SOME SUBREGULAR GERMS FOR p-ADIC $Sp_4(F)$

KIM YANGKON and SO KWANGHO

Department of Mathematics
Chonbuk National University
Chonju, Chonbuk 560-756, KOREA

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ABSTRACT. Shalika's unipotent regular germs were found by the authors in the case of $G = Sp_4(F)$. Next, subregular germs are also desirable, for at least $f(1)$ is constructible in another form for any smooth function f by using Shalika germs. Some of them were not so hard as expected although to find all of them is still not done explicitly.

KEY WORDS AND PHRASES: Shalika germs, unipotent orbits, orbital integrals, subregular germs.
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0. INTRODUCTION

Suppose that G is the set of F-points of a connected semi-simple algebraic group defined over a p-adic field F, that T is a Cartan subgroup of G, and that T' designate the set of regular elements in T. Let $d\dot{g}$ be a G-invariant measure on the quotient space $T\backslash G$, and let $C_c^\infty(G)$ be the set of smooth functions. Then it is known that for any $f \in C_c^\infty(G)$ and $t \in T'$ the orbital integral $\int_{T\backslash G} f(g^{-1}tg)d\dot{g}$ is convergent.

Next, let S_u be the set of unipotent conjugacy classes in G and let dx_0 be a G-invariant measure on $0 \in S_u$. It is also known that $A_0(f) = \int_0 f\, dx_0$ converges for any $f \in C_c^\infty(G)$.

Shalika, J. A. (see [14], p. 236) says that for any $t \in T'$ sufficiently close to 1, there exist germs $\Gamma_0(t)$ satisfying

$$\int_{T\backslash G} f(g^{-1}tg)d\dot{g} = \sum_{0 \in S_u} \Gamma_0(t)A_0(f)\,.$$

Shalika, J. A., Howe, R., Harish Chandra, Rogawski, J., and others contributed to the establishment of the germ associated to the trivial unipotent class. Recently Repka J. has found regular and subregualr germs for p-adic $GL_n(F)$ and $SL_n(F)$. The authors also found the regular germs for p-adic $Sp_4(F)$ in 1987. In this paper, the authors intend to find some subregular germs associated to some subregular conjugacy classes in $Sp_4(F)$.

Our result may in principle be seen elsewhere, but this paper gives an explicit formula in a special case.

1. NOTATIONS RELATED TO SYMPLECTIC GROUPS

Let $G = Sp_4(F) = (g \in SL_4(F) : {}^t gJg = J)$, where F is any p-adic field and

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

with 2×2 identity matrix I_2.

Let σ be the involution on $SL_4(F)$ defined by $\sigma(g) = J^{-1}(g)^{-1}J$ with $g \in SL_4(F)$. G may be interpreted as $SL_4(F)^{\sigma}$. G may also be expressed as the subgroup of $SL_4(F)$ generated by all the symplectic transvections whose most general forms are of the type

$$M_5(c,\alpha_1,\alpha_2,\beta_1,\beta_2) = \begin{pmatrix} 1+c\alpha_1\beta_1 & c\alpha_1\beta_2 & -\alpha_1^2c & -\alpha_1\alpha_2c \\ c\alpha_2\beta_1 & 1+c\alpha_2\beta_2 & -\alpha_1\alpha_2c & -\alpha_2^2c \\ c\beta_1^2 & c\beta_1\beta_2 & 1-c\alpha_1\beta_1 & -c\alpha_2\beta_1 \\ c\beta_1\beta_2 & c\beta_2^2 & -c\alpha_1\beta_2 & 1-c\alpha_2\beta_2 \end{pmatrix} \tag{1.0}$$

where $c \neq 0$, and α_i, β_j are arbitrary variables in a ground field F. So any symplectic element should be of the form

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad \text{with } M_{ij} \in M_2(F) \text{ satisfying}$$

$$\left.\begin{aligned} {}^tM_{11}M_{22} - {}^tM_{21}M_{12} = {}^tM_{22}M_{11} - {}^tM_{12}M_{21} = 1 \\ {}^tM_{11}M_{21} - {}^tM_{21}M_{11} = {}^tM_{22}M_{12} - {}^tM_{12}M_{22} = 0 \end{aligned}\right\} \tag{1.1}$$

Hereafter, we let F be a p-adic field of odd residual characteristic with ring of integers A; let P be the maximal ideal of A. Let $K = Sp_4(A)$, $K_1 = \{k \in K : k \equiv id \bmod P\}$, and let $diag(a,b,a^{-1},b^{-1})$ be denoted $d(a,b)$ for brevity. If $a = b$, denote $diag(a,b,a^{-1},b^{-1})$ simply by $d(a)$. Write char(s) for the characteristic polynomial of a matrix s, $c(s)$ for the pair consisting of the 2nd and 3rd coefficients of the characteristic polynomial of $s - id$, ignoring the signs that occur in the characteristic polynomial. Conjugating a matrix s by a matrix r means to produce $r^{-1}sr = s^r$ unless otherwise stated. Other symbols shall follow the standard convention.

2. UNIPOTENT ORBITS

G acts on itself by conjugation, so in particular on the set of all unipotent elements.

Referring to [5] §3, we may obtain the following.

PROPOSITION (2.0). Any unipotent orbit meets the set of all elements of the form

$$\begin{pmatrix} 1 & x & \alpha & \beta \\ 0 & 1 & \beta-\gamma x & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \tag{2.1}$$

where α, β, γ and $x \in F$.

If $x \neq 0$ in (2.1), we may calculate directly to see that the associated unipotent orbits meet the set of non-regular unipotent matrices or the set of regular unipotent matrices which as a G-set has representatives of the form

$$\begin{pmatrix} 1 & 1 & 0 & \delta \\ 0 & 1 & 0 & \delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \text{with } \delta \in F^{\times}/(F^{\times})^2. \tag{2.2}$$

If $x = 0$, however, in (2.1), it is not GL-conjugate to the element with all diagonal and super-diagonal entries equal to 1 and with all other entries equal to zero, i.e., not a regular unipotent element in short. Due to proposition (3.4) in [5] §3, (2.2) represents the orbits of the G-set consisting of all the regular unipotent elements of G.

On the other hand every subregular unipotent matrix, i.e., the matrix which is $GL(F)$-conjugate to the element with all diagonal entries equal to 1, with superdiagonal entries (1,0,1), and with all other entries equal to zero, must be conjugate to the matrix of the form

$$\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \gamma \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with } \alpha, \gamma \in F^\times .$$

By (1.1), we see easily that for $(a_{ij}) \in G$ and for two subregular unipotent matrices

$$\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \gamma \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{(a_{ij})} = \begin{pmatrix} 1 & 0 & \alpha' & 0 \\ 0 & 1 & 0 & \gamma' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{if and only if}$$

$$\left.\begin{aligned} &\frac{\alpha'}{\alpha}a_{11}^2 + \frac{\alpha'}{\gamma}a_{21}^2 = 1 \\ &\frac{1}{\alpha}a_{11}a_{12} + \frac{1}{\gamma}a_{21}a_{22} = 0 \\ &\frac{\gamma'}{\alpha}a_{12}^2 + \frac{\gamma'}{\gamma}a_{22}^2 = 1 \end{aligned}\right\} \tag{2.3}$$

holds. Without loss of generality, we may put $a_{21} \neq 0$; substituting $a_{22} = -\frac{\gamma}{\alpha}\frac{a_{11} \cdot a_{12}}{a_{21}}$ into the last equation in (2.3), we have

$$\left.\begin{aligned} &\frac{\alpha'}{\alpha}a_{11}^2 + \frac{\alpha'}{\gamma}a_{21}^2 = 1 \\ &\frac{\gamma'}{\alpha}a_{12}^2 + \frac{\gamma\gamma'}{\alpha^2} \cdot \frac{a_{11}^2 \cdot a_{12}^2}{a_{21}^2} = 1 \end{aligned}\right\}$$

From this we know that

$$\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \gamma \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is G-conjugate to the following analogous form

$$\begin{pmatrix} 1 & 0 & \dfrac{\alpha\gamma}{\alpha x_1^2 + \gamma x_2^2} & 0 \\ 0 & 1 & 0 & \dfrac{\alpha\gamma}{\alpha x_1^2 + \gamma x_2^2} \cdot \alpha\gamma x_3^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where x_i's are arbitrary so that denominators are nonzero. This, however, contains

$$\begin{pmatrix} 1 & 0 & \alpha x_3^2 & 0 \\ 0 & 1 & 0 & \gamma x_4^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & \gamma x_5^2 & 0 \\ 0 & 1 & 0 & \alpha x_6^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where x_j's are nonzero. Hence there exist at most $4 + \frac{12}{2} = 10$ representatives for this form in any case of F. In fact, a trivial computation shows that there are six, seven or eight classes.

Let

$$\bar{u}(\bar{\alpha},\bar{\gamma}) = \begin{pmatrix} 1 & 0 & \bar{\alpha} & 0 \\ 0 & 1 & 0 & \bar{\gamma} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with representative pairs $(\bar{\alpha},\bar{\gamma})$.

Now let $\bar{S}(\bar{\alpha},\bar{\gamma}) = (g \in K : g \equiv \bar{u}(\bar{\alpha},\bar{\gamma}) \bmod P)$. We may choose representative pairs $(\bar{\alpha},\bar{\gamma})$ with $|\bar{\alpha}| \geq |\bar{\gamma}| \geq 1$. By making use of $\bar{S}(\bar{\alpha},\bar{\gamma})$, we intend to compute the Shalika's germs associated to the unipotent classes of $\bar{u}(\bar{\alpha},\bar{\gamma})$. Any element of $\bar{S}(\bar{\alpha},\bar{\gamma})$ should be of the form

$$\begin{pmatrix} 1+p_{11} & p_{12} & \bar{\alpha}+p_{13} & p_{14} \\ x_{21} & 1+p_{22} & x_{23} & \bar{\gamma}+p_{24} \\ x_{31} & x_{32} & 1+p_{33} & p_{34} \\ p_{41} & x_{42} & x_{43} & 1+p_{44} \end{pmatrix} \tag{2.4}$$

where p_{ij} are arbitrary in P and $x_{ij} \in P$ are rational functions of p_{ij} with coefficients in A uniquely determined by (1.1). From this we obviously see that $\bar{S}(\bar{\alpha},\bar{\gamma}) \approx p^{10}$. We shall deal with the relationship between $\bar{u}(\bar{\alpha},\bar{\gamma})$ and $\bar{S}(\bar{\alpha},\bar{\gamma})$ in the upcoming proposition.

Here we shall practice conjugating by a succession of matrices in $Sp_4(F)$ to simplify $\bar{S}(\bar{x})$. Let $s \in \bar{S}(\bar{x})$. Any matrix of the form (2.4) may be changed into the analogous form with (1,4) entry and (2,3) entry equal by using some matrix of the form (2.1) with $|\alpha|$, $|\beta|$, $|\gamma|$ and $|x| \leq 1$. Over any p-adic field with odd residual characteristic, the Jacobian of these conjugation maps has modulus 1. Conjugating this by the matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with some $a \in P$ yields the form with (1,4) entry $=$ (2,3) entry $= 0$. Next conjugating this form by the matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}$$

with some $b \in P$ yields the form

$$\begin{pmatrix} * & * & * & 0 \\ * & * & 0 & * \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix}$$

with *'s as in (2.4), since the Jacobian of each of these conjugation maps has modulus 1. This form is then conjugate to the analogous form with (3,3) entry = 1 and with (1,4) entry = (2,3) entry = (3,4) entry = 0 by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with some $c \in P$, which may be transformed into the analogous form with (3,3) entry = (4,4) entry = 1 and with zeros as before by a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & d & 0 & 1 \end{pmatrix}$$

with some $d \in P$. Lastly it may be transformed into the form

$$\begin{pmatrix} 1+z_{11} & z_{12} & \bar{\alpha} & 0 \\ \frac{\bar{\gamma}}{\bar{\alpha}}z_{12} & 1+z_{22} & 0 & \bar{\gamma} \\ \frac{1}{\bar{\alpha}}z_{11} & \frac{1}{\bar{\alpha}}z_{12} & 1 & 0 \\ \frac{1}{\bar{\alpha}}z_{12} & \frac{1}{\bar{\gamma}}z_{22} & 0 & 1 \end{pmatrix} \tag{2.5}$$

with some $z_{ij} \in P$ by conjugating by $d(e,f)$ with $e = \sqrt{1+\frac{p_{13}}{\bar{\alpha}}}, f = \sqrt{1+\frac{p_{24}}{\bar{\gamma}}}$ for some $p_{13}, p_{24} \in P$. For later use, we let $\bar{S}_3(\bar{\alpha},\bar{\gamma})$ be the set of all matrices of the form (2.5), and let $\hat{P}$ be the composite map of the conjugations which take the form (2.4) to the form (2.5).

3. INTEGRAND FOR SHALIKA'S UNIPOTENT SUBREGULAR GERMS

If any of the form (2.5) may be a unipotent element, either $z_{12} = 0$ or $z_{11} = -z_{22}$ is obtained. The former result $z_{12} = 0$ implies $z_{11} = 0$, which again yields $z_{22} = 0$. The latter implies $\bar{\alpha}z_{11}^2 + \bar{\gamma}z_{12}^2 = 0$, so $\frac{-\bar{\gamma}}{\bar{\alpha}} \in (F^\times)^2$. Recall that an $n \times n$ matrix u is unipotent if and only if $(u-1)^m = 0$ for some $m \in Z^+$. So, we get the following considering (2.4) and the proof of [5] Proposition (3.8).

PROPOSITION (3.0). Let $\bar{x} = (\bar{\alpha},\bar{\gamma})$ be representative pairs with $-\frac{\bar{\alpha}}{\bar{\gamma}} \notin (F^\times)^2$. Then the only unipotent orbit intersecting $\bar{S}(\bar{x})$ is the class of $\bar{u}(\bar{x})$.

Now assume Θ to be a nonsquare in $F^\times$ and write $E^\Theta = F(\sqrt{\Theta})$. Then E_1^Θ being analogous to the unit circle in C, it becomes a compact group under multiplication. More precisely

$$E_1^\Theta = \{a + b\sqrt{\Theta} : a,b \in F \quad \text{and} \quad a^2 - \Theta b^2 = 1\}.$$

Supposing that T be the set of all matrices of the form

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ b\Theta_1 & 0 & a & 0 \\ 0 & \beta\Theta_2 & 0 & \alpha \end{pmatrix} \quad \text{with} \quad \Theta_1, \Theta_2 \in F^\times \backslash (F^\times)^2, \ i.e.$$

squarefree elements and with $a^2 - \Theta_1 b^2 = \alpha^2 - \Theta_2\beta^2 = 1$, we may see easily that T as a subgroup of G is isomporphic to $E_1^{\Theta 1} \times E_1^{\Theta 2}$ both algebraically and topologically, and that T becomes an elliptic torus as a Cartan subgroup.

According to the Shalika's theorem (see [14] p. 236) as we have mentioned earlier, we have a kind of expansion

$$\int_{T\backslash G} f(t^g)d\dot{g} = \sum_{j=1}^{n} \Gamma_j(t) \int_{Z(u_j)\backslash G} f(u_j^g)d\dot{g} \tag{3.1}$$

where $\{u_j\}$ is a finite set of representatives of the unipotent orbits, $f \in C_c^\infty(G)$, and t is any regular element sufficiently close to the identity, although "how close" depends on f. Here the functions Γ_j called Shalika's germs do not depend on f, but depend on a maximal torus T.

We intend to compute the functions $\Gamma_{\overline{u}(\overline{x})}(t)$ corresponding to the element $\overline{u}(\overline{x})$ of §2 by letting $f = \chi_{\overline{S}(\overline{x})}$ be the characteristic function of the set $\overline{S}(\overline{x})$ defined in §2. Thanks to proposition (3.0), the integrals on the right hand side of (3.1) all vanish in the case of $f = \chi_{\overline{S}(\overline{x})}$ with $\overline{x} = (\overline{\alpha}, \overline{\gamma})$ and $-\frac{\overline{\alpha}}{\overline{\gamma}} \notin (F^\times)^2$ except for that corresponding to $\overline{u}(\overline{x})$. This facilitates for us to compute the germs sought, but it may not be easy to calculate the others, i.e., those for the pairs with $-\frac{\overline{\alpha}}{\overline{\gamma}} \in (F^\times)^2$. This note deals with the former cases only.

4. CHANGE OF VARIABLES AND JACOBIANS

Let t be a regular element of T sufficiently close to the identity; write $t = x + d$, and assume that t is an element such that the nontrivial coefficients of the characteristic polynomial of $t - id$ are in P, i.e., $c(t) \in P^2$ according to our convention. By the way $char(t^g) = char(t) = det(t - \lambda \cdot 1) = \lambda^4 - 2(a+\alpha)\lambda^3 + 2(1+2a\alpha)\lambda^2 - 2(a+\alpha)\lambda + 1$, where a and α refer to the entries in the matrix in §3.

On the other hand, the characteristic polynomial of a matrix s in the form (2.6) turns out to be

$$char(s) = \lambda^4 + \lambda^3(-4 - z_{11} - z_{22}) + \lambda^2\left(6 + 2z_{11} + 2z_{22} + z_{11}z_{22} - z_{12}^2 \cdot \frac{\overline{\gamma}}{\overline{\alpha}}\right)$$
$$+ \lambda(-4 - z_{11} - z_{22}) + 1 .$$

So we obviously see that $c(s) \in P^2$. In case that s and t are conjugate, the corresponding coefficients of $char(s)$ and $char(t)$ must be the same, thus the following must hold:

$$\left.\begin{array}{l} z_{22} = 2(a+\alpha) - 4 - z_{11} \\ z_{11}z_{22} - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2 + 6 - 2(1+2a\alpha) + 4(a+\alpha) - 8 = 0 \end{array}\right\} \tag{4.0}$$

$$\Leftrightarrow \begin{cases} z_{22} = 2(a+\alpha) - 4 - z_{11} \\ z_{11}z_{22} - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2 - 4(a-1)(\alpha-1) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} z_{22} = 2(a+\alpha) - 4 - z_{11} \\ z_{11}^2 - 2\{(a-1) + (\alpha-1)\}z_{11} + 4(a-1)(\alpha-1) + \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2 = 0 . \end{cases}$$

The last equations are solvable if and only if $(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2 \in (F^\times)^2 \cup (0)$.

For any given matrix s of the form (2.5) subject to (4.0), we are going to determine whether we may find $g \in G$ satisfying $t^g = s$. But we see easily by trivial computation that: in case that

$s \in \overline{S}_3(\overline{\alpha},\overline{\gamma})$ is any element with the property $z_{12} \neq 0$, and with $(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2$ square, there exists $g \in G$ s.t.

$$\left.\begin{aligned} & t^g = s \in \overline{S}(\overline{x}) \quad \text{for} \quad \overline{x} = (\overline{\alpha},\overline{\gamma}) \\ & \Leftrightarrow \frac{2b\overline{\alpha}}{\overline{p}(\alpha - a)} \in N_F^{E^{\theta_1}}\left((E^{\theta_1})^x\right) \\ & \text{and } \frac{2\beta\overline{\alpha}}{\overline{Q}(a-\alpha)} \in N_F^{E^{\theta_2}}\left((E^{\theta_2})^x\right) \end{aligned}\right\} \tag{4.1}$$

For a fixed regular $t \in T$, let $c' : T\backslash G \to G$ be the continuous map given by $c'(g) = t^g$. Put $\overline{G}(t) = (c')^{-1}(\overline{S}(\overline{x}))$. The orbital integral of $f = \chi_{\overline{S}(\overline{x})}$ over the conjugacy class of t is just the measure of $\overline{G}(t)$. Define a mapping $P' : \overline{S}_3 \times P^7 \to P \times P^7$ via $P'((z_{11}, z_{12}, z_{22}), \ldots) = ((z_{12}), \ldots)$, which is obviously a projection. Now we construct the following composite map:

$$(T\backslash G \supset)\overline{G}(t) \xrightarrow{c'} \overline{S}(\overline{x}) \xrightarrow{\hat{P}} \overline{S}_3 \times p^7 \xrightarrow{P'} P \times P^7 .$$

Fig. 1

Here the middle arrow $\hat{P}$ in Fig. 1 arises as a homeomorphism which has shown up in §2. Due to the above description, if (2.5) satisfies (4.1), this composite map is bijective except at $z_{12} = 0$ and at z_{12} which does not make $(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2$ square. We want to find out the composite map's Jacobian so that we may compute the measure of $\overline{G}(t)$.

Let U be a neighborhood of a fixed $t \in T' \cap K_1$ chosen so that no two elements of U are conjugate. Let $\tilde{A} \subset T' \times T\backslash G$ be an open set $\tilde{A} = \{(t,g) : t \in U, t^g \in \overline{S}(\overline{x})\}$. Construct the following commuting diagram.

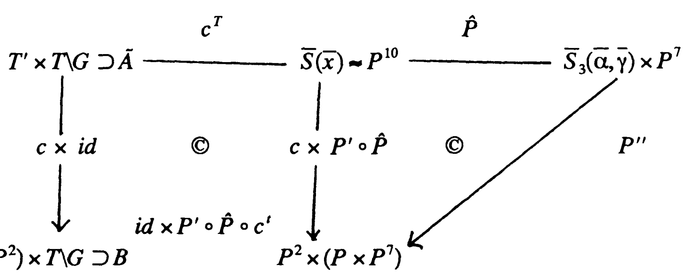

Fig. 2

The upper left mapping c^T is just the conjugation map taking $(t,g) \in T' \times T\backslash G$ to $t^g = g^{-1}tg$ and $B = c \times id(\tilde{A})$. The middle vertical map $c \times P' \circ \hat{P}$ denotes $c \times P' \circ \hat{P}(s) = (c(s), P' \circ \hat{P}(s)) \forall s \in \overline{S}(\overline{x})$. Specifically $c(s) = (c_1, c_2)$, where $c_1 = \text{trace}(s-1)$ and $c_2 =$ the coefficients of λ^2 appearing in $|s - 1 - \lambda \cdot 1|$. For $(s_3, p_1, \ldots, p_7) \in \overline{S}_3 \times P^7$, the opposite diagonal map P'' is defined as $P''(s_3, p_1, \ldots, p_7) = (c(s_3), z_{12}, p_1, \ldots, p_7)$.

Next we shall discuss the Jacobians of these maps. The Jacobian of the map c^T is just $D(t) = det(id - Ad(t))_{\mathcal{G}/\mathfrak{t}}$, where $\mathcal{G}$ and $\mathfrak{t}$ are the associated Lie algebras of G and T respectively (see [14] p. 231). It is not hard to know $|J(c)| = \left|(a-\alpha)\sqrt{a^2-1}\sqrt{\alpha^2-1}\right|$. Moreover, since $|J(\hat{P})| = |J(P'')| = 1$, we have

$$|J(P' \circ \hat{P} \circ c')| = \left| \frac{D(t)}{(a-\alpha)\sqrt{a^2-1}\sqrt{\alpha^2-1}} \right| \tag{4.2}$$

As in [5] §5, we have $|D(t)| = |\sqrt{a^2-1}\sqrt{\alpha^2-1}(a-\alpha)|^2$ and $|J(c \times P' \circ \hat{P})| = 1$. Hence

$$|J(P' \circ \hat{P} \circ c')| = \left|D(t)/(a-\alpha)\sqrt{a^2-1}\sqrt{\alpha^2-1}\right| = |D(t)|^{1/2}$$

5. ORBITAL INTEGRALS WITH NORMALIZATION OF MEASURES

We take the natural additive measure dx on F so that A has measure 1. As $T \cong E_1^{\Theta_1} \times E_1^{\Theta_2}$ and $(E^{\Theta_i})^{x}/F^{x} \supset E_1^{\Theta_i}/\{\pm 1\}$, choices of measures on $(E^{\Theta_i})^{x}$ and F^{x} determine a choice of measure on $E_1^{\Theta_i}$. On $(E^{\Theta_i})^{x}$ we may take the corresponding measure $d^{x}x = \frac{dx}{|x|_{E^{\Theta_1}}}$, and on F^{x} v $d^{x}s = \frac{ds}{|s|}$. Now select the measure on G whose restriction to K is an extension of the standard measure of $\overline{S}(\overline{x}) = P^{10}$. Since $|J(c \circ P' \circ \hat{P})| = 1$, Haar measure of $\overline{S}(\overline{x})$ must be the same as that of P^{10}. A choice of measure on $T\backslash G$ depends on that of G and T which also gives the natural measure on K and $\overline{S}(\overline{x})$.

Now recall

$$\overline{P} = 2-2a+z_{22}, \quad \overline{Q} = 2-2\alpha+z_{22},$$

i.e., explicitly

$$\overline{P} = \alpha - a \pm \sqrt{(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2}$$

$$\overline{Q} = a - \alpha \pm \sqrt{(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2}.$$

We put

$$X(a,\alpha,\overline{\alpha},\overline{\gamma}) = \left\{ z_{12} \in P : \alpha - a + \sqrt{(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2} \in N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^{x}\right) \right\},$$

$$X'(a,\alpha,\overline{\alpha},\overline{\gamma}) = \left\{ z_{12} \in P : \alpha - a - \sqrt{(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2} \in N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^{x}\right) \right\},$$

$$Y(a,\alpha,\overline{\alpha},\overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha + \sqrt{(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2} \in N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^{x}\right) \right\},$$

$$Y'(a,\alpha,\overline{\alpha},\overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha - \sqrt{(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2} \in N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^{x}\right) \right\},$$

$$\overline{X}(a,\alpha,\overline{\alpha},\overline{\gamma}) = \left\{ z_{12} \in P : \alpha - a + \sqrt{(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2} \in F^{x} \backslash N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^{x}\right) \right\},$$

$$\overline{X}'(a,\alpha,\overline{\alpha},\overline{\gamma}) = \left\{ z_{12} \in P : \alpha - a - \sqrt{(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2} \in F^{x} \backslash N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^{x}\right) \right\},$$

$$\overline{Y}(a,\alpha,\overline{\alpha},\overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha + \sqrt{(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2} \in F^{x} \backslash N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^{x}\right) \right\},$$

$$\overline{Y}'(a,\alpha,\overline{\alpha},\overline{\gamma}) = \left\{ z_{12} \in P : a - \alpha - \sqrt{(a-\alpha)^2 - \frac{\overline{\gamma}}{\overline{\alpha}} z_{12}^2} \in F^{x} \backslash N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^{x}\right) \right\}.$$

Let $\overline{m}$ be the Haar measure function on these sets. We have then the following orbital integrals.

PROPOSITION 5.0. (i) If $\frac{2b\bar{\alpha}}{\alpha-a} \in N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^x\right)$ and $\frac{2\beta\bar{\alpha}}{a-\alpha} \in N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^x\right)$, then

$$\int_{T\backslash G} \chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(t^g)d\dot{g} = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times q^{-7} \times |D(t)|^{-1/2}.$$

(ii) If $\frac{2b\bar{\alpha}}{\alpha-a} \in N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^x\right)$ and $\frac{2\beta\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^x\right)$, then

$$\int_{T\backslash G} \chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(t^g)d\dot{g} = \overline{m}((X \cap \overline{Y}) \cup (X' \cap \overline{Y}')) \times q^{-7} \times |D(t)|^{-1/2}.$$

(iii) If $\frac{2b\bar{\alpha}}{\alpha-a} \notin N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^x\right)$ and $\frac{2\beta\bar{\alpha}}{a-\alpha} \in N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^x\right)$, then

$$\int_{T\backslash G} \chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(t^g)d\dot{g} = \overline{m}((\overline{X} \cap Y) \cup (\overline{X}' \cap Y')) \times q^{-7} \times |D(t)|^{-1/2}.$$

(iv) If $\frac{2b\bar{\alpha}}{\alpha-a} \notin N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^x\right)$ and $\frac{2\beta\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^x\right)$, then

$$\int_{T\backslash G} \chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(t^g)d\dot{g} = \overline{m}((\overline{X} \cap \overline{Y}) \cup (\overline{X}' \cap \overline{Y}')) \times q^{-7} \times |D(t)|^{-1/2}.$$

PROOF. We have already seen the Jacobian of $P' \circ \hat{P} \circ c'$ is just $|D(t)|^{1/2}$ and that the measure of P is fixed to be q^{-1}. So we have our result considering the above remark and (4.1).

Now we must look for the orbital integral over the conjugacy class of $\bar{u}(\bar{\alpha},\bar{\gamma})$. To see this we need to specify the measure on the centralizer $Z(\bar{u}(\bar{\alpha},\bar{\gamma}))$. Any element of $Z(\bar{u}(1,1))$ should be of the form:

$$\begin{pmatrix} a_{11} & \pm\sqrt{1-a_{11}^2} & a_{13} & a_{14} \\ \pm\sqrt{1-a_{11}^2} & \mp a_{11} & a_{23} & a_{13} - \dfrac{a_{23}a_{11} \pm a_{11}a_{14}}{\sqrt{1-a_{11}^2}} \\ 0 & 0 & a_{11} & \pm\sqrt{1-a_{11}^2} \\ 0 & 0 & \pm\sqrt{1-a_{11}^2} & \mp a_{11} \end{pmatrix} \quad \text{if } a_{11} \neq \pm 1 \text{ and } \sqrt{1-a_{11}^2} \in F,$$

or

$$\begin{pmatrix} 1 & 0 & a_{13} & \mp a_{23} \\ 0 & \mp 1 & a_{23} & a_{24} \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & \mp 1 \end{pmatrix} \quad \text{if } a_{11} = \pm 1.$$

By the way $\bar{u}(\bar{\alpha},\bar{\gamma}) = d\left(\sqrt{\bar{\alpha}},\sqrt{\bar{\gamma}}\right)\bar{u}(1,1)d\left(\sqrt{\bar{\alpha}^{-1}},\sqrt{\bar{\gamma}^{-1}}\right)$ implies that

$$\begin{aligned} Z(\bar{u}(\bar{\alpha},\bar{\gamma})) &= Z\left(d\left(\sqrt{\bar{\alpha}},\sqrt{\bar{\gamma}}\right) \cdot \bar{u}(1,1) \cdot d\left(\sqrt{\bar{\alpha}^{-1}},\sqrt{\bar{\gamma}^{-1}}\right)\right) \\ &= d\left(\sqrt{\bar{\alpha}},\sqrt{\bar{\gamma}}\right) \cdot Z(\bar{u}(1,1)) \cdot d\left(\sqrt{\bar{\alpha}^{-1}},\sqrt{\bar{\gamma}^{-1}}\right) \end{aligned}$$

We decompose G into the form

$$G = B_{(\bar{\alpha},\bar{\gamma})} \cdot K = Z(\bar{u}(\bar{\alpha},\bar{\gamma})) \cdot \bar{P} \cdot K, \quad \text{where} \quad B_{(\bar{\alpha},\bar{\gamma})} = Z(\bar{u}(\bar{\alpha},\bar{\gamma})) \cdot \bar{P}$$

$$\text{and} \quad \bar{P} = \{d(b_{11}) : \forall\, b_{11} \in F^x\}.$$

Hence the integral over $Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\backslash G$ may be replaced by an integral over $\{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\backslash Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cdot\tilde{P}\}\cdot K$, and this coset space may be represented by a subset of $\tilde{P}K$, the measure of $\tilde{P}$ being just $d^{\times}b_{11}$, and dz being an appropriate Haar measure of $Z(\bar{u}(\bar{\alpha},\bar{\gamma}))$. Next, consider the following integral. For any $f\in C_c^{\infty}(G)$,

$$\int_G f(g)dg = \int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\backslash G}\int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))} f(zg)dz\,d\dot{g}$$

$$= \int_K\int_P\int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))} f(zpk)\cdot dz\cdot\bar{c}\cdot\frac{dp\,dk}{\Delta_{B(\bar{\alpha},\bar{\gamma})}(p)}$$

where $\bar{c}$ arises because $Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\times\tilde{P}\times K\rightarrow G$ given by $(z,p,k)\rightarrow z\cdot p\cdot k$ is not a topological isomorphism. We may figure out the constant $\bar{c}$ by calculating the measure of K. The modular function being trivial on $\tilde{P}\cap K$,

$$\int_K f(g)dg = \int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K\backslash K}\int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K} f(zg)dz\,d\dot{g}$$

$$= \int_K\int_{\tilde{P}\cap K}\int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K} f(zpk)\bar{c}\cdot dz\,dp\,dk\ .$$

The inner integrals must be the same after setting $f=\chi_K$, the characteristic function of K; so deleting these, we obtain

$$\int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K\backslash K} 1\cdot d\dot{g} = \int_K\int_{\tilde{P}\cap K}\bar{c}\,dp\,dk = \int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K\backslash K}\int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K}\int_{\tilde{P}\cap K}\bar{c}\,dp\,dz\,d\dot{g}\ .$$

So we have $\bar{c}=\left(1-\frac{1}{q}\right)^{-1}\cdot\overline{m}(Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K)^{-1}$. Hence the quotient measure of $Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\backslash G$ is obtained by writing $\dot{g}=pk$ with $p\in\tilde{P}$, $k\in K$ and putting $d\dot{g}=\left(1-\frac{1}{q}\right)^{-1}\times\overline{m}(Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K)^{-1}/\Delta_{B(\bar{\alpha},\bar{\gamma})}(p)dp\,dk$ since $B(\bar{\alpha},\bar{\gamma})$ is not unimodular although G, K, $\tilde{P}$ and $Z(\bar{u}(\bar{\alpha},\bar{\gamma}))$ are unimodular.

PROPOSITION 5.1. With the assumption of measures normalized as above, we have

$$\int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\backslash G}\chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(\bar{u}(\bar{\alpha},\bar{\gamma})^{g})d\dot{g}=q^{-7}$$

PROOF. The decomposition $G=B_{(\bar{\alpha},\bar{\gamma})}\cdot K$ assures that any element conjugate to $\bar{u}(\bar{\alpha},\bar{\gamma})$ is determined by $g=pk$ with $p\in\tilde{P}$ and $k\in K$. So, we have

$$\int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\backslash G}\chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(\bar{u}(\bar{\alpha},\bar{\gamma})^{g})d\dot{g}=\int_K\int_P\chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(\bar{u}(\bar{\alpha},\bar{\gamma})^{pk})\cdot\left(1-\frac{1}{q}\right)^{-1}\cdot\overline{m}(Z(\bar{u}(\bar{\alpha},\bar{\gamma})\cap K)^{-1}\cdot\frac{dp\,dk}{\Delta_{B(\bar{\alpha},\bar{\gamma})}(p)}$$

By the way $p^{-1}\bar{u}(\bar{\alpha},\bar{\gamma})p=ksk^{-1}$ for $k\in K$, $s\in\bar{S}(\bar{\alpha},\bar{\gamma})$ implies that $p\in\tilde{P}\cap K$. Hence it is not difficult to see that $k=p^{-1}k'$ with $k'\in(Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K)\cdot K_1$ if and only if $\bar{u}(\bar{\alpha},\bar{\gamma})^{pk}\in\bar{S}(\bar{\alpha},\bar{\gamma})$. Since the modular function is 1 for $p\in K$, we obtain

$$\int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\backslash G}\chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(\bar{u}(\bar{\alpha},\bar{\gamma})^{g})d\dot{g}=\int_{(Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K)\cdot K_1}\int_{\tilde{P}\cap K}\left(1-\frac{1}{q}\right)^{-1}\cdot\overline{m}(Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K)^{-1}dp\,dk\ .$$

Since the measure of $\tilde{P}\cap K$ is $1-\frac{1}{q}$ and the measure of $(Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K)\cdot K_1$ is just $q^{-7}\overline{m}(Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\cap K)$, we have

$$\int_{Z(\bar{u}(\bar{\alpha},\bar{\gamma}))\backslash G}\chi_{\bar{S}(\bar{\alpha},\bar{\gamma})}(\bar{u}(\bar{\alpha},\bar{\gamma})^{g})d\dot{g}=q^{-7}$$

as required.

Finally, we combine everything, in particular propositions (5.0) and (5.1) to yield our main result. Notice that Θ_j may belong to three nontrivial residue classes mod $(F^\times)^2$.

THEOREM 5.2. Suppose that we are given an elliptic torus T as in §3. Then the Shalika's unipotent subregular germs for G in the case of $-\frac{\bar{\alpha}}{\bar{\gamma}} \notin (F^\times)^2$ are obtained case by case as follows:

(i) If $\frac{2b\bar{\alpha}}{\alpha-a} \in N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^\times\right)$ and $\frac{2\beta\bar{\alpha}}{a-\alpha} \in N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^\times\right)$, then

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times |D(t)|^{-1/2}.$$

(ii) If $\frac{2b\bar{\alpha}}{\alpha-a} \in N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^\times\right)$ and $\frac{2\beta\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^\times\right)$, then

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \overline{m}((X \cap \overline{Y}) \cup (X' \cap \overline{Y}')) \times |D(t)|^{-1/2}.$$

(iii) If $\frac{2b\bar{\alpha}}{\alpha-a} \notin N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^\times\right)$ and $\frac{2\beta\bar{\alpha}}{a-\alpha} \in N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^\times\right)$, then

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \overline{m}((\overline{X} \cap Y) \cup (\overline{X}' \cap Y')) \times |D(t)|^{-1/2}.$$

(iv) If $\frac{2b\bar{\alpha}}{\alpha-a} \notin N_F^{E^{\Theta_1}}\left(\left(E^{\Theta_1}\right)^\times\right)$ and $\frac{2\beta\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\Theta_2}}\left(\left(E^{\Theta_2}\right)^\times\right)$, then

$$\Gamma_{\bar{u}(\bar{\alpha},\bar{\gamma})} = \overline{m}((\overline{X} \cap \overline{Y}) \cup (\overline{X}' \cap \overline{Y}')) \times |D(t)|^{-1/2}.$$

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