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EFFECT OF GRAVITY ON VISCO-ELASTIC SURFACE WAVES IN SOLIDS INVOLVING TIME RATE OF STRAIN AND STRESS OF HIGHER ORDER

TAPAN KUMAR DAS and P. R. SENGUPTA

Department of Mathematics, University of Kalyani
Kalyani, West Bengal, India

and

LOKENATH DEBNATH
Department of Mathematics, University of Central Florida
Orlando, Florida 23816, U.S.A.

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ABSTRACT. A study is made of the surface waves in a higher order visco-elastic solid involving time rate of change of strain and stress under the influence of gravity. A fairly general equation for the wave velocity is derived. This equation is used to examine various kinds of surface waves including Rayleigh waves, Love waves and Stoneley waves. It is shown that the corresponding classical results follow from this analysis in the absence of gravity and viscosity.

KEY WORDS AND PHRASES: Surface waves, effects of gravity and viscosity.
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1. INTRODUCTION

Considerable literature including Bullen [1], Flugge [2] and Stoneley [3] is available on the theory of surface waves in an isotropic homogeneous elastic solid medium. However, the effects of gravity, viscosity and curvature, although important, are not included in the classical problems. Biot [4] has first investigated the effect of gravity on Rayleigh waves on the surface of an elastic solid based on the assumption that gravity produces a type of initial stress of hydrostatic in nature. Subsequently, Biot's theory has been used by several authors including De and Sengupta [5,6] to study problems of waves and vibrations in solids under the initial stress in various configurations. Further, Sengupta and his associates [7-9] have made an attempt to study the problems of surface waves in solids involving time rate of strain and viscosity. In spite of these studies, relatively less attention has been given to surface wave problems in a higher order visco-elastic solid involving time rate of strain and stress under the influence of gravity. The main purpose of this paper is to study such problems. A fairly general equation for the wave velocity is derived. This equation is utilized to examine various kinds of surface waves including Rayleigh waves, Love waves, and Stoneley waves. It is shown that the corresponding classical results follow from this analysis in the absence of viscosity and gravity.

2. FORMULATION OF THE PROBLEM AND BOUNDARY CONDITIONS

Let M_1 and M_2 be two homogeneous general visco-elastic solid media involving time rate of strain and stress of higher order in welded contact under the influence of gravity at their common surface of separation. We suppose that the media are separated by a plane horizontal boundary extending to infinitely great distance from the origin, M_2 being above M_1. We introduce a set of orthogonal Cartesian coordinate axes $0x_1x_2x_3$ in the semi-infinite isotropic visco-elastic media, with the origin at the common boundary surface and the x_3-axis is normal to M_1. We consider the possibility of a type of wave travelling in the direction of $0x_1$ in such a manner that the disturbance is largely confined to the neighborhood of the boundary and at any instant all particles on any line parallel to $0x_2$ have equal displacements. Hence the wave is a surface wave and all partial derivatives with respect to the coordinate x_2 are zero. Then the components of displacement u_1 and u_3 at any point may be expressed in the form [1]

$$u_1 = \frac{\partial\phi}{\partial x_1} - \frac{\partial\psi}{\partial x_3}, \quad u_3 = \frac{\partial\phi}{\partial x_3} + \frac{\partial\psi}{\partial x_1}; \tag{2.1ab}$$

where ϕ and ψ are the functions of x_1, x_3 and t and

$$\nabla^2\phi = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \Delta, \quad \nabla^2\psi = \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}. \tag{2.2ab}$$

Thus the introduction of the functions ϕ and ψ enables us to separate out the purely dilational and rotational disturbances associated with the components u_1 and u_3. The component u_2, of course, is associated with purely distortional movement. Thus ϕ, ψ and u_2 are associated respectively with P-waves, SV-waves and SH-waves, as used by Bullen [1].

The stress-strain relations are

$$D_\eta \sigma_{ij} = D_\lambda \Delta \delta_{ij} + 2D_\mu e_{ij}, \tag{2.3}$$

where

$$D_\eta = \sum_{k=0}^{n} \eta_k \frac{\partial^k}{\partial t^k}, \quad D_\lambda = \sum_{k=0}^{n} \lambda_k \frac{\partial^k}{\partial t^k}, \quad D_\mu = \sum_{k=0}^{n} \mu_k \frac{\partial^k}{\partial t^k} \tag{2.4abc}$$

where η_0, λ_0 and μ_0 are the elastic constants and η_k, λ_k and $\mu_k (k = 1, 2, \ldots, n)$ are the effects of viscosity, e_{ij} is the strain tensor and δ_{ij} is the Kronecker symbol.

The displacement equations of motion in the higher order general visco-elastic medium, under the influence of gravity, are

$$(D_\lambda + D_\mu)\frac{\partial\Delta}{\partial x_1} + D_\mu \nabla^2 u_1 + \rho g D_\eta \frac{\partial u_3}{\partial x_1} = \rho D_\eta \frac{\partial^2 u_1}{\partial t^2}, \tag{2.5}$$

$$D_\mu \nabla^2 u_2 = \rho D_\eta \frac{\partial^2 u_2}{\partial t^2}, \tag{2.6}$$

$$(D_\lambda + D_\mu)\frac{\partial\Delta}{\partial x_3} + D_\mu \nabla^2 u_3 - \rho g D_\eta \frac{\partial u_1}{\partial x_1} = \rho D_\eta \frac{\partial^2 u_3}{\partial t^2}, \tag{2.7}$$

where $\rho, \eta_k, \lambda_k, \mu_k (k = 0, 1, 2, \ldots, n)$ denote the properties of the medium M_1 and those with dashes the properties of the medium M_2. Substituting (2.1*ab*) in equations (2.5)-(2.7), we obtain the wave equations in M_1 satisfied by ϕ, ψ and u_2, as

$$\frac{\partial^2\phi}{\partial t^2} = \mathcal{D}_T \nabla^2\phi + g\frac{\partial\psi}{\partial x_1}, \tag{2.8}$$

$$\frac{\partial^2\psi}{\partial t^2} = \mathcal{D}_S \nabla^2\psi - g\frac{\partial\phi}{\partial x_1}, \tag{2.9}$$

$$\frac{\partial^2 u_2}{\partial t^2} = \mathcal{D}_S \nabla^2 u_2, \tag{2.10}$$

where

$$V_{kT}^2 = (\lambda_k + 2\mu_k)/\rho\,, \quad V_{kS}^2 = \mu_k/\rho \tag{2.11ab}$$

and

$$\mathcal{D}_T = \sum_{k=0}^{n} V_{kT}^2 \frac{\partial^k}{\partial t^k} \Big/ \sum_{k=0}^{n} \eta_k \frac{\partial^k}{\partial t^k}\,, \quad \mathcal{D}_S = \sum_{k=0}^{n} V_{kS}^2 \frac{\partial^k}{\partial t^k} \Big/ \sum_{k=0}^{n} \eta_k \frac{\partial^k}{\partial t^k}\,, \tag{2.12ab}$$

and similar relations in M_2 with $\rho, \eta_k, \lambda_k, \mu_k$ replaced by $\rho', \eta_k', \lambda_k', \mu_k'$ and so on (where $k = 0, 1, 2, \ldots, n$).

The boundary conditions are

(i) The components of displacement at the boundary surface between the media M_1 and M_2 must be continuous at all times and distances.

(ii) The stresses $\sigma_{31}, \sigma_{32}, \sigma_{33}$ are

$$D_\eta \sigma_{31} = D_\mu \left(2\frac{\partial^2 \phi}{\partial x_1 \partial x_3} + \frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_3^2} \right), \tag{2.13}$$

$$D_\eta \sigma_{32} = D_\mu \frac{\partial u_2}{\partial x_3}\,, \tag{2.14}$$

$$D_\eta \sigma_{33} = D_\lambda \nabla^2 \phi + 2D_\mu \left(\frac{\partial^2 \phi}{\partial x_3^2} + \frac{\partial^2 \psi}{\partial x_1 \partial x_3} \right), \tag{2.15}$$

and similar expressions for M_2, across the boundary surface between M_1 and M_2 must be continuous at all times and distances.

3. SOLUTION OF THE PROBLEM

To solve equations (2.8)-(2.10), we put

$$(\phi, \psi, u_2) = [\bar{\phi}(x_3), \bar{\psi}(x_3), \bar{u}_2(x_3)]\, e^{i(\eta x_1 - \omega t)}\,, \tag{3.1}$$

for medium M_1 and similar solutions for M_2, the functions $\bar{\phi}, \bar{\psi}, \bar{u}_2$ being replaced by $\bar{\phi}', \bar{\psi}', \bar{u}_2'$.

Introducing (3.1) in (2.8)-(2.10), we have for the medium M_1:

$$\left[\frac{d^2}{dx_3^2} - (\eta^2 - \omega^2 \eta_k^* / V_{kT}^{*2}) \right] \bar{\phi} = -ig\eta \bar{\psi} \eta_k^* / V_{kT}^{*2}\,, \tag{3.2}$$

$$\left[\frac{d^2}{dx_3^2} - (\eta^2 - \omega^2 \eta_k^* / V_{kS}^{*2}) \right] \bar{\psi} = ig\eta \bar{\phi} \eta_k^* / V_{kS}^{*2}\,, \tag{3.3}$$

$$\left[\frac{d^2}{dx_3^2} - (\eta^2 - \omega^2 \eta_k^* / V_{kS}^{*2}) \right] \bar{u}_2 = 0\,, \tag{3.4}$$

where

$$\eta_k^* = \sum_{k=0}^{n} (-i\omega)^k \eta_k\,, \quad V_{kT}^{*2} = \sum_{k=0}^{n} (-i\omega)^k V_{kT}^2\,, \quad V_{ks}^{*2} = \sum_{k=0}^{n} (-i\omega)^k V_{ks}^2\,. \tag{3.5abc}$$

Similar relations for M_2 can be obtained by replacing $\bar{\phi}, \bar{\psi}, \bar{u}_2, \eta_k, V_{kT}, V_{kS}, \eta_k^*, V_{kT}^*, V_{kS}^*, \lambda_k, \mu_k, \rho$ by the same symbols with dashes. Here $\rho', \eta_k', \lambda_k', \mu_k'$ $(k = 0, 1, 2, \ldots, n)$ are the physical properties of the medium M_2.

We assume that ϕ, ψ and u_3 represent exponentially decaying solutions in the medium M_1 as $x_3 \to \infty$ so that they can be expressed in the form:

$$\phi = \left[A_1 e^{-x_3\sqrt{\eta^2 - \zeta_1^2}} + A_2 e^{-x_3\sqrt{\eta^2 - \zeta_2^2}} \right] e^{i(\eta x_1 - \omega t)} \tag{3.6}$$

$$\psi = \left[B_1 e^{-x_3\sqrt{\eta^2 - \zeta_1^2}} + B_2 e^{-x_3\sqrt{\eta^2 - \zeta_2^2}} \right] e^{i(\eta x_1 - \omega t)} \tag{3.7}$$

$$u_2 = \left[C e^{-x_3\sqrt{\eta^2 - \omega^2 \eta_k^* / V_{kS}^{*2}}} \right] e^{i(\eta x_1 - \omega t)} \tag{3.8}$$

and similar solutions in M_2 can be obtained replacing ϕ, ψ, u_2, A_1, A_2, B_1, B_2, C, η_k^*, V_{kS}^*, ζ_1, ζ_2 the same symbols with dashes in solutions (3.6)-(3.8). Here ζ_j^2 and ζ_j' $(j = 1,2)$ are respectively the roots of the equations

$$[\omega^2 - \zeta^2 V_{kS}^{*2}/\eta_k^*][\omega^2 - \zeta^2 V_{kT}^{*2}/\eta_k^*] - g^2\eta^2 = 0 \tag{3.9}$$

$$[\omega^2 - \zeta'^2 V_{kS}^{*'2}/\eta_k^{*'}][\omega^2 - \zeta'^2 V_{kT}^{*'2}/\eta_k^{*'}] - g^2\eta^2 = 0 \tag{3.10}$$

and

$$B_1 = \alpha_1 A_1, \quad B_2 = \alpha_2 A_2; \quad B_1' = \alpha_1' A_1', \quad B_2' = \alpha_2' A_2'$$

where

$$\alpha_j = ig\eta/(\omega^2 - \zeta_j^2 V_{kS}^{*2}/\eta_k^*), \quad \alpha_j' = ig\eta/(\omega^2 - \zeta_j'^2 V_{kS}^{*'2}/\eta_k^{*'}) \qquad (j = 1,2)$$

In evaluating quantities like $\sqrt{\eta^2 - \zeta^2}$ and $\sqrt{\eta^2 - \omega^2\eta_k^*/V_{kS}^{*2}}$, the root with positive real part must be taken in each case.

Using boundary conditions (i) and (ii), we obtain

$$[1 - i\alpha_1 Q_1]A_1 + [1 - i\alpha_2 Q_2]A_2 = [1 + i\alpha_1' Q_1']A_1' + [1 + i\alpha_2' Q_2']A_2' \tag{3.11a}$$

$$C = C' \tag{3.11b}$$

$$[\alpha_1 + iQ_1]A_1 + [\alpha_2 + iQ_2]A_2 = [\alpha_1' - iQ_1']A_1' + [\alpha_2' - iQ_2']A_2' \tag{3.11c}$$

$$\rho(V_{kS}^{*2}/\eta_k^*)[\{2iQ_1 + (1 + Q_1^2)\alpha_1\}A_1 + \{2iQ_2 + (1 + Q_2^2)\alpha_2\}A_2] =$$
$$\rho'(V_{kS}^{*'2}/\eta_k^{*'})[\{-2iQ_1' + (1 + Q_1'^2)\alpha_1' + \{-2iQ_2' + (1 + Q_2'^2)\alpha_2'\}A_2'] \tag{3.11d}$$

$$-\rho(V_{kS}^{*2}/\eta_k^*)\sqrt{\eta^2 - \omega^2\eta_k^*/V_{kS}^{*2}}\, C = \rho'\left(V_{kS}^{*'2}/\eta_k^{*'}\right)\sqrt{\eta^2 - \omega^2\eta_k^{*'}/V_{kS}^{*'2}}\, C' \tag{3.11e}$$

$$(\rho/\eta_k^*)[\{V_{kT}^{*2}(Q_1^2 - 1) + 2V_{kS}^{*2}(1 - i\alpha_1 Q_1)\}A_1 + \{V_{kT}^{*2}(Q_2^2 - 1) + 2V_{kS}^{*2}(1 - i\alpha_2 Q_2)\}A_2]$$
$$= (\rho'/\eta_k^{*'})[\{V_{kT}^{*'2}(Q_1'^2 - 1) + 2V_{kS}^{*'2}(1 + i\alpha_1' Q_1')\}A_1' + \{V_{kT}^{*'2}(Q_2'^2 - 1) + 2V_{kS}^{*'2}(1 + i\alpha_2' Q_2')\}A_2']\,. \tag{3.11f}$$

It follows from equations (3.11b) and (3.11e) that $C = C' = 0$. Thus there is no propagation of displacement u_2. Hence there are no *SH*-waves in this case.

From equations (3.11*a*), (3.11*c*), (3.11*d*) and (3.11*f*), we eliminate the constants A_1, A_2, A_1', A_2', to get equation for the wave velocity in determinant form

$$|M_{ij}| = 0\,, \quad (i,j = 1,2,3,4)\,, \tag{3.12}$$

where

$$M_{1\,m} = [1 - i\alpha_m Q_m], \quad M_{1\,m+2} = -[1 + i\alpha_m' Q_m'];$$

$$M_{2\,m} = [\alpha_m + iQ_m], \quad M_{2\,m+2} = -[\alpha_m' - iQ_m'];$$

$$M_{3\,m} = (\rho V_{kS}^{*2}/\eta_k^*)[2iQ_m + (1 + Q_m^2)\alpha_m],$$

$$M_{3\,m+2} = (-\rho' V_{kS}^{*'2}/\eta_k^{*'})[-2iQ_m' + (1 + Q_m'^2)\alpha_m'];$$

$$M_{4\,m} = (\rho/\eta_k^*)[V_{kT}^{*2}(Q_m^2 - 1) + 2V_{kS}^{*2}(1 - i\alpha_m Q_m)],$$

$$M_{4\,m+2} = (-\rho'/\eta_k^{*'})[V_{kT}^{*'2}(Q_m'^2 - 1) + 2V_{kS}^{*'2}(1 + i\alpha_m' Q_m')]$$

and where $m = 1,2$.

$$Q_m = \sqrt{1 - \zeta_m^2/\eta^2}, \quad Q_m' = \sqrt{1 - \zeta_m'^2/\eta^2}.$$

Equation (3.12) gives the wave velocity for the surface waves in the common boundary, and the strain rate and the stress rate of higher order in the presence of gravity and viscosity are included in (3.12).

4. PARTICULAR CASES

(i) Rayleigh Waves

In order to investigate the possibility of Rayleigh waves, we take the plane boundary as a free surface with M_2 replaced by a vacuum. Obviously, there are no *SH*-waves in this case. In view of (3.11*d*) and (3.11*f*), we obtain

$$\{2iQ_1 + (1 + Q_1^2)\alpha_1\}A_1 + \{2iQ_2 + (1 + Q_2^2)\alpha_2\}A_2 = 0, \tag{4.1}$$

and

$$\{V_{kT}^{*2}(Q_1^2 - 1) + 2V_{kS}^{*2}(1 - i\alpha_1 Q_1)\}A_1 + \{V_{kT}^{*2}(Q_2^2 - 1) + 2V_{kS}^{*2}(1 - i\alpha_2 Q_2)\}A_2 = 0. \tag{4.2}$$

Eliminating the constants A_1 and A_2 from equations (4.1)-(4.2), we get

$$|M_{ij}'| = 0 \qquad (i,j = 1,2) \tag{4.3}$$

where

$$M_{1r}' = [2iQ_r + (1 + Q_r^2)\alpha_r]; \quad M_{2r}' = [V_{kT}^{*2}(Q_r^2 - 1) + 2V_{kS}^{*2}(1 - i\alpha_r Q_r)]; \quad (r = 1,2). \tag{4.4ab}$$

Equation (4.3) is the required wave velocity equation for visco-elastic Rayleigh waves including the strain rate and stress rate of higher order under gravitational field. When the effects of viscosity and gravity are neglected, this equation reduces to the classical result as discussed by Bullen [1].

(ii) Love Waves

In this case we consider a layered semi-infinite medium in which M_2 is bounded by two horizontal plane surfaces at a finite distance H-apart, while M_1 remains infinite as it was. In this case, we consider the displacement component u_2 only.

For the medium M_2 we write down the full solution, since the displacement in M_2 may no longer diminish with the increasing distance from common boundary $x_3 = 0$ and for the medium M_1 the solutions are the same as it was in the general case.

Therefore, for the medium M_2 we write

$$u_2' = \left[C_1' e^{x_3\sqrt{\eta^2 - \omega^2\eta_k^{*'}/V_{kS}^{*'2}}} + C_2' e^{-x_3\sqrt{\eta^2 - \omega^2\eta_k^{*'}/V_{kS}^{*'2}}}\right] e^{i(\eta x_1 - \omega t)}, \tag{4.5}$$

where the restriction that the real part of $\sqrt{\eta^2 - \omega^2 \eta_k^{*\prime}/V_{kS}^{*\prime 2}}$ is positive is not required for M_2.

In this case the boundary conditions are

(i) u_2 and σ_{32} are continuous at $x_3 = 0$ and (ii) $\sigma_{32}^{\prime} = 0$ at $x_3 = -H$.

Applying these boundary conditions and using (3.8) and (4.5), we find

$$C = C_1^{\prime} + C_2^{\prime} \tag{4.6}$$

$$(-\rho V_{kS}^{*2}/\eta_k^{*})\sqrt{\eta^2 - \omega^2 \eta_k^{*}/V_{kS}^{*2}} = [C_1^{\prime} - C_2^{\prime}](\rho^{\prime} V_{kS}^{*\prime 2}/\eta_k^{*\prime})\sqrt{\eta^2 - \omega^2 \eta_k^{*\prime}/V_{kS}^{*\prime 2}} \tag{4.7}$$

$$C_1^{\prime} e^{-H\sqrt{\eta^2 - \omega^2 \eta_k^{*\prime}/V_{kS}^{*\prime 2}}} - C_2^{\prime} e^{H\sqrt{\eta^2 - \omega^2 \eta_k^{*\prime}/V_{kS}^{*\prime 2}}} = 0 . \tag{4.8}$$

Eliminating $C, C_1^{\prime}$ and $C_2^{\prime}$ from the equations (4.6)-(4.8), we obtain

$$(\rho V_{kS}^{*2}/\eta_k^{*})\sqrt{1 - c^2 \eta_k^{*}/V_{kS}^{*2}} + (\rho^{\prime} V_{kS}^{*\prime 2}/\eta_k^{*\prime})\sqrt{(c^2 \eta_k^{*\prime}/V_{kS}^{*\prime 2}) - 1}\ \tan \eta H \sqrt{(c^2 \eta_k^{*\prime}/V_{kS}^{*\prime 2}) - 1} = 0 \tag{4.9}$$

where $c = \omega/\eta$. This is the required wave velocity equation for higher order visco-elastic Love waves involving the strain rate and stress rate under the influence of gravity. It is important to note that Love waves are not affected by gravity but by viscosity. When $\eta_0 = 1$ and $\eta_k = \eta_k^{\prime} = \lambda_k = \lambda_k^{\prime} = \mu_k = \mu_k^{\prime} = 0$ $(k = 1, 2, \ldots, n)$ then equation (4.9) is in agreement with the corresponding classical result [1] in a perfectly elastic medium.

(iii) Stoneley Waves

In the classical theory, the Stoneley waves are a generalized from of Rayleigh waves propagating along the common boundary of M_1 and M_2. In the influence of gravity, Stoneley waves along the common boundary of the general visco-elastic solid media M_1 and M_2 involving the strain rate and stress rate of higher order, are therefore determined by the roots of the frequency equation (3.12). In the absence of these effects, this equation also agrees with the corresponding classical result.

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