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ARBITRARY SQUEEZE FLOW BETWEEN TWO DISKS

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ABSTRACT. A viscous incompressible fluid is contained between two parallel disks with arbitrarily shrinking width $h(\tau)$. The solution is obtained as a power series in a single nondimensional parameter (squeeze number) S, for small values of S in contrast to the "multifold" series solution obtained by Ishizawa in terms of an infinite set of nondimensional parameters. The gap width $h(\tau)$ is obtained for different states: when the top disk moves with constant velocity, constant force or constant power.

KEY WORDS AND PHRASES. Unsteady squeezing, Navier-Stokes equation, squeeze number.

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1. INTRODUCTION.

The problem of unsteady squeezing of a viscous incompressible fluid between two parallel disks in motion normal to their own surfaces independent of each other and arbitrary with respect to time is a fundamental type of unsteady flow which is met frequently in many hydrodynamical machines and apparatus. The similarity solution obtained by Wang [1], when a viscous fluid is squeezed between two parallel disks which are spaced a distance $a\sqrt{1-t/T}$ apart, is restricted to a special time dependent motion, namely, the distance of separation of the disks is taken to be $a\sqrt{1-t/T}$. It is unlikely that the distance would behave as $a\sqrt{1-t/T}$ in reality, since the pressures are found to approach infinity as $t \to T$. Also, the solution presented by Ishizawa [2] for the unsteady laminar flow of an incompressible fluid in a narrow gap between two parallel disks of varying width $h(\tau)$ is a "multifold" series of an infinite set of nondimensional time-dependent parameters $\frac{h}{\nu}\ \frac{dh}{dt}, \frac{h^3}{\nu^2}\ \frac{d^2h}{dt^2}, \frac{h^5}{\nu^3}\ \frac{d^3h}{dt^3} \cdots$. The present paper studies the arbitrary symmetric squeezing of a viscous incompressible fluid from a gap between two parallel disks of varying width $2ah(\tau), \tau = t/T$, which in general, does not lead to similarity solutions and the solution is obtained as a power series in a single nondimensional parameter (squeeze number) $S = a^2/2\nu T$ for small values of S, where ν is the kinematic viscosity, $2a$ is the width of the gap between the disks at $t = 0$ and T is a characteristic time. The gap width $h(\tau)$ is obtained when the top disk moves with constant velocity, constant force or constant power.

2. MATHEMATICAL FORMULATION.

Let the position of the two disks be at $Z = \pm ah(\tau)$, where $\tau = t/T$ is a normalized time. We assume that the length $2L$ of the channel is much larger than the gap width $2ah(\tau)$ at any time such that the end effects can be neglected. Let u and w be the velocity components in the r and z directions respectively. The axisymmetric flow of a viscous incompressible fluid between the parallel disks is governed by the unsteady Navier-Stokes equations.

$$u_t + uu_r + wu_z = -\frac{1}{\rho}\, p_r + \nu\left(u_{rr} + \frac{1}{r}\, u_r + u_{zz} - \frac{u}{r^2}\right) \tag{2.1}$$

$$w_t + uw_r + ww_z = -\frac{1}{\rho}\, p_z + \nu\left(w_{rr} + \frac{1}{r}\, w_r + w_{zz}\right) \tag{2.2}$$

$$u_r + \frac{u}{r} + w_z = 0 \tag{2.3}$$

The boundary conditions are

$$u = 0, w = \frac{a}{T} h_\tau \text{ on } z = ah(\tau) \tag{2.4}$$

$$w = 0, u_z = 0 \text{ on } z = 0 \tag{2.5}$$

The use of the transformations

$$\eta = \frac{z}{ah(\tau)},\ u = \frac{-h_\tau}{2Th(\tau)} r\ f_\eta\ (\eta,\tau),\ w = \frac{a}{T} h_\tau\ f(\eta,\tau) \tag{2.6}$$

in (2.2) yields

$$p_\eta = \frac{\mu h_\tau \partial}{hT\partial\eta}(f_\eta) - \rho a h w_t - \frac{\rho a^2}{2T^2} h_\tau^2 \frac{\partial}{\partial\eta}(f^2) \tag{2.7}$$

from which it follows that

$$p_{\eta r} = 0 \tag{2.8}$$

Substituting (2.6) in (2.1), we obtain

$$\frac{\mu h_\tau r}{2a^2h^3T}\left(S\left\{-\frac{2h^2h_{\tau\tau}f_\eta}{h_\tau} + 2hh_\tau f_\eta - 2h^2 f_{\eta\tau} + 2hh_\tau \eta f_{\eta\eta} + hh_\tau {f_\eta}^2 - 2hh_\tau f f_{\eta\eta}\right\} + f_{\eta\eta\eta}\right) = -p_r \tag{2.9}$$

which along with (2.8) implies

$$p_r = -\frac{\mu h_\tau r}{2a^2h^3T} A(\tau) \tag{2.10}$$

so that

$$S\left(-\frac{2h^2h_{\tau\tau}f_\eta}{h_\tau} + 2hh_\tau f_\eta - 2h^2 f_{\eta\tau} + 2hh_\tau \eta f_{\eta\eta} + hh_\tau {f_\eta}^2 - 2hh_\tau f f_{\eta\eta}\right) + f_{\eta\eta\eta} = A(\tau) \tag{2.11}$$

From (2.7), (2.8) and (2.10) we obtain the pressure as

$$p(\eta,\tau) = P_0 + \frac{\mu h_\tau f_\eta}{hT} - \int_{ah}^{z} \rho a h w_t d\eta - \frac{\rho a^2 {h_\tau}^2}{2T^2}(f^2-1) - \frac{\mu h_\tau A(\tau)}{4a^2h^3T}(r^2 - L^2) \tag{2.12}$$

where P_o is the pressure at the top edge of the upper disk, ρ the density, $A(\tau), f(\eta,\tau)$ are functions to be determined. In terms of the function $f(\eta,\tau)$, the boundary conditions are

$$f(0,\tau) = 0, f(1,\tau) = 1, f_{\eta\eta}(0,\tau) = 0, f_\eta(1,\tau) = 0 \tag{2.13}$$

When $h(\tau) = \sqrt{1-\tau}$, the equation (2.11) reduces to the similarity ordinary differential equation

$$S[\eta f''' + 3f'' - ff'''] = f'''' \tag{2.14}$$

given by Wang [1].

3. SERIES SOLUTION FOR SMALL SQUEEZE NUMBERS.

When $S < < 1$, we expand the unknown functions in terms of the squeeze number S as

$$f(\eta,\tau) = f_0(\eta,\tau) + Sf_1(\eta,\tau) + S^2 f_2(\eta,\tau) + \cdots\cdots \tag{3.1}$$

$$A(\tau) = A_0(\tau) + SA_1(\tau) + S^2A_2(\tau) + \cdots\cdot \tag{3.2}$$

The equation (2.11) yields successively

$$\begin{aligned} f_{0_{\eta\eta\eta}} &= A_0(\tau) \\ f_{1_{\eta\eta\eta}} &= A_1(\tau) + 2h^2\left(\frac{h_{\tau\tau}}{h_\tau} f_{0_\eta} + f_{0_{\eta\tau}}\right) - hh_\tau(2f_{0_\eta} + 2\eta f_{0_{\eta\eta}} + {f_{0_\eta}}^2 - 2f_0 f_{0_{\eta\eta}}) \\ f_{2_{\eta\eta\eta}} &= A_2(\tau) + 2h^2\left(\frac{h_{\tau\tau}}{h_\tau} f_{1_\eta} + f_{1_{\eta\tau}}\right) - hh_\tau(2f_{1_\eta} + 2\eta f_{1_{\eta\eta}} + 2f_{0_\eta}f_{1_\eta} - 2f_0f_{1_{\eta\eta}} - 2f_1f_{0_{\eta\eta}}) \end{aligned} \tag{3.3}$$

The solutions are

$$f_0 = (1/2)(3\eta - \eta^3), A_0 = -3$$

$$f_1(\eta,\tau) = \frac{\alpha}{30}(\eta^5 - 2\eta^3 + \eta) - \frac{\beta}{210}(\eta^7 + 21\eta^5 - 45\eta^3 + 23\eta)$$

$$A_1(\tau) = \frac{8\alpha}{5} - \frac{40\beta}{7}, \alpha = -\frac{3h^2h_{\tau\tau}}{2h_\tau}, \beta = -\frac{3}{4}hh_\tau$$

$$f_2(\eta,\tau)=\left(\frac{2h^2h_{\tau\tau}}{h_\tau}+2h^2\frac{\partial}{\partial\tau}\right)\left(\frac{\alpha}{30}\left(\frac{\eta^7}{42}-\frac{\eta^5}{10}+\frac{\eta^3}{6}-\frac{4}{105}\eta^3-\frac{11}{210}\eta\right)\right)$$

$$+\left(\frac{2h^2h_{\tau\tau}}{h_\tau}+2h^2\frac{\partial}{\partial\tau}\right)\left(\frac{\beta}{210}\left(-\frac{\eta^9}{72}-\frac{\eta^7}{2}+\frac{9}{4}\eta^5-\frac{53}{18}\eta^3+\frac{29}{24}\eta\right)\right)$$

$$-2hh_\tau\left(\frac{\alpha}{30}\left(\frac{11}{1008}\eta^9-\frac{\eta^7}{420}-\frac{\eta^5}{8}+\frac{5\eta^3}{12}-\frac{6571}{20160}\eta^3+\frac{519}{20160}\eta\right)\right)$$

$$-2hh_\tau\left(\frac{\beta}{210}\left(-\frac{3\eta^{11}}{220}-\frac{2\eta^9}{9}+\frac{\eta^7}{14}+\frac{14}{5}\eta^5-\frac{67321}{13860}\eta^3+\frac{30792}{13860}\eta\right)\right)$$

$$A_2(\tau)=\left(\frac{2h^2h_{\tau\tau}}{h_\tau}+2h^2\frac{\partial}{\partial\tau}\right)\left(-\frac{4}{525}\alpha+\frac{16}{630}\beta\right)$$

$$+2hh_\tau\left(\frac{6571}{100800}\alpha-\frac{2047}{148050}\beta\right) \tag{3.4}$$

Thus, given the motion $h(\tau)$, we can obtain the velocities and pressure from (2.6), (2.12), (3.1) and (3.2).

The pressure force exerted by the walls of the disks on the fluid is

$$F=2\pi\int_{-ah(\tau)}^{ah(\tau)}\int_0^L r(p-p_0)_{\eta=1}\,dr\,dz=\frac{\pi\mu L^4}{4aT}\frac{h_\tau}{h^2}A(\tau) \tag{3.5}$$

and the power imparted by the walls of the disks on the fluid is

$$R=2\pi\int_{-ah(\tau)}^{ah(\tau)}\int_0^L r(p-p_0)_{\eta=1}(-w)_{\eta=1}\,dr\,dz=-\frac{\pi\mu L^4}{4T^2}\frac{h_\tau{}^2}{h^2}A(\tau) \tag{3.6}$$

In what follows, we shall consider the cases: (i) constant velocity squeezing, (ii) constant force squeezing and (iii) constant power squeezing and obtain inversely the channel width, since many biological and mechanical devices are mostly limited to any of the three cases mentioned above.

4. CONSTANT VELOCITY SQUEEZING.

Suppose the top disk is moving vertically with constant velocity V. Then the time scale is $T=a/V$. Thus, $h(\tau)=1\mp\tau$ where the top sign is for squeezing. From equation (3.5), the force is

$$F=\frac{3\pi\mu L^4}{4aT}\left(\pm\alpha^{-2}+\frac{10S}{7}\alpha^{-1}+S^2\left(-\frac{2047}{296100}+\frac{4}{315}\alpha^{-2}\right)+O(S^3)\right) \tag{4.1}$$

where $\alpha=1\mp\tau$.

5. CONSTANT FORCE SQUEEZING.

Suppose that a constant force F is applied to the top disk. We wish to compute the unknown gap width $h(\tau)$. The characteristic time T is defined by

$T=(3\pi\mu L^4)/(4a\,|\,F\,|\,)$

The equation (3.5) becomes

$$\pm 3h^2=h_\tau A(\tau) \tag{5.1}$$

Since S is small, we set

$$h(\tau)=h_0(\tau)+Sh_1(\tau)+S^2h_2(\tau)+\cdots. \tag{5.2}$$

and solving the resulting equations, we obtain the solutions as

$$h_0=\alpha^{-1}$$

$$h_1=\mp\frac{3}{35}(\alpha^{-4}-\alpha^{-2})$$

$$h_2=-\frac{3187607}{3948000}\alpha^{-7}+\frac{712}{3675}\alpha^{-5}+\frac{9}{1225}\alpha^{-3}+\frac{5585323}{9212000}\alpha^{-2} \tag{5.3}$$

where $\alpha=(1\pm\tau)$ with top sign for squeezing.

6. CONSTANT POWER SQUEEZING.

In the case of constant power squeezing, we define the time scale by

$$T = \sqrt{\frac{3\pi\mu L^4}{4R}}$$

and so, from equation (3.6) we obtain

$$3h^2 = -A(\tau)h_\tau{}^2, h(0) = 1 \tag{6.1}$$

An expansion similar to (5.2) yields

$$h_0 = exp(\mp\tau)$$
$$h_1 = \mp\frac{11}{70}[exp(\mp 3\tau) - exp(\mp\tau)]$$
$$h_2 = -\frac{6668}{98700}exp(\mp 5\tau) + \frac{143}{2940}exp(\mp 3\tau) + \frac{4357}{230300}exp(\mp\tau) \tag{6.2}$$

where the top sign is for squeezing.

7. CONCLUSION.

The highly nonlinear unsteady axisymmetric flow equations (2.1)-(2.3) offer a solution in the case of arbitrary squeezing of a channel by the use of transformation equations (2.4). The resulting nonlinear partial differential equation (2.11) in two independent variables η and τ is solved for arbitrary squeezing $h(\tau)$ by an expansion in terms of a single nondimensional parameter S, for small values of S. When $S = 0$, the solution is

$$f_0 = (1/2)(3\eta - \eta^3)$$

which is exactly the quasi-steady poiseuille flow between two parallel disks. The higher order terms are corrections due to inertial effects.

We have obtained the gap width for three different states. The following table compares the eading terms ($S = 0$) of distance, force of squeezing and power of squeezing. We observe that these motions are basically different.

	Squeezing with constant velocity	Squeezing with constant force	Squeezing with constant power
distance between disks	$(1-\tau)$	$(1+\tau)^{-1}$	$e^{-\tau}$
force on top disk	$(1-\tau)^{-2}$	1	e^{τ}
power on top disk	$(1-\tau)^{-2}$	$(1+\tau)^{-2}$	1

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