on R^1 and has the equilibrium states $x_{e1} = -1$, $x_{e2} = 0$ and $x_{e3} = +1$ (see Example 1). Let, again, $U = \{x : x \in R^1, |x| < \alpha\} =]-\alpha,+\alpha[$ so that $u(x) = |x|$. From (5.1a) we get

$$D^{+}v(x) = -|x|, \quad x \in N.$$

Integrating this equation along motions of the system (6.3) we derive

$$v(x) = \ln(1-|x|), \quad x \in N,$$

which is negative definite on N and, thus, does not satisfy the necessary and sufficient conditions for asymptotic stability of $x = 0$ of the system (6.3). Hence, $x = 0$ of the system (6.3) is not asymptotically stable and does not have the asymptotic stability domain.

Example 3. Let $n = 2$ and

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_1(1+|x_1|x_2^2)(1-|x_1|)\\ -x_2(1-x_1^2|x_2|)(1-|x_2|)\end{bmatrix} = f(x). \tag{6.4}$$

The function f is globally Lipschitz continuous. The system has Strong Smoothness Property on R^2. The set S_e of its equilibrium states is determined by

$$S_e = \{x : x \in R^2, (x = 0) \text{ or } (|x_1| = 1, |x_2| = 1)\}.$$

This suggests $S = \{x : x \in R^2, |x_1| < 1, |x_2| < 1\}$. The system (6.4) has Weak Smoothness Property on S. Let $U = \{x : x \in R^2, |x_1| + |x_2| < \alpha\}$, $\alpha \in]0,1[$, so that U is o-uniquely bounded set with the generating function u on R^2 defined by $u(x) = |x_1| + |x_2|$, which together with (5.1) and (6.4) yields

$$v(x) = -\ln[(1-|x_1|)(1-|x_2|)].$$

The function v and the set $N = S$ obey all the conditions of Theorem 2. Therefore, $x = 0$ of the system (6.4) is asymptotically stable with the domain D of its asymptotic stability obtained as $D = N = S$, that is that

$$D = \{x : x \in R^2, |x_1| < 1, |x_2| < 1\}.$$

7. CONCLUSION

The necessary and sufficient conditions for asymptotic stability of the zero equilibrium state and for a set to be the domain of its asymptotic stability are proved in an algorithmic form that enables accurate construction of a system Lyapunov function. If a function v obtained from $D^+v = -u$ for an arbitrarily chosen u, which is a generating function of an o-uniquely bounded set, is not positive definite then the zero state is not asymptotically stable. There is no sense to try with another function u. However, if so derived function v is positive definite then the zero state is asymptotically stable. In this way the problem of an algorithm to construct accurately and directly a system Lyapunov function has been solved. However, it imposes other very complex mathematical problems: the problem of finding conditions on u guaranteeing existence of well defined and continuous v satisfying (5.1) on anyhow small neighborhood $\overline{B}_\mu$ of $x = 0$, and the problem of solving (5.1). These problems have not been solved.

Theorems of the paper open and initiate new directions in the Lyapunov stability analysis.

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