COORDINATE d-DIMENSION PRINTS

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ABSTRACT. We define the coordinate d-dimension print to distinguish sets of same fractal dimension, and investigate its geometrical properties.

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1. INTRODUCTION

Sets of very different geometric characteristics may have same Hausdorff dimension. In [6], C. A. Rogers introduced the notion of Hausdorff dimension print to distinguish such sets. It is not easy to obtain the Hausdorff dimension print of a given set, even though it may be highly regular. In particular, we recognized that it is extremely difficult to find the Hausdorff dimension print of a nowhere differentiable continuous function. We introduce the coordinate d-dimension print, evolved from the d-dimension to deal with above difficulties. The coordinate d-dimension print of a given set informs us of its geometric characteristics, its d-dimension, and the d-dimension of its projection to x-axis.

We investigate coordinate d-dimension prints for the graphs of nowhere differentiable continuous functions and for regular sets.

At the end, we shows how the coordinate d-dimension print can be used for calculating the d-dimension of a set related to the aforementioned graph.

2. PRELIMINARIES

We restrict our attention to subsets of $\mathbf{R}^2$ for simplicity.

We mean a (b, a) coordinate rectangle by the set of the form $[mb, (m+1)b] \times [na, (n+1)a]$ where m and n are integers.

For $s, t \geq 0$, we define a pre-measure for a set $E \subset \mathbf{R}^2$ by

$$CD^{(s,t)}(E) = \lim_{\delta \to 0} \inf_{\delta > 0} \{N_E(a, b) a^s b^t : 0 < b \leq a \leq \delta\}$$

where $N_E(a, b)$ is the number of (b, a) coordinate rectangles that interest E.

We obtain an outer measure using Method I by Munroe [4]

$$cd^{(s,t)}(E) = \inf\{\sum_{n=1}^{\infty} CD^{(s,t)}(E_n) : E = \cup_{n=1}^{\infty} E_n\}$$

for a set $E \subset \mathbf{R}^2$.

Now, we recall lower box dimension ([2]) (lower capacity ([5])), $\underline{\dim}_B(E)(= \underline{\text{Cap}}\ (E)) = \liminf_{a\to 0} \frac{\log N(E,a)}{-\log a}$, and modified lower box dimension ([2]) (d-dimension ([3])), $\underline{\dim}_{MB}(E)(= d-\dim(E)) = \inf\{\sup_n \underline{\dim}_B(E_n) : E = \cup_{n=1}^{\infty} E_n\}$, where $N(E,a) = N_E(a,a)$ for $E \subset \mathbf{R}^2$ and $N(E,a)$ is the number of intervals of the form $[ma,(m+1)a]$ that interest E if $E \subset \mathbf{R}^1$.

It is not difficult to show that

$$\sup\{s > 0 : CD^{(s,0)}(E) > 0\} = \underline{\text{Cap}}(E)$$

and

$$\sup\{s > 0 : cd^{(s,0)}(E) > 0\} = d - \dim(E)$$

for $E \subset \mathbf{R}^2$ (cf. [2], [3]). (Note that the smallest number of squares of side a that cover $E \leq \inf_{0<b\leq a} N_E(a,b) \leq N_E(a,a)$.) Note that $CD^{(s,0)}$ and $cd^{(s,0)}$ are only the variations of D^s-premeasure and d^s-measure in [3] respectively.

Hence we just write $CD^{(s,0)}(E) = D^s(E)$ and

$$cd^{(s,0)}(E) = d^s(E) \text{ for } E \subset \mathbf{R}^2.$$

For $E \subset \mathbf{R}^1$, similarly we define

$$D^s(E) = \liminf_{a\to 0} N(E,a)a^s$$

and

$$d^s(E) = \inf\{\sum_{n=1}^{\infty} D^s(E_n) : E = \cup E_n\}.$$

Then we have

$$\sup\{s > 0 : D^s(E) > 0\} = \underline{\text{Cap}}(E)$$

and

$$\sup\{s > 0 : d^s(E) > 0\} = d - \dim(E).$$

We define *the coordinate d-dimension print* of $E \subset \mathbf{R}^2$ by

$$\text{cd - Print } (E) = \{(s,t) : cd^{(s,t)}(E) > 0\}.$$

Plainly the coordinate d-dimension print is monotonic; cd-Print $(E_1) \subset$ cd- Print (E_2) for $E_1 \subset E_2$. Further it is additive, in the sense that

$$\text{cd - Print } (\cup_{n=1}^{\infty} E_n) = \cup_{n=1}^{\infty} \text{cd - Print}(E_n).$$

3. PROPERTIES OF cd-PRINTS

By the definition, we have the following straightforward propositions.

PROPOSITION 1. $\mathcal{H}$-Print $(E) \subset$ cd-Print (E) for $E \subset \mathbf{R}^2$. (Here, $\mathcal{H}$-Print means Hausdorff dimension print [6])

PROOF. Comparing the families of rectangles that cover E in the definitions, we have $\mathcal{H}^{(s,t)}(E) \leq cd^{(s,t)}(E)$ for $E \subset \mathbf{R}^2$.

If a point (s,t) is in the coordinate d-dimension print of E, then $\{(s',t') : s'+t' < s+t, t' < t\}$ is contained in the coordinate d-dimension print of E

PROPOSITION 2. If $cd^{(s,t)}(E) > 0$, then $cd^{(s',t')}(E) = \infty$ for $s'+t' < s+t$ and $t' \leq t$.

PROOF. Suppose $cd^{(s,t)}(E) > 0$. Then for every sequence $\{E_n\}$ of subsets that cover E, $\sum_{n=1}^{\infty} CD^{(s,t)}(E_n) > 0$. So there exists E_{n_0} such that $CD^{(s,t)}(E_{n_0}) > \alpha > 0$. Thus $\inf\{N_{E_{n_0}}(a,b)a^s b^t : 0 < b \leq a \leq \delta_0\} > \alpha$ for some δ_0.

For $s'+t' < s+t$, $t' \leq t$, and $b \leq a \leq \delta \leq \delta_0$

$$\begin{aligned}
N_{E_{n_0}}(a,b)a^{s'}b^{t'} &= N_{E_{n_0}}(a,b)a^s b^t a^{s'-s} b^{t'-t} \\
&\geq N_{E_{n_0}}(a,b)a^s b^t a^{s'-s} a^{t'-t} \\
&\geq N_{E_{n_0}}(a,b)a^s b^t \delta^{(s'+t')-(s+t)} \\
&> \alpha\delta^{(s'+t')-(s+t)}
\end{aligned}$$

Therefore,

$$CD^{(s',t')}(E_{n_0}) = \infty \quad \text{for } s'+t' < s+t \text{ and } t' \leq t.$$

Hence $cd^{(s',t')}(E) = \infty$.

REMARK 3. Let $C_{\frac{1}{4}} = \left\{ \sum_{n=1}^{\infty} a_n 4^{-n} : a_n \in \{0,3\} \right\}$.

By Example 2 in [6] and Proposition 1

$$cd^{(\frac{1}{2},\frac{1}{2})}(C_{\frac{1}{4}} \times C_{\frac{1}{4}}) > 0.$$

It follows from Proposition 2 that $cd^{(0,\frac{1}{2})}(C_{\frac{1}{4}} \times C_{\frac{1}{4}}) = \infty$.

However we note that $d^{\frac{1}{2}}(C_{\frac{1}{4}}) = 1$. (Compare this with Theorem 8.)

PROPOSITION 4. $cd^{(s',t')}(E) \geq cd^{(s,t)}(E)$ for $s'+t' \leq s+t$ and $t' \leq t$.

PROOF. It follows from similar argument with the proof of Proposition 2.

COROLLARY 5. If $cd^{(0,t)}(E) < \infty$, then $cd^{(s,t)}(E) = 0$ for any $s > 0$, and $cd^{(0,t')}(E) = 0$ for $t' > t$.

PROOF. It follows immediately from Proposition 2.

To deal with the geometric characteristics of cd-Print, we need the following two simple but interesting lemmas.

LEMMA 6. If $d - \dim(A) = \alpha$ for $A \subset \mathbf{R}^1$, then $cd^{(0,\beta)}(A \times \mathbf{R}) = 0$ for $\beta > \alpha$. In particular, if $d - \dim\,(\mathrm{Proj}_x E) = \alpha$ for $E \subset \mathbf{R}^2$, then $cd^{(0,\beta)}(E) = 0$ for $\beta > \alpha$. (Here $\mathrm{Proj}_x E$ denotes the projection of E to x-axis.)

PROOF. Since $d-\dim(A)=\alpha$, $d^{\beta'}(A)=0$ for $\beta>\beta'>\alpha$. By the definition of d-measure, there exists a sequence $\{A_n^0\}_{n=1}^{\infty}$ such that $\cup_{n=1}^{\infty}A_n^0=A$ and $\sum_{n=1}^{\infty}D^{\beta'}(A_n^0)<1$. Thus, for any integer m and n and $\beta>\beta'$,

$$\begin{aligned} CD^{(0,\beta)}(A_n^0\times[m,m+1]) &= \lim_{\delta\to 0}\inf\{N_{A_n^0\times[m,m+1]}(a,b)a^0b^{\beta}:0<b\le a\le\delta\} \\ &\le \lim_{\delta\to 0}(\frac{1}{\delta}+2)\inf\{N(A_n^0,b)b^{\beta}:0<b\le\delta\} \\ &\le \lim_{\delta\to 0}(\frac{1}{\delta}+2)\lim_{\rho\to 0}\inf_{0<b\le\rho}N(A_n^0,b)b^{\beta} \\ &= \lim_{\delta\to 0}(\frac{1}{\delta}+2)D^{\beta}(A_n^0)=0. \end{aligned}$$

Therefore

$$\begin{aligned} cd^{(0,\beta)}(A\times\mathbf{R}) &\le \inf\{\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}CD^{(0,\beta)}(A_n\times[m,m+1]):A=\cup_{n=1}^{\infty}A_n\} \\ &\le \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}CD^{(0,\beta)}(A_n^0\times[m,m+1])=0. \end{aligned}$$

LEMMA 7. If $d-\dim\,(\mathrm{Proj}_x E)=\alpha$ for $E\subset\mathbf{R}^2$, then $cd^{(0,\gamma)}(E)=\infty$ for $\gamma<\alpha$.

PROOF. Since $d^{\gamma}\,(\mathrm{Proj}_x E)=\infty$ for $\gamma<\alpha$, for any cover $\{E_n\}$ of E,

$$\begin{aligned} &\sum_{n=1}^{\infty}CD^{(0,\gamma)}(E_n) \\ &\ge \sum_{n=1}^{\infty}D^{\gamma}(Proj_x E_n) \\ &\ge d^{\gamma}(Proj_x E)=\infty. \end{aligned}$$

As $d-\dim(E)=\alpha$ doesn't always mean $d^{\alpha}(E)>0$, it is natural to consider cl[cd-Print (E)], the closure of cd-Print (E) in the space $\{(s,t):s\ge 0,t\ge 0\}$.

There is very interesting connection between d-dimension of the projection of a set to x-axis and its cd-Print.

THEOREM 8. For $E\subset\mathbf{R}^2$, $d-\dim\,(\mathrm{Proj}_x E)=\alpha$ iff cl[cd - Print $(E)]\cap y-axis=[0,\alpha]$.

PROOF. If $d-\dim\,(\mathrm{Proj}_x E)=\alpha$, then cl [cd- Print $(E)]\cap\, y-axis=[0,\alpha]$ by Lemma 6 and 7.

Suppose that cl [cd - Print $(E)]\cap y-axis=[0,\alpha]$. Then $cd^{(0,\gamma)}(E)=\infty$ for $\gamma<\alpha$, and $cd^{(0,\beta)}(E)=0$ for $\beta>\alpha$ by Proposition 4 and Corollary 5.

It follows from Lemma 6 that $d-\dim\,(\mathrm{Proj}_x E)\ge\alpha$. And $d-\dim\,(\mathrm{Proj}_x E)$ $\le\alpha$ follows from Lemma 7.

Next theorem tells us a geometrical connection between d-dimension and cd-Print.

THEOREM 9. For $E\subset\mathbf{R}^2$, $d-\dim(E)=\beta$ iff cl[cd - Print $(E)]\cap x-axis=[0,\beta]$.

PROOF. It follows from $\sup\{s>0:cd^{(s,0)}(E)>0\}=d-\dim(E)$.

COROLLARY 10. If $d-\dim(\mathrm{Proj}_x E) = \alpha$ and $d-\dim(E) = \beta$ for $E \subset \mathbf{R}^2$, then cd-Print $(E) \subset \{(s,t) : s+t \le \beta, t \le \alpha\}$. In particular, cd-Print $(E) \subset \{(s,t) : s+t \le 2, t \le 1\}$ for $E \subset \mathbf{R}^2$.

PROOF. It is immediate from Proposition 4, Theorems 8 and 9.

4. CALCULATION OF *cd*-PRINTS

Now, we calculate the cd-Print of some specific subsets of $\mathbf{R}^2$. Let $|J|$ denote the length of interval J.

THEOREM 11. Let $\varphi : [0,1] \to \mathbf{R}$ be a function such that $C_1|J|^\alpha \le \sup_{x,y\in J}|\varphi(x) - \varphi(y)| \le C_2|J|^\alpha$ for any interval $J \subset [0,1]$, where some constants $C_1, C_2 > 0$ and $0 < \alpha \le 1$. Let $A(\subset [0,1])$ be a compact set such that $\underline{\mathrm{Cap}}(A\cap(a-\delta,a+\delta)) \ge \underline{\mathrm{Cap}}(A) = \beta$ for any $a \in A$ and $\delta > 0$, with $0 \le \beta \le 1$. Then for the graph of φ on A, $\mathcal{G}_A$

$$cl[\text{cd - Print}(\mathcal{G}_A)] = \{(s,t) \cdot s+t \le 1+\beta-\alpha, t \le \beta - \alpha s\}.$$

PROOF. Since $\mathcal{G}_A$ is a closed subset of $\mathbf{R}^2$, $\cup_{n=1}^\infty \overline{G_n} = \mathcal{G}_A$ for each sequence $\{G_n\}$ such that $\cup_{n=1}^\infty G_n = \mathcal{G}_A$, where $\overline{G_n}$ is the closure of G_n in $\mathbf{R}^2$. Further, by Baire Category theorem, there exists integer n_0 such that $\overline{G_{n_0}}$ contains $\mathcal{G}_A \cap B_r(x)$ for some $x \in \overline{G_{n_0}} \subset \mathcal{G}_A$ and some $r > 0$.

By continuity of φ, we can choose $\delta > 0$ such that $\{(b,(\varphi(b)) : b \in (a-\delta, a+\delta), (a,\varphi(a)) = x\} \subset B_r(x)$.

Therefore we have $\{(b,\varphi(b)) : b \in (a-\delta,a+\delta) \cap A, (a,\varphi(a)) = x\} \subset \mathcal{G}_A \cap B_r(x)$. And, we note that $\underline{\mathrm{Cap}}((a-\delta,a+\delta)\cap A) = \beta$ and $d-\dim(A) = \beta$ ([7]).

Hence we only need to show that

(1) $CD^{(s,t)}(\mathcal{G}_A) = \infty$ for $s+t < 1+\beta-\alpha$ and $t < \beta - \alpha s$.

(2) $CD^{(s,t)}(\mathcal{G}_A) = 0$ for $s+t < 1+\beta-\alpha$ or $t > \beta - \alpha s$.

(1): Suppose that $s+t < 1+\beta-\alpha$ and $t < -\alpha s + \beta$. Then we can find $\varepsilon > 0$ satisfying $s+t < 1+\beta-\alpha-\varepsilon$ and $t < -\alpha s + \beta - \varepsilon$. Since $\underline{\mathrm{Cap}}(A) = \beta$, there is b_0 such that for all positive $b \le b_0$,

$$N(A,b) \ge b^{-\beta+\varepsilon}.$$

Now, consider $\delta \le b_0$ and b, a such that $0 < b \le a \le \delta$.

In case that $C_1 b^\alpha > a$, we have

$$N_{\mathcal{G}_A}(a,b)a^s b^t \ge (C_1 b^\alpha/a) b^{-\beta+\varepsilon} a^s b^t$$
$$\ge C_1 b^{\alpha-\beta+\varepsilon+t} a^{s-1}.$$

If $\alpha - \beta + \varepsilon - t < 0$, then

$$N_{\mathcal{G}_A}(a,b)a^s b^t \ge C_1 a^{\alpha-\beta+\varepsilon-1+s+t}.$$

Otherwise,

$$N_{\mathcal{G}_A}(a,b)a^s b^t \ge C_1[(a/C_1)^{\frac{1}{\alpha}}]^{\alpha-\beta+\varepsilon+t} a^{s-1}$$

$$= (\frac{1}{C_1})^{\frac{-\beta+\varepsilon+t}{\alpha}} a^{\frac{-\beta+\varepsilon+\alpha s+t}{\alpha}}$$

In case that $C_1 b^\alpha \leq a$, we have

$$N_{\mathcal{G}_A}(a,b)a^s b^t \geq b^{-\beta+\varepsilon} a^s b^t.$$

Since $t < \beta - \varepsilon$,

$$N_{\mathcal{G}_A}(a,b)a^s b^t \geq a^s[(\frac{a}{C_1})^{\frac{1}{\alpha}}]^{-\beta+\varepsilon+t}$$

$$= (\frac{1}{C_1})^{\frac{-\beta+\varepsilon+t}{\alpha}} a^{\frac{-\beta+\varepsilon+\alpha s+t}{\alpha}}$$

Thus

$$\lim_{\delta\to 0} \inf\{N_{\mathcal{G}_A}(a,b)a^s b^t : 0 < b \leq a \leq \delta\} = \infty.$$

(2) : Suppose that $s+t > 1+\beta-\alpha$ or $t > -\alpha s+\beta$. Then we can find $\varepsilon > 0$ such that $s+t > 1+\beta-\alpha+\varepsilon$ or $t > -\alpha s+\beta+\varepsilon$. Since $\underline{\mathrm{Cap}}(A) = \beta$, for such $\varepsilon > 0$, there exist infinitely many n such that $N(A, \frac{1}{n}) \leq (\frac{1}{n})^{-\beta-\varepsilon}$. Hence,

$$N_{\mathcal{G}_A}(C_2(\frac{1}{n})^\alpha, \frac{1}{n})[C_2(\frac{1}{n})^\alpha]^s(\frac{1}{n})^t$$

$$\leq 2(\frac{1}{n})^{-\beta-\varepsilon}[C_2(\frac{1}{n})^\alpha]^s(\frac{1}{n})^t$$

$$= 2C_2^s(\frac{1}{n})^{-\beta-\varepsilon+\alpha s+t}.$$

(Here, we may assume $C_2 > 1$). And,

$$N_{\mathcal{G}_A}(\frac{1}{n}, \frac{1}{n})(\frac{1}{n})^s(\frac{1}{n})^t$$

$$\leq (\frac{1}{n})^{-\beta-\varepsilon}[C_2(\frac{1}{n})^\alpha/(\frac{1}{n}) + 2](\frac{1}{n})^{s+t}$$

$$\leq C_2(\frac{1}{n})^{\alpha-\beta-\varepsilon-1+s+t} + 2(\frac{1}{n})^{-\beta-\varepsilon+s+t}$$

Hence

$$\lim_{\delta\to 0} \inf\{N_{\mathcal{G}_A}(a,b)a^s b^t : 0 < b \leq a \leq \delta\} = 0.$$

COROLLARY 12. Let $\varphi : [0,1] \to \mathbf{R}^1$ be a function satisfying $C_1|J|^\alpha \leq \sup_{x,y\in J} |\varphi(x)-\varphi(y)| \leq C_2|J|^\alpha$ for any interval $J \subset [0,1]$, where some $0 < \alpha \leq 1$ and some constants C_1 and $C_2 > 0$. Let $A(\subset [0,1])$ be a symmetric Cantor set with a sequence of contracting ratios $\{a_n\}$([7]). Then

$$cl[\text{cd-Print}(\{(x,\varphi(x) : x \in A\})]$$

$$= \{(s,t) : s+t \leq 1 + \liminf \frac{-n\log 2}{\log a_n} - \alpha$$

$$\text{and} \quad t \leq \max\{\liminf \frac{-n\log 2}{\log a_n} - \alpha s, 0\}\}.$$

PROOF. It is easy to show

$$\underline{\mathrm{Cap}}(A \cap (a-\delta, a+\delta)) = \liminf \frac{-n\log 2}{\log a_n} = \underline{\mathrm{Cap}}(A)$$

for any $a \in A$, $\delta > 0$, and A is a compact set ([7]). It follows immediately from Theorem 11 and the fact above.

REMARK 13. There are many examples of Theorem 11. Kiesswetter's curve([1]) is one of them and the closure of its cd-Print is $\{(s,t) : s+t \le \frac{3}{2} \text{ and } t \le 1 - \frac{1}{2}s\}$.

Now, we introduce a technique to gain the closure of cd-Print of some particular sets.

PROPOSITION 14. Let $E \subset \mathbf{R}^2$ and $d-\dim(\mathrm{Proj}_x E) = \alpha$, $d-\dim(E) = \alpha+\beta$ and $cd^{(s,t)}(E) = \infty$ for $s+t < \alpha+\beta$ and $t < \alpha$. Then $cl\,[\text{cd-Print}\,(E)] = \{(s,t) : s+t \le \alpha+\beta, t \le \alpha\}$.

PROOF. If follows from Proposition 4, Theorem 8 and 9.

COROLLARY 15. Let A, B be compact subsets of $\mathbf{R}^1$ with $d-\dim(A \times B) = \alpha+\beta$, satisfying

$$\underline{\mathrm{Cap}}(A \cap (a-\delta, a+\delta)) \ge \underline{\mathrm{Cap}}(A) = \alpha \text{ for any } a \in A \text{ and any } \delta > 0$$

and

$$\underline{\mathrm{Cap}}(B \cap (b-\delta, b+\delta)) \ge \underline{\mathrm{Cap}}(B) = \beta \text{ for any } b \in B \text{ and any } \delta > 0.$$

Then

$$cl[\text{cd-Print}(A \times B)] = \{(s,t) : s+t \le \alpha+\beta \text{ and } t \le \alpha\}.$$

PROOF. Recalling the argument in the proof of Theorem 11, we only need to show $CD^{(s,t)}(A \times B) = \infty$ to prove $cd^{(s,t)}(A \times B) = \infty$ for $s+t < \alpha+\beta$ and $t < \alpha$.

Suppose that $s+t < \alpha+\beta$ and $t < \alpha$. Then there is $\varepsilon > 0$ such that $s+t < \alpha+\beta-2\varepsilon$ and $t < \alpha-\varepsilon$. Since $\underline{\mathrm{Cap}}(A) = \alpha$ and $\underline{\mathrm{Cap}}(B) = \beta$, there exists b_0 such that for all positive $b \le b_0$, $N(A,b) \ge b^{-\alpha+\varepsilon}$ and there is a_0 such that for all positive $a \le a_0$, $N(B,a) \ge a^{-\beta+\varepsilon}$. For $0 < b \le a \le \min\{a_0, b_0\}$,

$$\begin{aligned} N_{A\times B}(a,b)a^s b^t &\ge b^{-\alpha+\varepsilon} a^{-\beta+\varepsilon} a^s b^t \\ &= b^{-\alpha+\varepsilon+t} a^{-\beta+\varepsilon+s}. \end{aligned}$$

Since $-\alpha+\varepsilon+t < 0$,

$$N_{A\times B}(a,b)a^s b^t \ge a^{-\alpha-\beta+2\varepsilon+s+t}.$$

Since $-\alpha-\beta+2\varepsilon+s+t < 0$,

$$\lim_{\delta\to 0} \inf\{N_{A\times B}(a,b)a^s b^t : 0 < b \le a \le \delta \le \min\{a_0, b_0\}\} = \infty.$$

Noting $d-\dim(\mathrm{Proj}_x A \times B) = d-\dim(A) = \alpha$ ([7]) and $d-\dim(A \times B) = \alpha+\beta$, we have our conclusion from Proposition 14.

EXAMPLES 16. Let A and B be symmetric Cantor sets in $\mathbf{R}$ with sequences of contracting ratios $\{a_n\}$ and $\{b_n\}$ respectively. Let $\liminf_{n\to\infty} \frac{-n\log 2}{\log a_n} = \alpha$, and $\lim_{n\to\infty} \frac{-n\log 2}{\log b_n} = \beta$.

Then $d-\dim(A \times B) = \alpha+\beta = d-\dim(B \times A)$ ([7]).

Clearly A and B satisfy the assumptions of Corollary 15.

Hence

$$cl[\text{cd-Print}(A \times B)] = \{(s,t) : s+t \le \alpha+\beta \text{ and } t \le \alpha\}.$$

Also

$$cl[\text{cd-Print}(B \times A)] = \{(s,t) : s+t \le \alpha+\beta \text{ and } t \le \beta\}.$$

EXAMPLE 17. Using Proposition 14 or Corollary 15, we easily see the following fact.

(a) If E is a smooth curve and $\mathrm{Proj}_x E$ is not a singleton, then cl [cd-Print (E)] $= \{(s,t) : s+t \leq 1\}$.

(b) If E is any set in $\mathbf{R}^2$ with non-empty interior, then

$$cl[\text{cd-Print}(E)] = \{(s,t) : s+t \leq 2, t \leq 1\}.$$

(c) Let A be a symmetric Cantor set with a sequence of contracting ratios $\{a_n\}$ and $\liminf_{n\to\infty} \frac{-n\log 2}{\log a_n} = \alpha$. Then

$$cl[\text{cd - Print}(A \times \mathbf{R})] = \{(s,t) : s+t \leq 1+\alpha, t \leq \alpha\}.$$

and

$$cl[\text{cd -Print}(\mathbf{R} \times A)] = \{(s,t) : s+t \leq 1+\alpha, t \leq 1\}.$$

EXAMPLE 18. Let E_θ denote the rotation of E with the angle θ with respect to origin and let $C_{\frac{1}{4}}$ be as in Remark 3. Then for almost all $\theta \in (0,\pi)$,

$$cl[\text{cd - Print}(C_{\frac{1}{4}} \times C_{\frac{1}{4}})_\theta] = \{(s,t) : s+t \leq 1\}$$

since Hausdorff dimension of $\mathrm{Proj}_x[(C_{\frac{1}{4}} \times C_{\frac{1}{4}})_\theta]$ is the Hausdorff dimension of $C_{\frac{1}{4}} \times C_{\frac{1}{4}}$, 1, almost all $\theta \in (0,\pi)$ ([2] Theorem 6.1).

We note that

$$cl[\text{cd - Print}(C_{\frac{1}{4}} \times C_{\frac{1}{4}})] = \{(s,t) : s+t \leq 1 \text{ and } t \leq \frac{1}{4}\},$$

and

$$cl[\text{cd-Print}(C_{\frac{1}{4}} \times C_{\frac{1}{4}})_{\frac{\pi}{4}}] = \{(s,t) : s+t \leq 1 \text{ and } t \leq \frac{\log 3}{\log 4}\}.$$

Now, we introduce an application of cd-Print.

THEOREM 19. Let $\varphi : [0.1] \to \mathbf{R}$ be a function such that

$$C_1|J|^\alpha \leq \sup_{x,y\in J} |\varphi(x) - \varphi(y)| \leq C_2|J|^\alpha$$

for any interval $J \subset [0,1]$, where some constants $C_1, C_2 > 0$ and $0 < \alpha \leq 1$. Then $d-\dim(\{x \in [0,1] : \varphi(x)$ is not an algebraic number $\}) = 1$.

PROOF. Let $K = \{x \in [0,1] : \varphi(x) \notin \mathcal{A}\}$, where $\mathcal{A}$ is the set of algebraic numbers. Suppose that $d-\dim(K) < 1$. Then $d-\dim(K) < \beta < 1$ for some $\beta > 0$. Now,

$$\{(x,y) : y - \varphi(x) \in \mathcal{A}\} = \{(x,y) : \varphi(x) \in \mathcal{A}, y - \varphi(x) \in \mathcal{A}\}$$

$$\cup \{(x,y) : \varphi \notin \mathcal{A}, y - \varphi(x) \in \mathcal{A}\}$$

$$= \{(x,y) : y \in \mathcal{A}\} \cup \{(x,y) : x \in K, y - \varphi(x) \in \mathcal{A}\}.$$

Since $\mathcal{A}$ is a countable set, we can enumerate $\mathcal{A} = \{a_n\}_{n=1}^\infty$. Therefore

$$\{(x,y) : y - \varphi(x) \in \mathcal{A}\} = \cup_{n=1}^{\infty}\{(x,\varphi(x) + a_n) : x \in [0,1]\}.$$

By the additivity of cd-Print and Theorem 11,

$$cl[\text{cd - Print}(\{(x,y) : y - \varphi(x) \in \mathcal{A}\})] = \{(s,t) : s + t \leq 2 - \alpha, t \leq 1 - \alpha s\}.$$

But

$$cl[\text{cd - Print}(\{(x,y) : x \in K, y - \varphi(x) \in \mathcal{A}\})] \subset \{(s,t) : s + t \leq 2 - \alpha, t \leq \beta\}$$

and

$$\begin{aligned} & cl[\text{cd - Print}(\{(x,y) : y \in \mathcal{A})] \\ & \subset cl[\text{cd - Print}(\cup_{n=1}^{\infty}\{(x,a_n) : x \in [0,1]\})] \\ & = \{(s,t) : s + t \leq 1\}. \end{aligned}$$

A contradiction arises by the monotonicity of cd-Print.

REMARK 20. Let $A \subset \mathbf{R}$ be a compact set such that $\underline{\text{Cap}}(A \cap (a-\varepsilon, a+\varepsilon)) = \underline{\text{Cap}}(A) = \alpha$ for any $a \in A$ and any $\varepsilon > 0$. Then $d-\dim(\{x \in A : \varphi(x) \notin \mathcal{A}\}) = \alpha$, where $\mathcal{A}$ is the set of algebraic numbers.

REMARK 21. There might be several different way to get cl [cd-Print $(\mathbf{R} \times \mathbf{R})$] $= \{(s,t) : s + t \leq 2, t \leq 1\}$.

For example, we could consider countable graphs of φ_n in Theorem 11 with $\alpha = \frac{1}{n}$.

Another method is to use countable sets of $[0,1] \times K_n$, where K_n is a symmetric Cantor set with a sequence of contracting ratios $\{a_{n,k}\}_{k=1}^{\infty}$ satisfying

$$\lim_{k\to\infty} \frac{-k\log 2}{\log a_{n,k}} = 1 - \frac{1}{n}.$$

CONJECTURE 22. Let $E \subset \mathbf{R}^2$ with $d - \dim(E) = \alpha > 1$. We conjecture

$$cl[\text{cd -Print}(E_\theta)] = \{(s,t) : s + t \leq \alpha, t \leq 1\}$$

for almost all $\theta \in [0,\pi)$.

REMARK 23. While the Hausdorff dimension print is invariant under linear transformations, our coordinate d-dimension print is not so. For example, a straight line parallel to y-axis has a smaller coordinate d-dimension print than a line at 45^o to y-axis. Nevertheless our coordinate d-dimension print is particularly useful for the study of sets, such as the graphs of functions or Cartesian products which naturally have special relationship to the coordinate axes. (We thank our referee for pointing out the previous fact.) It would be interesting to investigate how the coordinate d-dimension print changes according to linear transformations.

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