Internat. J. Math. & Math. Sci.
VOL. 17 NO. 4 (1994) 817-820

SEVERAL IDENTITIES IN STATISTICAL MECHANICS

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(Received March 16, 1993)

ABSTRACT. In an earlier paper concerning a solvable model in statistical mechanics, Miwa and Jimbo state a theta-function identity which they have checked to the 200th power, but of which they do not have a proof. The main objective of this note is to provide such a proof.

KEY WORDS AND PHRASES. q-series identity, Jacobi's triple product, sums and products.
1991 AMS SUBJECT CLASSIFICATION CODE(S). 83B23.

1. INTRODUCTION.

In their paper [1], Miwa and Jimbo state a theta function identity, which they have checked to the 2000th power, but of which they do not have a proof. They also state, and prove, two other theta function identities. The object of this note is to provide a proof of the first identity, and to make the observation that the other two identities are special cases of an identity of mine that generalizes the quintuple-product identity.

2. MAIN RESULTS.

Our only tool in proving the first identity is Jacobi's triple product identity,

$$(-aq;q^2)_\infty(-a^{-1}q;q^2)_\infty(q^2;q^2)_\infty = \sum a^n q^{n^2}. \tag{2.1}$$

[Here, $(a;q)_\infty = \prod_{n\geq 1}(1-aq^{n-1})$, and throughout this paper all sums are taken from $-\infty$ to ∞.]

The identity to be proved ([1], (B.13)) is

$$\begin{aligned}&\left(\sum q^{(3n^2-3n)/2}\right)\left(\sum(-1)^n q^{6n^2-4n+1}\right)+\left(\sum q^{(3n^2-n)/2}\right)\left(\sum(-1)^n q^{6n^2}\right)\\ &=\left(\sum q^{n^2}\right)\left(\sum(-1)^n q^{6n^2-n}+\sum(-1)^n q^{6n^2-5n+1}\right)\end{aligned} \tag{2.2}$$

Our first step is to make use of (2.1) to express all the sums in (2.2) as products.

Thus

$$\sum q^{(3n^2-3n)/2} = 2(-q^3;q^3)^2_\infty(q^3;q^3)_\infty$$

$$\sum(-1)^n q^{6n^2-4n} = (q^2;q^{12})_\infty(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty$$

$$\sum q^{(3n^2-n)/2} = (-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty$$

$$\sum(-1)^n q^{6n^2} = (q^6;q^{12})^2_\infty(q^{12};q^{12})_\infty$$

$$\sum q^{n^2} = (-q;q^2)^2_\infty (q^2;q^2)_\infty$$

and

$$\begin{aligned}\sum(-1)^n q^{6n^2-n} + \sum(-1)^n q^{6n^2-5n+1} &= \sum(-1)^n(-q)^{(3n^2-n)/2} \\ &= \prod_{n\ge 1}(1-(-q)^n) \\ &= (-q;q^2)_\infty (q^2;q^2)_\infty.\end{aligned}$$

So (2.2) is equivalent to

$$\begin{aligned}&(-q;q^2)^3_\infty (q^2;q^2)^2_\infty \\ &= (-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty(q^6;q^{12})^2_\infty(q^{12};q^{12})_\infty \\ &+ 2q(-q^3;q^3)^2_\infty(q^3;q^3)_\infty(q^2;q^{12})_\infty(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty \\ &= (-q;q^6)_\infty(-q^2;q^6)_\infty(q^3;q^6)_\infty(-q^4;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty\cdot \\ &\qquad\cdot(q^6;q^{12})^2_\infty(q^{12};q^{12})_\infty \\ &+ 2q(-q^3;q^6)^2_\infty(q^3;q^6)_\infty(-q^6;q^6)^2_\infty(q^6;q^6)_\infty\cdot \\ &\qquad\cdot(q^2;q^{12})_\infty(q^{10};q^{12})(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty.\end{aligned} \tag{2.3}$$

If we put $-q$ for q, we see that (2.3) is equivalent to

$$\begin{aligned}&(q;q^2)^3_\infty(q^2;q^2)^2_\infty \\ &= (q;q^6)_\infty(-q^2;q^6)_\infty(-q^3;q^6)_\infty(-q^4;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty \\ &\qquad\cdot(q^3;q^6)^2_\infty(-q^3;q^6)^2_\infty(q^6;q^6)_\infty(-q^6;q^6)_\infty \\ &- 2q(q^3;q^6)^2_\infty(-q^3;q^6)_\infty(-q^6;q^6)^2_\infty(q^6;q^6)_\infty\cdot \\ &\qquad\cdot(q;q^6)_\infty(-q;q^6)_\infty(q^5;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty(-q^6;q^6)_\infty\end{aligned} \tag{2.4}$$

Next divide by

$$(q;q^2)_\infty = (q;q^6)_\infty(q^3;q^6)_\infty(q^5;q^6)_\infty$$

and use the fact that

$$(q;q^2)_\infty(-q;q^2)_\infty(-q^2;q^2)_\infty = (q;q^2)_\infty(-q;q)_\infty = (q;q^2)_\infty\frac{(q^2;q^2)_\infty}{(q;q)_\infty} = \frac{(q;q)_\infty}{(q;q)_\infty} = 1$$

with q^3 for q, and we see that (2.4) is equivalent to

$$\begin{aligned}(q;q)^2_\infty &= (-q^2;q^6)_\infty(-q^4;q^6)_\infty(q^6;q^6)_\infty(-q^3;q^6)^2_\infty(q^6;q^6)_\infty \\ &- 2q(-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty(-q^6;q^6)^2_\infty(q^6;q^6)_\infty.\end{aligned} \tag{2.5}$$

We now prove (2.5)

$$(q;q)^2_\infty = \sum(-1)^{r+s}q^{(3r^2-r+3s^2-s)/2}.$$

Split this sum into two, according to whether $r+s$ is even or odd. If $r+s$ is even, let $r=m+n, s=m-n$; if $r+s$ is odd, let $r=m+n+1, s=m-n$.

We obtain

$$\begin{aligned}(q;q)^2_\infty &= \sum q^{(3(m+n)^2-(m+n)+3(m-n)^2-(m-n))/2} \\ &- \sum q^{(3(m+n+1)^2-(m+n+1)+3(m-n)^2-(m-n))/2}\end{aligned}$$

$$= \sum q^{3m^2 - m + 3n^2} - \sum q^{3m^2 + 2m + 3n^2 - 3n + 1}$$
$$= (-q^2;q^6)_\infty(-q^4;q^6)_\infty(q^6;q^6)_\infty(-q^3;q^6)^2_\infty(q^6;q^6)_\infty$$
$$-2q(-q;q^6)_\infty(-q^5;q^6)_\infty(q^6;q^6)_\infty(-q^6;q^6)^2_\infty(q^6;q^6)_\infty$$

as required.

The other two identities stated and proved by Miwa and Jimbo are ([1], (B.12))

$$\Big(\frac{1}{2}\sum q^{(3n^2-3n)/2}\Big)\Big(\sum q^{12n^2}\Big) + \Big(\sum q^{(3n^2-n)/2}\Big)\Big(\sum q^{12n^2-8n+1}\Big) = \Big(\sum q^{2n^2-n}\Big)\Big(\sum q^{4n^2-2n}\Big) \tag{2.6}$$

and

$$\Big(\frac{1}{2}\sum q^{(3n^2-3n)/2}\Big)\Big(\sum q^{12n^2-12n+3}\Big) + \Big(\sum q^{(3n^2-n)/2}\Big)\Big(\sum q^{12n^2-4n}\Big) = \Big(\sum q^{2n^2-n}\Big)\Big(\sum q^{4n^2-2n}\Big). \tag{2.7}$$

These can be written, respectively, as

$$\Big(\sum q^{2n^2-n}\Big)\Big(\sum q^{4n^2-2n}\Big) = \Big(\sum q^{6n^2-3n}\Big)\Big(\sum q^{12n^2}\Big) + \Big(\sum q^{6n^2-n} + \sum q^{6n^2-5n+1}\Big)\Big(\sum q^{12n^2-8n+1}\Big) \tag{2.8}$$

and

$$\Big(\sum q^{2n^2-n}\Big)\Big(\sum q^{4n^2-2n}\Big) = \Big(\sum q^{6n^2-3n}\Big)\Big(\sum q^{12n^2-12n+3}\Big) + \Big(\sum q^{6n^2-n} + \sum q^{6n^2-5n+1}\Big)\Big(\sum q^{12n^2-4n}\Big). \tag{2.9}$$

In product form, these become

$$\begin{aligned}
&(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty(-q^2;q^8)_\infty(-q^6;q^8)_\infty(q^8;q^8)_\infty \\
&= (-q^3;q^{12})_\infty(-q^9;q^{12})_\infty(q^{12};q^{12})_\infty(-q^{12};q^{24})^2_\infty(q^{24};q^{24})_\infty \\
&+ q(-q^5;q^{12})_\infty(-q^7;q^{12})_\infty(q^{12};q^{12})_\infty(-q^4;q^{24})_\infty(-q^{20};q^{24})_\infty(q^{24};q^{24})_\infty \\
&+ q^2(-q;q^{12})_\infty(-q^{11};q^{12})_\infty(q^{12};q^{12})_\infty(-q^4;q^{24})_\infty(-q^{20};q^{24})_\infty(q^{24};q^{24})_\infty
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty(-q^2;q^8)_\infty(-q^6;q^8)_\infty(q^8;q^8)_\infty \\
&= (-q^5;q^{12})_\infty(-q^7;q^{12})_\infty(q^{12};q^{12})_\infty(-q^8;q^{24})_\infty(-q^{16};q^{24})_\infty(q^{24};q^{24})_\infty \\
&+ q(-q;q^{12})_\infty(-q^{11};q^{12})_\infty(q^{12};q^{12})_\infty(-q^8;q^{24})_\infty(-q^{16};q^{24})_\infty(q^{24};q^{24})_\infty \\
&+ 2q^3(-q^3;q^{12})_\infty(-q^9;q^{12})_\infty(q^{12};q^{12})_\infty(-q^{24};q^{24})^2_\infty(q^{24};q^{24})_\infty
\end{aligned} \tag{2.11}$$

Now, consider the identity ([2], (2))

$$\begin{aligned}
&(-aq;q^2)_\infty(-a^{-1}q;q^2)_\infty(q^2;q^2)_\infty(-bq^2;q^4)_\infty(-b^{-1}q^2;q^4)_\infty(q^4;q^4)_\infty \\
&= (-ab^{-1}q^3;q^6)_\infty(-a^{-1}bq^3;q^6)_\infty(q^6;q^6)_\infty(-a^2bq^6;q^{12})_\infty(-a^{-2}b^{-1}q^6;q^{12})_\infty(q^{12};q^{12})_\infty \\
&+ aq(-ab^{-1}q^5;q^6)_\infty(-a^{-1}bq;q^6)_\infty(q^6;q^6)_\infty(-a^2bq^{10};q^{12})_\infty(-a^{-2}b^{-1}q^2;q^{12})_\infty(q^{12};q^{12})_\infty \\
&+ a^{-1}q(-ab^{-1}q;q^6)_\infty(-a^{-1}bq^5;q^6)_\infty(q^6;q^6)_\infty(-a^2bq^2;q^{12})_\infty(-a^{-2}b^{-1}q^{10};q^{12})_\infty(q^{12};q^{12})_\infty
\end{aligned} \tag{2.12}$$

If in (2.12), we set q^2 for q, then $a = q^{-1}, b = q^2$, we obtain (2.10), while if we set q^2 for q then $a = q^{-1}, b = q^{-2}$, we obtain (2.11).

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