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SOME RESULTS ON BIORTHOGONAL POLYNOMIALS

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ABSTRACT. Some biorthogonal polynomials of Hahn and Pastro are derived using a polynomial modification of the Lebesgue measure $d\theta$ combined with analytic continuation. A result is given for changing the measures of biorthogonal polynomials on the unit circle by the multiplication of their measures by certain Laurent polynomials.

KEY WORDS AND PHRASES: Biorthogonal polynomials, a formula of Christoffel, change of weight, unit circle, determinant.
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1. INTRODUCTION

In [9], Pastro introduced a pair of polynomial sets which are biorthogonal on the unit circle with respect to the weight function

$$\Omega(z;q^2)=\frac{(q^2;q^2)_\infty(abq^2;q^2)_\infty(qz;q^2)_\infty(qz^{-1};q^2)_\infty}{(aq^2;q^2)_\infty(bq^2;q^2)_\infty(qaz;q^2)_\infty(qbz^{-1};q^2)_\infty},\quad z=e^{i\theta}$$

where

$$(t;q^2)_n=\prod_{k=0}^{n-1}(1-tq^{2k}),\quad (t;q^2)_\infty=\prod_{k=0}^{\infty}(1-tq^{2k}).$$

To be precise, he showed that if $\{p_n(z)\}$ and $\{q_n(z)\}$ are defined by

$$p_n(z)=p_n(z,a,b)=\sum_{k=0}^{n}\frac{(aq^2;q^2)_k(b;q^2)_{n-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}}(q^{-1}z)^k$$

and

$$q_n(z)=q_n(z,a,b)=p_n(z,\bar{b},\bar{a})$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}p_n(z)\overline{q_m(z)}\,\Omega(z;q^2)d\theta=\frac{(abq^2;q^2)_n}{(q^2;q^2)_n}q^{-2n}\delta_{mn},\quad z=e^{i\theta}. \tag{1.1}$$

Pastro assume the parameters a and b are real but as Al-Salam and Ismail note [1] they do not have to be. These polynomials generalize those of Askey ($a=b$, both real), and Szegö ($a=b=0$), see [10]. A weight equivalent to $\Omega(z;q^2)$ was considered earlier by Hahn [4] in the case of real coefficients.

Throughout this paper we assume that q is real and, for convergence of the infinite products, $|q|<1$. Considering the denominator of $\Omega(z;q^2)$ we also want $|qaz|<1$ and $|qbz^{-1}|<1$, that is

$$|bq|<|z|<|aq|^{-1}.$$

We also require that $|aq|<1$ and $|bq|<1$. Note that these restrictions, besides ensuring convergence and existence, make both sides of equation (1.1) analytic in the parameters a and b. This we will need in Section 3.

In section 2 we state, in determinant form, a pair of polynomial sets which are biorthogonal on the unit circle with respect to the measure

$$d\nu(\theta) = z^{-m}(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_h)d\theta\,, \quad z = e^{i\theta}$$

assuming that no α_j is zero and that $0 \le m \le h$.

In Section 3 we show how these yield Pastro's polynomials in the special case $a = q^{2r}$, $b = q^{2s}$. The full result follows by analytic continuation.

Pastro also gave in [9] explicit examples of Laurent orthogonal polynomials, making concrete the earlier work of Jones and Thron [7] in which such "polynomials" were introduced. (They are not actually polynomials, they contain both positive and negative powers of their variable.) More than this, he states an interesting connection between biorthogonal polynomials and orthogonal Laurent polynomials.

There is a well-known formula of Christoffel for modifying the measure $d\alpha(x)$ by polynomial multiplication. That is, let

$$\rho(x) = k(x-x_1)(x-x_2)\ldots(x-x_r)$$

be a polynomial which is non-negative on $[a,b]$ and let $\{q_n(x)\}$ be the polynomials orthogonal with respect to the new measure $\rho(x)d\alpha(x)$ on $[a,b]$. Then the polynomials $\{q_n(x)\}$ can be represented in terms of the polynomials $\{p_n(x)\}$ by

$$\rho(x)q_n(x) = c_n det\begin{pmatrix} p_n(x) & p_{n+1}(x) & \cdots & p_{n+r}(x) \\ p_n(x_1) & p_{n+1}(x_1) & \cdots & p_{n+r}(x_1) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p_n(x_r) & p_{n+1}(x_r) & \cdots & p_{n+r}(x_r) \end{pmatrix}$$

for suitable constants c_n.

Both this formula of Christoffel and a related formula of Uvarov carry over to polynomials orthogonal on the unit circle. See Godoy and Marcellan [3] or Ismail and Ruedemann [6]. The natural question is, does this formula of Christoffel have an analogue for biorthogonal polynomials on the unit circle? In Section 4 we show how a trivial modification of the result in [6] yields a result for biorthogonal polynomials, at least for certain cases. Unfortunately, we only allow certain modifications and must assume that certain determinants do not vanish. Actually, this assumption of nonzero determinants is common to biorthogonality (see the work of Baxter [2]).

In the remainder of this paper we adopt the following notation. For $\rho_r(z)$ a polynomial of degree r we define $\rho_r^*(z) = z^r\,\bar{\rho}_r(z^{-1})$. For nonzero complex numbers α, α^* denotes $1/\bar{\alpha}$. Finally, z denotes $e^{i\theta}$ in the integrals presented.

2. A PAIR OF BIORTHOGONAL POLYNOMIAL SETS

In this section we consider a pair of polynomial sets which are biorthogonal on the unit circle with respect to the measure

$$d\nu(\theta) = z^{-m}(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_h)d\theta\,, \quad z = e^{i\theta}$$

assuming that no α_j is zero and that $0 \le m \le h$. First we need two lemmas.

LEMMA 1. Assume that $0 \le m \le h$ and define $\psi_n(z)$ by

$(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_h)\psi_n(z) =$

$$det\begin{pmatrix} z^{n+h} & z^{n+h-1} & \dots & z^{n+m} & z^{m-1} & z^{m-2} & \dots & z & 1 \\ \alpha_1^{n+h} & \alpha_1^{n+h-1} & \dots & \alpha_1^{n+m} & \alpha_1^{m-1} & \alpha_1^{m-2} & \dots & \alpha_1 & 1 \\ \alpha_2^{n+h} & \alpha_2^{n+h-1} & \dots & \alpha_2^{n+m} & \alpha_2^{m-1} & \alpha_2^{m-2} & \dots & \alpha_2 & 1 \\ . & . & & . & . & . & & . & . \\ . & . & & . & . & . & & . & . \\ . & . & & . & . & . & & . & . \\ \alpha_h^{n+h} & \alpha_h^{n+h-1} & \dots & \alpha_h^{n+m} & \alpha_h^{m-1} & \alpha_h^{m-2} & \dots & \alpha_h & 1 \end{pmatrix} \quad (2.1)$$

Then if $\rho_{n-1}(z)$ is a polynomial of degree at most $n-1$ we have

$$\int_{-\pi}^{\pi} \psi_n(z)\overline{\rho_{n-1}(z)}[z^{-m}(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_h)]d\theta = 0 .$$

LEMMA 2. Still assuming that $0 \le m \le h$ we define $\phi_n(z)$ by

$(z-\alpha_1^*)(z-\alpha_2^*)\dots(z-\alpha_h^*)\phi_n(z) =$

$$det\begin{pmatrix} z^{n+h} & z^{n+h-1} & \dots & z^{n+h-m} & z^{h-m-1} & z^{h-m-2} & \dots & z & 1 \\ \alpha_1^{*n+h} & \alpha_1^{*n+h-1} & \dots & \alpha_1^{*n+h-m} & \alpha_1^{*h-m-1} & \alpha_1^{*h-m-2} & \dots & \alpha_1^* & 1 \\ \alpha_2^{*n+h} & \alpha_2^{*n+h-1} & \dots & \alpha_2^{*n+h-m} & \alpha_2^{*h-m-1} & \alpha_2^{*h-m-2} & \dots & \alpha_2 & 1 \\ . & . & & . & . & . & & . & . \\ . & . & & . & . & . & & . & . \\ . & . & & . & . & . & & . & . \\ \alpha_h^{*n+h} & \alpha_h^{*n+h-1} & \dots & \alpha_h^{*n+h-m} & \alpha_h^{*h-m-1} & \alpha_h^{*h-m-2} & \dots & \alpha_h^* & 1 \end{pmatrix} \quad (2.2)$$

where no α_j is zero and $\alpha_j^* = 1/\overline{\alpha_j}$. Then if $\rho_{n-1}(z)$ is a polynomial of degree at most $n-1$ we have

$$\int_{-\pi}^{\pi} \rho_{n-1}(z)\overline{\phi_n(z)}[z^{-m}(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_h)]d\theta = 0 .$$

THEOREM 1. Let the polynomial sets $\{\psi_n(z)\}$ and $\{\phi_n(z)\}$ be defined as in the above two lemmas. Assume, moreover, that for each n, $\psi_n(z)$ and $\phi_n(z)$ are of precise degree n. (This is equivalent to assuming certain subdeterminants in equations (2.1) and (2.2) are nonzero.) Then, provided that for each n we have,

$$\int_{-\pi}^{\pi} \psi_n(z)\overline{\phi_n(z)}[z^{-m}(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_h)]d\theta \ne 0 ,$$

these polynomial sets are biorthogonal on the unit circle with respect to the measure

$$dv(\theta) = z^{-m}(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_h)d\theta , \quad z = e^{i\theta}$$

where $0 \le m \le h$ and no α_j is zero.

3. APPLICATION TO THE POLYNOMIALS OF PASTRO

In this section we will consider the weight

$$w(z) = \frac{(qz;q^2)_\infty(qz^{-1};q^2)_\infty}{(aqz;q^2)_\infty(bqz^{-1};q^2)_\infty}, \quad z = e^{i\theta}$$

and derive Pastro's biorthogonal polynomials using Theorem 1 above and the same idea behind Ismail's [5] proof of Ramanujan's ${}_1\psi_1$-summation. Namely, we choose appropriate values for the parameters a and b, and then use analytic continuation to get the full result.

For the choice of $a = q^{2r}$ and $b = q^{2s}$ we have

$$w(z) = \frac{(qz;q^2)_\infty (qz^{-1};q^2)_\infty}{(q^{2r+1}z;q^2)_\infty (q^{2s+1}z^{-1};q^2)_\infty}$$

$$= (qz;q^2)_r (qz^{-1};q^2)_s$$

$$= [(1-qz)(1-q^3z)\dots(1-q^{2r-1}z)][(1-qz^{-1})(1-q^3z^{-1})\dots(1-q^{2s-1}z^{-1})]$$

$$= q^{r^2}(-1)^r z^{-s}[(z-q^{-(2r-1)})(z-q^{-(2r-3)})\dots(z-q^{-3})(z-q^{-1})][(z-q)(z-q^3)\dots(z-q^{2s-1})]$$

Note that the zeros of $w(z)$ increase by factors of q^2 and, moreover, the conjugation $\overline{w(z)}$ merely switches the roles of r and s. We are now ready to apply our lemmas.

Let $h = r+s$ and $m = s$ in Lemma 1 and let

$$\alpha_1 = q^{-(2r-1)}, \alpha_2 = q^{-(2r-3)}, \dots, \alpha_{r+s-1} = q^{2s-3}, \alpha_{r+s} = q^{2s-1}.$$

Define $\psi_n(z)$ by

$$\psi_n(z) = \frac{1}{\xi(z)} det \begin{pmatrix} z^{n+r+s} & z^{n+r+s-1} & \dots & z^{n+s} & z^{s-1} & z^{s-2} & \dots & z & 1 \\ \alpha_1^{n+r+s} & \alpha_1^{n+r+s-1} & \dots & \alpha_1^{n+s} & \alpha_1^{s-1} & \alpha_1^{s-2} & \dots & \alpha_1 & 1 \\ \alpha_2^{n+r+s} & \alpha_2^{n+r+s-1} & \dots & \alpha_2^{n+s} & \alpha_2^{s-1} & \alpha_2^{s-2} & \dots & \alpha_2 & 1 \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \alpha_{r+s}^{n+r+s} & \alpha_{r+s}^{n+r+s-1} & \dots & \alpha_{r+s}^{n+s} & \alpha_{r+s}^{s-1} & \alpha_{r+s}^{s-2} & \dots & \alpha_{r+s} & 1 \end{pmatrix} \tag{3.1}$$

where $\xi(z)$ denotes the Vandermonde determinant, or difference product, on

$$\{z, q^{-(2r-1)}, q^{-(2r-3)}, \dots, q^{2s-3}, q^{2s-1}\}.$$

Let h_k denote the complete symmetric function on $\{z, q^{-(2r-1)}, q^{-(2r-3)}, \dots, q^{2s-3}, q^{2s-1}\}$ and let j_k denote the complete symmetric function on $\{q^{-(2r-1)}, q^{-(2r-3)}, \dots, q^{2s-3}, q^{2s-1}\}$. We set $h_0 = j_0 = 1$ and $h_{-k} = j_{-k} = 0$ for $k > 0$.

Note that

$$h_k = zh_{k-1} + j_k \tag{3.2}$$

for all integers k.

By use of the Jacobi-Trudi identity, equation (3.4) in [8], we may write

$$\psi_n(z) = det \begin{pmatrix} h_0 & h_1 & \dots h_{s-1} & h_{n+s} & h_{n+s+1} & \dots h_{n+r+s} \\ h_{-1} & h_0 & \dots h_{s-2} & h_{n+s-1} & h_{n+s} & \dots h_{n+r+s-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{-s+1} & h_{-s+2} & \dots h_0 & h_{n+1} & h_{n+2} & \dots h_{n+r+1} \\ h_{-s} & h_{-s+1} & \dots h_{-1} & h_n & h_{n+1} & \dots h_{n+r} \\ h_{-s-1} & h_{-s} & \dots h_{-2} & h_{n-1} & h_n & \dots h_{n+r-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ h_{-s-r} & h_{-s-r+1} & \dots h_{-r-1} & h_{n-r} & h_{n-r+1} & \dots h_n \end{pmatrix}$$

Thus

$$\psi_n(z) = det\begin{pmatrix} h_n & h_{n+1} & h_{n+2} & \dots h_{n+r} \\ h_{n-1} & h_n & h_{n+1} & \dots h_{n+r-1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ h_{n-r} & h_{n-r+1} & h_{n-r+2} & \dots h_n \end{pmatrix}$$

and using (3.2) repeatedly we get

$$\psi_n(z) = A_{n,n}z^n + A_{n,n-1}z^{n-1} + \dots + A_{n,1}z + A_{n,0}$$

where

$$A_{n,k} = det\begin{pmatrix} j_{n-k} & j_{n+1} & j_{n+2} & \dots j_{n+r} \\ j_{n-(k+1)} & j_n & j_{n+1} & \dots j_{n+r-1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ j_{n-(k+r)} & j_{n-r+1} & j_{n-r+2} & \dots j_n \end{pmatrix}$$

The problem now is to evaluate $A_{n,k}$ in general. We have $r+s$ zeros in our weight function but $A_{n,k}$ is only a $(r+1)$ by $(r+1)$ determinant. We "fill out" $A_{n,k}$ and use the Jacobi-Trudi identity in reverse. That is,

$$A_{n,k} = det\begin{pmatrix} j_0 & j_1 & \dots j_{s-2} & j_{n-k+s-1} & j_{n+s} & \dots j_{n+s+1} & \dots j_{n+r+s-1} \\ j_{-1} & j_0 & \dots j_{s-3} & j_{n-k+s-2} & j_{n+s-1} & \dots j_{n+s} & \dots j_{n+r+s-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ j_{-s+2} & j_{-s+3} & \dots j_0 & j_{n-k+1} & j_{n+2} & j_{n+3} & \dots j_{n+r+1} \\ j_{-s+1} & j_{-s+2} & \dots j_{-1} & j_{n-k} & j_{n+1} & j_{n+2} & \dots j_{n+r} \\ j_{-s} & j_{-s+1} & \dots j_{-2} & j_{n-k-1} & j_n & j_{n+1} & \dots j_{n+r-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ j_{-s-r+1} & j_{-s-r+2} & \dots j_{-r-1} & j_{n-k-r} & j_{n-r+1} & j_{n-r+2} & \dots j_n \end{pmatrix}$$

Now setting

$$\sigma_k = -(2r-1)[0+1+\dots+(s-2)+(n-k+s-1)+(n+s)+(n+s+1)+\dots+(n+r+s-1)],$$

we find

$$A_{n,k} = q^{\sigma_k}\frac{\xi(1,q^2,q^4,\dots,q^{2(s-2)},q^{2(n-k+s-1)},q^{2(n+s)},q^{2(n+s+1)},\dots,q^{2(n+r+s-1)})}{\xi(q^{-(2r-1)},q^{-(2r-3)},\dots,q^{2s-3},q^{2s-1})}$$

and straightforward but rather tedious calculations yield

$$\frac{A_{n,k+1}}{A_{n,k}} = q^{-1}\frac{(1-q^{2(n-k)})(1-q^{2(k+r+1)})}{(1-q^{2(n-k+s-1)})(1-q^{2(k+1)})}.$$

Thus

$$\frac{A_{n,k+1}}{A_{n,k}} = q^{-1}\left(\left[\frac{(q^2;q^2)_{n-k}}{(q^2;q^2)_{n-k-1}}\right]\left[\frac{(q^{2r}q^2;q^2)_{k+1}}{(q^{2r}q^2;q^2)_k}\right]\right) + \left(\left[\frac{(q^{2s};q^2)_{n-k}}{(q^{2s};q^2)_{n-k-1}}\right]\left[\frac{(q^2;q^2)_{k+1}}{(q^2;q^2)_k}\right]\right)$$

so that

$$\frac{A_{n,k+1}}{A_{n,k}} = q^{-1}\left(\frac{(aq^2;q^2)_{k+1}(b;q^2)_{n-(k+1)}}{(q^2;q^2)_{k+1}(q^2;q^2)_{n-(k+1)}}\right) + \left(\frac{(aq^2;q^2)_k(b;q^2)_{n-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}}\right)$$

where $a = q^{2r}$ and $b = q^{2s}$. At this point we know that for a suitable constant c_n

$$\psi_n(z) = c_n \sum_{k=0}^{n} \frac{(aq^2;q^2)_k(b;q^2)_{n-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}}(q^{-1}z)^k .$$

To sum things up, if we define as Pastro does,

$$p_n(z,a,b) = \sum_{k=0}^{n} \frac{(aq^2;q^2)_k(b;q^2)_{n-k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}}(q^{-1}z)^k$$

then

$$\int_{-\pi}^{\pi} p_n(z,a,b)\overline{\rho_{n-1}(z)}\,w(z)d\theta = 0, \quad z = e^{i\theta}$$

for our particular choice of a and b.

We could use Lemma 2 to find the other set of polynomials required for biorthogonality but, as noted previously, the conjugation of the weight function $w(z)$ merely switches the roles of r and s and hence those of a and b as well. Thus the polynomials $q_n(z,a,b) := p_n(z,b,a)$ satisfy

$$\int_{-\pi}^{\pi} \rho_{n-1}(z)\overline{q_n(z,a,b)}\,w(z)d\theta = 0, \quad z = e^{i\theta}$$

At this point we have the biorthogonality of the polynomial sets $\{p_n(z)\}$ and $\{q_n(z)\}$. We still must compute the value of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p_n(z)\overline{q_n(z)}\,w(z)d\theta .$$

In fact this poses no great problem. It is fairly easy to see that the monic versions of the polynomials in Theorem 1, call them $\{\Psi_n(z)\}$ and $\{\Phi_n(z)\}$, satisfy

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi_n(z)\overline{\Phi_n(z)}[z^{-m}(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_h)]d\theta$$

$$= (-1)^{h-m}\frac{\det\begin{pmatrix} \alpha_1^{n+h} & \alpha_1^{n+h-1} & \dots & \alpha_1^{n+m+1} & \alpha_1^{m-1} & \alpha_1^{m-2} & \dots & \alpha_1 & 1 \\ \alpha_2^{n+h} & \alpha_2^{n+h-1} & \dots & \alpha_2^{n+m+1} & \alpha_2^{m-1} & \alpha_2^{m-2} & \dots & \alpha_2 & 1 \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \alpha_h^{n+h} & \alpha_h^{n+h-1} & \dots & \alpha_h^{n+m+1} & \alpha_h^{m-1} & \alpha_h^{m-2} & \dots & \alpha_h & 1 \end{pmatrix}}{\det\begin{pmatrix} \alpha_1^{n+h-1} & \alpha_1^{n+h-2} & \dots & \alpha_1^{n+m} & \alpha_1^{m-1} & \alpha_1^{m-2} & \dots & \alpha_1 & 1 \\ \alpha_2^{n+h-1} & \alpha_2^{n+h-2} & \dots & \alpha_2^{n+m} & \alpha_2^{m-1} & \alpha_2^{m-2} & \dots & \alpha_2 & 1 \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \alpha_h^{n+h-1} & \alpha_h^{n+h-2} & \dots & \alpha_h^{n+m} & \alpha_h^{m-1} & \alpha_h^{m-2} & \dots & \alpha_h & 1 \end{pmatrix}}$$

Now if we let

$$\alpha_1 = q^{-(2r-1)}, \alpha_2 = q^{-(2r-3)}, \dots, \alpha_{r+s-1} = q^{2s-3}, \alpha_{r+s} = q^{2s-1}$$

the right hand side of the previous equation becomes a power of q times a quotient of Vandermonde determinants. To be precise, let $P_n(z)$ and $Q_n(z)$ denote the monic versions of Pastro's $p_n(z)$ and $q_n(z)$ respectively. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(z)\overline{Q_n(z)}\,[z^{-s}(z-q^{-(2r-1)})(z-q^{-(2r-3)})\ldots(z-q^{2s-3})(z-q^{2s-1})]d\theta$$

$$=(-1)^{2r-1}q^{-(2r-1)(r)}\frac{\xi(q^{2(n+r+s)},q^{2(n+r+s-1)},\ldots,q^{2(n+s+1)},q^{2(s-1)},q^{2(s-2)},\ldots,1)}{\xi(q^{2(n+s)},q^{2(n+r+s-1)},\ldots,q^{2(n+s+1)},q^{2(s-1)},q^{2(s-2)},\ldots,1)}$$

$$=(-1)^r q^{-r^2}\frac{(1-q^{2n+2r+2})(1-q^{2n+2r+4})\ldots(1-q^{2n+2r+2s})}{(1-q^{2n+2})(1-q^{2n+4})\ldots(1-q^{2n+2s})}$$

so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(z)\overline{Q_n(z)}\,(qz;q^2)_r(qz^{-1};q^2)_s d\theta=\frac{(1-q^{2n+2r+2})(1-q^{(2n+2r+4)})\ldots(1-q^{2n+2r+2s})}{(1-q^{2n+2})(1-q^{2n+4})\ldots(1-q^{2n+2s})}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p_n(z)\overline{q_n(z)}\,(qz;q^2)_r(qz^{-1};q^2)_s d\theta=$$

$$\left[\frac{(q^{2r}q^2;q^2)_n}{(q^2;q^2)_n}q^{-n}\right]\left[\frac{(q^{2s}q^2;q^2)_n}{(q^2;q^2)_n}q^{-n}\right]\left[\frac{(1-q^{2n+2r+2})(1-q^{2n+2r+4})\ldots(1-q^{2n+2r+2s})}{(1-q^{2n+2})(1-q^{2n+4})\ldots(1-q^{2n+2s})}\right].$$

Finally, define

$$\Omega(z;q^2)=\frac{(q^2;q^2)_\infty(abq^2;q^2)_\infty(qz;q^2)_\infty(qz^{-1};q^2)_\infty}{(aq^2;q^2)_\infty(bq^2;q^2)_\infty(qaz;q^2)_\infty(qbz^{-1};q^2)_\infty},\quad a=q^{2r},\quad b=q^{2s}.$$

We get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p_n(z)\overline{q_n(z)}\,\Omega(z;q^2)d\theta=q^{-2n}\left[\frac{(q^{2r}q^2;q^2)_n(q^{2s}q^2;q^2)_n}{(q^2;q^2)_n(q^2;q^2)_n}\right]$$

$$\times\left[\frac{(q^2;q^2)_\infty(q^{2r+2s+2};q^2)_\infty}{(q^{2r+2};q^2)_\infty(q^{2s+2};q^2)_\infty}\right]\left[\frac{(1-q^{2n+2r+2})(1-q^{2n+2r+4})\ldots(1-q^{2n+2r+2s})}{(1-q^{2n+2})(1-q^{2n+4})\ldots(1-q^{2n+2s})}\right]$$

$$=\frac{q^{-2n}}{(q^2;q^2)_n}(q^{2r+2s+2};q^2)_n$$

$$=\frac{(abq^2;q^2)_n}{(q^2;q^2)_n}q^{-2n}\quad\text{where}\quad a=q^{2r},\quad b=q^{2s}.$$

The full result follows by analytic continuation. Actually, two analytic continuations are needed: first with respect to the parameter a with b fixed at a q^{2s}, then with respect to the parameter b.

4. MODIFICATION OF MEASURES BY LAURENT POLYNOMIALS

In this section we start with a measure $d\nu(\theta)$ which is not necessarily positive on $z=e^{i\theta}$. From Baxter [2] we know that if certain Toeplitz determinants are nonzero then there exists a unique pair of polynomial sets which are biorthogonal on the unit circle. We will call this pair $\{\phi_n(z)\}$ and $\{\hat{\phi}_n(z)\}$. That is, for any polynomial $\rho_{n-1}(z)$ of degree at most $n-1$ we have

$$\int_{-\pi}^{\pi} \phi_n(z)\overline{\rho_{n-1}(z)}\,dv(\theta) = \int_{-\pi}^{\pi} \overline{\hat{\phi}_n(z)}\,\rho_{n-1}(z)dv(\theta) = 0\,,$$

and that for each n,

$$\int_{-\pi}^{\pi} \phi_n(z)\overline{\hat{\phi}_n(z)}\,dv(\theta) \neq 0\,.$$

What we want to do is multiply the complex measure $dv(\theta)$ by a Laurent polynomial and get determinant formulas for the new biorthogonal polynomials, $\{\psi_n(z)\}$ and $\{\hat{\psi}_n(z)\}$, in terms of the old polynomials, $\{\phi_n(z)\}$ and $\{\hat{\phi}_n(z)\}$. Actually, we are going to restrict ourselves to two types of Laurent polynomials, those of the forms $R_{2m}(z) = z^{-m}G_{2m}(z)$ and $R_{2m+1}(z) = z^{-(m+1)}G_{2m+1}(z)$, where $G_{2m}(z)$ and $G_{2m+1}(z)$ are polynomials having precise degrees $2m$ and $2m+1$ respectively. Furthermore, we shall require that neither $G_{2m}(z)$ or $G_{2m+1}(z)$ have z as a factor. We have two cases: the even case and the odd case.

THEOREM 2. (even case) Let $\{\psi_n(z)\}$ be given by

$$G_{2m}(z)\psi_n(z) = det\begin{pmatrix} \hat{\phi}^*(z) & z\hat{\phi}^*(z) & \ldots z^{m-1}\hat{\phi}^*(z) & \phi(z) & z\phi(z) & \ldots z^{m}\phi(z) \\ \hat{\phi}^*(\alpha_1) & \alpha_1\hat{\phi}^*(\alpha_1) & \ldots \alpha_1^{m-1}\hat{\phi}^*(\alpha_1) & \phi(\alpha_1) & \alpha_1\phi(\alpha_1) & \ldots \alpha_1^{m}\phi(\alpha_1) \\ \hat{\phi}^*(\alpha_2) & \alpha_2\hat{\phi}^*(\alpha_2) & \ldots \alpha_2^{m-1}\hat{\phi}^*(\alpha_2) & \phi(\alpha_2) & \alpha_2\phi(\alpha_2) & \ldots \alpha_2^{m}\phi(\alpha_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hat{\phi}^*(\alpha_{2m}) & \alpha_{2m}\hat{\phi}^*(\alpha_{2m}) & \ldots \alpha_{2m}^{m-1}\hat{\phi}^*(\alpha_{2m}) & \phi(\alpha_{2m}) & \alpha_{2m}\phi(\alpha_{2m}) & \ldots \alpha_{2m}^{m}\phi(\alpha_{2m}) \end{pmatrix} \tag{4.1}$$

where the zeros of $G_{2m}(z)$ are $\{\alpha_1, \alpha_2, \ldots, \alpha_{2m}\}$, $\phi(z)$ denotes $\phi_{n+m}(z)$, and $\hat{\phi}(z)$ denotes $\hat{\phi}_{n+m}(z)$. Let $\{\hat{\psi}_n(z)\}$ be given by

$$G_{2m}^*(z)\hat{\psi}_n(z) = det\begin{pmatrix} \phi^*(z) & z\phi^*(z) & \ldots z^{m-1}\phi^*(z) & \hat{\phi}(z) & z\hat{\phi}(z) & \ldots z^{m}\hat{\phi}(z) \\ \phi^*(\alpha_1^*) & \alpha_1^*\phi^*(\alpha_1^*) & \ldots \alpha_1^{*m-1}\phi^*(\alpha_1^*) & \hat{\phi}(\alpha_1^*) & \alpha_1^*\hat{\phi}(\alpha_1^*) & \ldots \alpha_1^{*m}\hat{\phi}(\alpha_1^*) \\ \phi^*(\alpha_2^*) & \alpha_2^*\phi^*(\alpha_2^*) & \ldots \alpha_2^{*m-1}\phi^*(\alpha_2^*) & \hat{\phi}(\alpha_2^*) & \alpha_2^*\hat{\phi}(\alpha_2^*) & \ldots \alpha_2^{*m}\hat{\phi}(\alpha_2^*) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi^*(\alpha_{2m}^*) & \alpha_{2m}^*\phi^*(\alpha_{2m}^*) & \ldots \alpha_{2m}^{*m-1}\phi^*(\alpha_{2m}^*) & \hat{\phi}(\alpha_{2m}^*) & \alpha_{2m}^*\hat{\phi}(\alpha_{2m}^*) & \ldots \alpha_{2m}^{*m}\hat{\phi}(\alpha_{2m}^*) \end{pmatrix} \tag{4.2}$$

Here we are assuming that the zeros of $G_{2m}(z)$, $\{\alpha_1, \alpha_2, \ldots, \alpha_{2m}\}$, are pairwise distinct. (For zeros of multiplicity $s, s > 1$, we replace the corresponding rows in the determinant by the derivatives of order $0, 1, 2, \ldots, s-1$ of the polynomials in the first row, evaluated at that zero.) Furthermore, we shall assume that $\psi_n(z)$ and $\hat{\psi}_n(z)$ are both of precise degree n. This is equivalent to assuming certain subdeterminants in equations (4.1) and (4.2) are nonzero.

Then for any polynomial $\rho_{n-1}(z)$ of degree at most $n-1$ we have

$$\int_{-\pi}^{\pi} \psi_n(z)\overline{\rho_{n-1}(z)}\,z^{-m}G_{2m}(z)dv(\theta) = \int_{-\pi}^{\pi} \overline{\hat{\psi}_n(z)}\,\rho_{n-m}(z)z^{-m}G_{2m}(z)dv(\theta) = 0\,,$$

and, assuming that

$$\int_{-\pi}^{\pi} \psi_n(z)\overline{\hat{\psi}_n(z)}\,z^{-m}G_{2m}(z)dv(\theta) \neq 0\,,$$

then the polynomial sets $\{\psi_n(z)\}$ and $\{\hat{\psi}_n(z)\}$ are biorthogonal on the unit circle with respect to the measure $z^{-m}G_{2m}(z)d\nu(\theta)$.

THEOREM 3. (odd case) Set $h = 2m+1$ and let $\{\psi_n(z)\}$ be given by

$$G_h(z)\psi_n(z) = det\begin{pmatrix} \hat{\phi}^*(z) & z\hat{\phi}^*(z) & \dots z^m\hat{\phi}^*(z) & z\phi(z) & z^2\phi(z) & \dots z^{m+1}\phi(z) \\ \hat{\phi}^*(\alpha_1) & \alpha_1\hat{\phi}^*(\alpha_1) & \dots\alpha_1^m\hat{\phi}^*(\alpha_1) & \alpha_1\phi(\alpha_1) & \alpha_1^2\phi(\alpha_1) & \dots\alpha_1^{m+1}\phi(\alpha_1) \\ \hat{\phi}^*(\alpha_2) & \alpha_2\hat{\phi}^*(\alpha_2) & \dots\alpha_2^m\hat{\phi}^*(\alpha_2) & \alpha_2\phi(\alpha_2) & \alpha_2^2\phi(\alpha_2) & \dots\alpha_2^{m+1}\phi(\alpha_2) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hat{\phi}^*(\alpha_h) & \alpha_h\hat{\phi}^*(\alpha_h) & \dots\alpha_h^m\hat{\phi}^*(\alpha_h) & \alpha_h\phi(\alpha_h) & \alpha_h^2\phi(\alpha_h) & \dots\alpha_h^{m+1}\phi(\alpha_h) \end{pmatrix} \tag{4.3}$$

where the zeros of $G_h(z)$ are $\{\alpha_1,\alpha_2,\dots,\alpha_h\}$, $\phi(z)$ denotes $\phi_{n+m}(z)$, and $\hat{\phi}(z)$ denotes $\hat{\phi}_{n+m}(z)$. Let $\{\hat{\psi}_n(z)\}$ be given by

$$zG_h^*(z)\hat{\psi}_n(z) = det\begin{pmatrix} \phi^*(z) & z\phi^*(z) & \dots z^m\phi^*(z) & \hat{\phi}(z) & z\hat{\phi}(z) & \dots z^{m+1}\hat{\phi}(z) \\ \phi^*(0) & 0 & \dots 0 & \hat{\phi}(0) & 0 & \dots 0 \\ \phi^*(\alpha_1^*) & \alpha_1^*\phi^*(\alpha_1^*) & \dots\alpha_1^{*m}\phi^*(\alpha_1^*) & \hat{\phi}(\alpha_1^*) & \alpha_1^*\hat{\phi}(\alpha_1^*) & \dots\alpha_1^{*m+1}\hat{\phi}(\alpha_1^*) \\ \phi^*(\alpha_2^*) & \alpha_2^*\phi^*(\alpha_2^*) & \dots\alpha_2^{*m}\phi^*(\alpha_2^*) & \hat{\phi}(\alpha_2^*) & \alpha_2^*\hat{\phi}(\alpha_2^*) & \dots\alpha_2^{*m+1}\hat{\phi}(\alpha_2^*) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi^*(\alpha_h^*) & \alpha_h^*\phi^*(\alpha_h^*) & \dots\alpha_h^{*m}\phi^*(\alpha_h^*) & \hat{\phi}(\alpha_h^*) & \alpha_h^*\hat{\phi}(\alpha_h^*) & \dots\alpha_h^{*m+1}\hat{\phi}(\alpha_h^*) \end{pmatrix} \tag{4.4}$$

Unlike the previous three determinants in (4.1) to (4.3), here in (4.4) $\phi(z)$ denotes $\phi_{n+m+1}(z)$, and $\hat{\phi}(z)$ denotes $\hat{\phi}_{n+m+1}(z)$. We are assuming that the zeros of $zG_h(z)$, $\{0,\alpha_1,\alpha_2,\dots,\alpha_h\}$, are pairwise distinct. (We take care of zeros of multiplicity $s, s > 1$ as usual.) Furthermore, we shall assume that $\psi_n(z)$ and $\hat{\psi}_n(z)$ are both of precise degree n. This is equivalent to assuming certain subdeterminants in equations (4.3) and (4.4) are nonzero.

Then for any polynomial $\rho_{n-1}(z)$ of degree at most $n-1$ we have

$$\int_{-\pi}^{\pi} \psi_n(z)\overline{\rho_{n-1}(z)}z^{-(m+1)}G_{2m+1}(z)d\nu(\theta) = \int_{-\pi}^{\pi} \overline{\hat{\psi}_n(z)}\rho_{n-1}(z)z^{-(m+1)}G_{2m+1}(z)d\nu(\theta) = 0\,,$$

and, assuming that

$$\int_{-\pi}^{\pi} \psi_n(z)\overline{\hat{\psi}_n(z)}z^{-(m+1)}G_{2m+1}(z)d\nu(\theta) \neq 0\,,$$

then the polynomial sets $\{\psi_n(z)\}$ and $\{\hat{\psi}_n(z)\}$ are biorthogonal on the unit circle with respect to the measure $z^{-(m+1)}G_{2m+1}(z)d\nu(\theta)$. The unusual form of the determinant in (4.4) comes about as we are writing $z^{-m}G_{2m+1}^*$ in the form $z^{-(m+1)}[zG_{2m+1}^*(z)]$ so that the same idea behind Theorem 2 applies in a sense.

5. PROOFS

PROOF OF (2.1). We want to show that if $\psi_n(z)$ is defined as in equation (2.1) then for any polynomial $\rho_{n-1}(z)$ of degree at most $n-1$ we have

$$\int_{-\pi}^{\pi} \psi_n(z)\overline{\rho_{n-1}(z)}[z^{-m}(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_h)]d\theta = 0.$$

However, multiplying both sides of (2.1) by z^{-m} and then expanding the determinant along the first row we find that

$$\psi_n(z)[z^{-m}(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_h)] = \sum_{k=n}^{k=n+h-m} c_k z^k + \sum_{k=-1}^{k=-m} c_k z^k .$$

As each of the terms in the above sums are orthogonal to any $\rho_{n-1}(z)$ the result follows immediately.

PROOF OF (2.2). We want to show that if $\phi_n(z)$ is defined as in equation (2.2) then for any polynomial $\rho_{n-1}(z)$ of degree at most $n-1$ we have

$$\int_{-\pi}^{\pi} \rho_{n-1}(z)\overline{\phi_n(z)}[z^{-m}(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_h)]d\theta = 0.$$

Conjugating both sides of this equation we see it is equivalent to showing that

$$\prod_{k=1}^{h}(-\overline{\alpha_k}) \int_{-\pi}^{\pi} \overline{\rho_{n-1}(z)}\,\phi_n(z)[z^{-(h-m)}(z-\alpha_1^*)(z-\alpha_2^*)\dots(z-\alpha_h^*)]d\theta = 0.$$

and (2.2) follows as an instance of (2.1) with m replaced by $h-m$ and the α_k's replaced by α_k^*'s.

PROOF OF (4.1). We want to show that if $\psi_n(z)$ is defined as in equation (4.1) then for any polynomial $\rho_{n-1}(z)$ of degree at most $n-1$ we have

$$\int_{-\pi}^{\pi} \psi_n(z)\overline{\rho_{n-1}(z)}\,z^{-m}G_{2m}(z)dv(\theta) = 0.$$

We only have two types of polynomials in the first row of the determinant in (4.1). We consider each separately. Let $\rho_{n-1}(z)$ be any polynomial of degree at most $n-1$.

(i) Then, for the polynomials $z^l\phi_{n+m}(z)$, where $l = 0,1,2,\dots,m$, we have

$$\int_{-\pi}^{\pi} \frac{z^l\phi_{n+m}(z)}{z^m}\overline{\rho_{n-1}(z)}\,dv(\theta) = \int_{-\pi}^{\pi} \phi_{n+m}(z)\overline{z^{m-l}\rho_{n-1}(z)}\,dv(\theta) = 0.$$

(ii) For the polynomials $z^l\hat{\phi}^*_{n+m}(z)$ we have

$$\int_{-\pi}^{\pi}\left(\frac{z^l\hat{\phi}^*_{n+m}(z)}{z^m}\right)\overline{\rho_{n-1}(z)}\,dv(\theta) = \int_{-\pi}^{\pi} z^{l-m}z^{n+m}\overline{\hat{\phi}_{n+m}}(1/z)\overline{\rho_{n-1}}(1/z)dv(\theta)$$

$$= \int_{-\pi}^{\pi} z^{l+1}\overline{\hat{\phi}_{n+m}(z)}\,z^{n-1}\overline{\rho_{n-1}}(1/z)dv(\theta)$$

$$= \int_{-\pi}^{\pi} \overline{\hat{\phi}_{n+m}(z)[z^{l+1}\rho^*_{n-1}(z)]}dv(\theta)$$

$$= 0$$

for $l = -1,0,1,2,\dots,m-1$. Hence from (i), (ii) and (4.1) we see that

$$\int_{-\pi}^{\pi}\left(\frac{G_{2m}(z)\psi_n(z)}{z^m}\right)\overline{\rho_{n-1}(z)}\,dv(\theta) = 0,$$

but this is simply

$$\int_{-\pi}^{\pi} \psi_n(z)\overline{\rho_{n-1}(z)}\, z^{-m} G_{2m}(z) d\nu(\theta) = 0 .$$

[Note: for the polynomials $z^l \hat{\phi}^*_{n+m}(z)$ we will use the choice $l = -1$ in our proof of (4.3). Also, in (4.1) we may allow 0 values for some α_j's. In fact, we make use of this choice in (4.4). The only reason we are restricting their values here to be nonzero is because of the α_j^*'s in (4.2).]

PROOF OF (4.2). For $\hat{\psi}_n(z)$ as defined by (4.2) we want to show that

$$\int_{-\pi}^{\pi} \overline{\hat{\psi}_n(z)}\, \rho_{n-1}(z) z^{-m} G_{2m}(z) d\nu(\theta) = 0 .$$

Equivalently, we wish to show that

$$\int_{-\pi}^{\pi} \hat{\psi}_n(z)\overline{\rho_{n-1}(z)}\, z^{-m} G^*_{2m}(z)\, \overline{d\nu(\theta)} = 0$$

and this we get simply by applying (4.1) to the modification of the measure $\overline{d\nu(\theta)}$ by the Laurent polynomial $z^{-m}G^*_{2m}(z)$. Note that having $\overline{d\nu(\theta)}$ rather than $d\nu(\theta)$ simply switches the roles of $\phi(z)$ and $\hat{\phi}(z)$ in (4.1).

PROOF OF (4.3). We want to show that if $\psi_n(z)$ is defined as in equation (4.3) then

$$\int_{-\pi}^{\pi} \psi_n(z)\overline{\rho_{n-1}(z)}\, z^{-(m+1)} G_{2m+1}(z) d\nu(\theta) = 0 .$$

Considering the first row of the determinant in (4.3) we see that this is equivalent to showing that

$$\int_{-\pi}^{\pi}\left(\frac{z^l \phi_{n+m}(z)}{z^{m+1}}\right)\overline{\rho_{n-1}(z)}\, d\nu(\theta) = \int_{-\pi}^{\pi}\left(\frac{z^{l-1} \phi_{n+m}(z)}{z^{m}}\right)\overline{\rho_{n-1}(z)}\, d\nu(\theta) = 0$$

for $l = 1, 2, \ldots, m+1$ and that

$$\int_{-\pi}^{\pi}\left(\frac{z^l \hat{\phi}^*_{n+m}(z)}{z^{m+1}}\right)\overline{\rho_{n-1}(z)}\, d\nu(\theta) = \int_{-\pi}^{\pi}\left(\frac{z^{l-1} \hat{\phi}^*_{n+m}(z)}{z^{m}}\right)\overline{\rho_{n-1}(z)}\, d\nu(\theta) = 0$$

for $l = 0, 1, 2, \ldots, m$. However, these statements are equivalent to (i) and (ii) in our proof of (4.1).

PROOF OF (4.4). Finally, we want to show that if $\hat{\psi}_n(z)$ is defined as in equation (4.4) then

$$\int_{-\pi}^{\pi} \overline{\hat{\psi}_n(z)}\, \rho_{n-1}(z) z^{-(m+1)} G_{2m+1}(z) d\nu(\theta) = 0$$

that is, we want

$$\int_{-\pi}^{\pi} \hat{\psi}_n(z)(\overline{\rho_{n-1}(z)})(\overline{z^{-(m+1)} G_{2m+1}(z)})(\overline{d\nu(\theta)}) = 0 .$$

Here the problem is that

$$\overline{z^{-(m+1)} G_{2m+1}(z)} \neq (constant) \times z^{-(m+1)} G^*_{2m+1}(z)$$

so that we cannot use (4.3) to get (4.4). In fact,

$$\overline{z^{-(m+1)} G_{2m+1}(z)} = (constant) \times z^{-m} G^*_{2m+1}(z) .$$

However, we get around this by applying (4.1) to the modification of $\overline{d\nu(\theta)}$ by $z^{-(m+1)}H_{2(m+1)}(z)$ where

$$H_{2(m+1)}(z) := zG^{*}_{2m+1}(z).$$

[This is what accounts for the unusual form of (4.4).]

6. REMARKS

We have found we may modify the Lebesgue measure $d\theta$ on the unit circle by multiplication by any Laurent polynomial whose zeros we know, provided certain determinants were nonzero. When we passed to the more general problem, as we did in section 4, of multiplying an unknown measure $d\nu(\theta)$ by Laurent polynomials, we restricted which Laurent polynomials we could use. This made the proofs for that section straightforward. However, this restriction is unsatisfying—at least to the author—but at the present time it is still unresolved.

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