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CR-HYPERSURFACES OF COMPLEX PROJECTIVE SPACE

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ABSTRACT. We consider compact n-dimensional minimal foliate CR-real submanifolds of a complex projective space. We show that these submanifolds are great circles on a 2-dimensional sphere provided that the square of the length of the second fundamental form is less than or equal to $n-1$.

KEY WORDS AND PHRASES. Kaehler manifold, CR-submanifold, mixed foliate, hypersurfaces of complex projective space.
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1. INTRODUCTION.

CR-submanifolds of a Kaehlerian manifold have been defined by A. Bejancu [1]. These manifolds have then been studied by several authors. Among these are B.Y. Chen [2],[3], K. Yano, M. Kon, K. Sekigawa, and A. Ross [4].

In particular CR-submanifolds isometrically immersed in complex projective space have been considered by K. Yano and M. Kon [6]. They studied CR-submanifolds isometrically immersed in complex projective space with geometric properties such as semi-flat normal connection or parallel mean curvature. In this paper we consider minimal proper CR-hypersurfaces of a complex projective space. for such submanifolds we have obtained the following:

THEOREM 1. Let M be a compact n-dimensional minimal foliate CR-real hypersurface of a complex projective space. If the square of the length of the second fundamental form is $\leq (n-1)$, then M is a totally real submanifold of dimension 1. In fact M is a great circle on S^2.

2. PRELIMINARIES.

A submanifold M of a Kaehler manifold is called a CR-submanifold if there is a differentiable distribution $D: x \to D_x \subseteq T_xM$ on M satisfying the following conditions:

(a) D is holomorphic i.e., $JD = D$ for each $x \in M$, where J is the almost complex structure.

(b) The complementary orthogonal distribution $\overset{\perp}{D}: x \to \overset{\perp}{D}_x \subseteq T_xM$ is totally real i.e., $J\overset{\perp}{D} \subseteq \overset{\perp}{T}_xM$ where $\overset{\perp}{T}M$ is the normal bundle. If $\dim \overset{\perp}{D}_x = 0$ (respectively, $\dim D_x = 0$), M is called a complex (respectively totally real) submanifold. A CR-submanifold is said to be proper if it is neither complex nor totally real. The normal bundle T_xM splits as $T_xM = J\overset{\perp}{D} \oplus \mu$, where μ is invariant sub-bundle of T_xM under J.

Now let $\bar{M}$ be the complex projective space, which is a Kaehler manifold with constant holomorphic sectional curvature 4. Let g be the Hermitian metric tensor field of $\bar{M}$. Suppose that M is an n-dimensional CR-hypersurface of $\bar{M}$. We denote by the same g the Riemannian metric tensor field induced on M from that of $\bar{M}$. Let ∇, $\bar{\nabla}$, $\overset{\perp}{\nabla}$ be the Riemannian connections on M, $\bar{M}$ and the normal bundle respectively. Then we have Gauss formula and Weingarten formula;

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.1}$$

$$\bar{\nabla}_X N = -A_N X, \qquad N \in \overset{\perp}{T} M \tag{2.2}$$

where $h(X,Y)$ and $A_N X$ are the second fundamental forms which are related by

$$g(h(X,Y),N) = g(A_N X, Y) \tag{2.3}$$

where X and Y are vector fields on M.

We also have the following Gauss equation

$$R(X,Y;Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W)$$
$$+ 2g(X,JY)g(JZ,W) + g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)) \tag{2.4}$$

where $R(X,Y;Z,W)$ is the Riemannian curvature tensor of type $(0,4)$.
Let $H = \frac{1}{n}$ (trace h) be the mean curvature vector. Then M is said to be minimal if $H = 0$.
A CR-submanifold is said to be mixed foliate if
(a) the holomorphic distribution D is integrable.
(b) $h(X,\xi) = 0$ for $X \in D$ and $\xi \in \overset{\perp}{D}$.
For mixed foliate submanifolds of a complex space form $\bar{M}(c)$ (i.e., a Kaehler manifold of constant holomorphic sectional curvature c), the following result is well known

THEOREM 2.[3] If M is a mixed foliate proper CR-submanifold of a complex space form $\bar{M}(c)$, then we have $c \leq 0$.

3. CR-HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE.

We consider an n-dimensional proper CR-hypersurface M of a complex projective space $\bar{M}$. Then it follows that $dim\ \overset{\perp}{D} = 1$. Now assume that M is minimal and the holomorphic distribution D is integrable. If $(\bar{e}_i), i = 1,....,2p$ is an orthonormal basis for D, where $2p = dim\ \overset{\perp}{D}$, then $\sum_{i=1}^{2p} h(e_i,e_i) = \underline{0}$. Since M is minimal we get $h(\xi,\xi) = 0$ for ξ a unit vector in $\overset{\perp}{D}$. Note that $\nabla_X \xi \in D$. Then using the equation $\bar{\nabla}_X J\xi = J\bar{\nabla}_X \xi$ and equations (2.1) and (2.2) we have for $X \in D$

$$\nabla_X \xi = JAX - h(X,\xi) \tag{3.1}$$

Also the equation $\bar{\nabla}_\xi J\xi = J\bar{\nabla}_\xi \xi$ with $h(\xi,\xi) = 0$ and equations (2.1) and (2.2) yields

$$\nabla_\xi \xi = JA\xi \tag{3.2}$$

Let $(e_i), i = 1,....,n$ be an orthonormal basis for M, where $e_i = \bar{e}_i$ for $i = 1,....,2p$ and $e_n = \xi$. $n = 2p+1$. Since A is symmetric and J is skew symmetric we get

$$g(JAe_i,e_i) = -g(JAJe_i,Je_i). \tag{3.3}$$

Then using (3.1), (3.2), and (3.3) we compute

$$div\xi = \sum_{i=1}^{n} g(\nabla_{e_i}\xi,e_i) = \sum_{i=1}^{2p} g(\nabla_{e_i}\xi,e_i) = \sum_{i=1}^{p} \{g(JAe_i,e_i) + g(JAJe_i,Je_i)\} = 0 \tag{3.4}$$

For any vector field X on M we have [5]

$$div(\nabla_X X) - div(divX)X) = S(X,X) + \frac{1}{2}|L_X g|^2 - |\nabla X|^2 - (divX)^2 \tag{3.5}$$

where S is the Ricci tensor and $L_X g$ is the Lie differentiation with respect to a vector field X, defined by

$$(L_X g)(Y,Z) = g(\nabla_X Y, Z) + g(\nabla_X Z, Y)$$

Using (3.4) in (3.5) with $X = \xi$ we get

$$div(\nabla_\xi \xi) = S(\xi,\xi) + \frac{1}{2}|L_\xi g|^2 - |\nabla \xi|^2 \tag{3.6}$$

From Gauss equation (2.4) and the fact that $h(\xi,\xi) = 0$ we have

$$S(\xi,\xi) = (n-1)g(\xi,\xi) - \sum_{i=1}^{n} g(h(e_i,\xi),h(e_i,\xi)) = (n-1) - \sum_{i=1}^{n} g(h(e_i,\xi),J\xi)g(h(e_i,\xi),J\xi)$$

$$= (n-1) - \sum_{i=1}^{n} g(A\xi,e_i)g(A\xi,e_i) = (n-1) - g(A\xi,A\xi) = (n-1) - g(A^2\xi,\xi) \tag{3.7}$$

Using (3.1) and (3.2) we also have

$$|\nabla\xi|^2 = \sum_i g(\nabla_{e_i}\xi, \nabla_{e_i}\xi) = \sum_{i,j} g(\nabla_{e_i}\xi,e_j)g(\nabla_{e_i}\xi,e_j) = \sum_{i,j} g(JAe_i,e_j)g(JAe_i,e_j)$$

$$= \sum_i g(JAe_i,JAe_i) - \sum_i g(JAe_i,J\xi)\, g(JAe_i,J\xi) = traceA^2 - \sum_i g(A\xi,e_i)g(A\xi,e_i)$$

$$= traceA^2 - g(A\xi,A\xi) = traceA^2 - g(A^2\xi,\xi) \tag{3.8}$$

From (3.6), (3.7), and (3.8) we obtain

$$div(\nabla_\xi \xi) = (n-1) - traceA^2 + \frac{1}{2}|L_\xi g|^2 \tag{3.9}$$

PROOF. Using equation (3.9) and the assumption that M is compact we have

$$2\int_M [(n-1) - trA^2]dv = -\int_M |L_\xi g|^2 dv \tag{3.10}$$

From the hypothesis of Theorem and equation (3.10), we have $|L_\xi g| = 0$. Hence

$$0 = (L_\xi g)(JX,\xi) = g(\nabla_{JX}\xi,\xi) + g(\nabla_\xi \xi, JX) = g(\nabla_\xi \xi, JX)$$

Using equation (3.2) in the above equation we get $h(X,\xi) = 0$ i.e., M is mixed foliate. Since the holomorphic sectional curvature c of the complex projective space $\bar{M}$ equals 4, then by theorem (2) M cannot be proper mixed foliate. Therefore M is either totally real or holomorphic. But since $dim \overset{\perp}{D} = 1, M$ cannot be holomorphic. Therefore M is totally real. Since M is a hypersurface this implies that $dim\ M = 1$ and $dim\ \bar{M} = 2$. Now using the assumption that $tr.A^2 \leq n-1$ and $dimM = 1$ we have $tr.A^2 = 0$ i.e., M is totally geodesic. Since $dim\ \bar{M} = 2$ i.e., $\bar{M}$ is $S^2(\equiv CP)$, then M totally geodesic implies that M is a great circle S^1 on S^2.

NOTE: It has been pointed out to us that the result in this theorem might be in conflict with Proposition 2.3 of Maeda, Y., "On real hypersurfaces of a complex projective space," J. Math. Soc. Japan, Vol. 28, No. 3.3 (1976), 529-540. We could not detect any mistakes in our proof, but we shall investigate this point later.

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