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# WELL-POSEDNESS AND REGULARITY RESULTS FOR A DYNAMIC VON KÁRMÁN PLATE

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**Abstract.** We consider the problem of well-posedness and regularity of solutions for a dynamic von Kármán plate which is clamped along one portion of the boundary and which experiences boundary damping through "free edge" conditions on the remainder of the boundary We prove the existence of unique strong solutions for this system

**Key Words.** von Kármán plate, strong regularity, well-posedness

**AMS(MOS) subject classification.** 35B65, 47N20, 73K10

**1. INTRODUCTION.** In this paper, we consider the well-posedness of the von Kármán system given by

$$
(1.1)\qquad
\begin{array}{rll}
w_{tt} - \gamma^2 \Delta w_{tt} + \Delta^2 w = [w, F(w)] & \text{in} & Q = \Omega \times (0,T) \\
w(0,\cdot) = w_0; w_t(0,\cdot) = w_1 & \text{in} & \Omega \\
w = \dfrac{\partial}{\partial \nu} w = 0 & \text{on} & \Sigma_0 = \Gamma_0 \times (0,T) \\
\Delta w + (1-\mu) B_1 w = -\dfrac{\partial}{\partial \nu} w_t & \text{on} & \Sigma_1 = \Gamma_1 \times (0,T) \\
\dfrac{\partial}{\partial \nu}\Delta w + (1-\mu) B_2 w - \gamma^2 \dfrac{\partial}{\partial \nu} w_{tt} = w_t - \dfrac{\partial^2}{\partial \tau^2} w_t & \text{on} & \Sigma_1,
\end{array}
$$

where we assume $\Omega \subset \mathbf{R}^2$, with sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, $0 < \mu < \frac{1}{2}$ represents Poisson's ratio and the boundary operators $B_1$ and $B_2$ are given by

$$
(1.1)(b)\qquad
\left.
\begin{array}{l}
B_1 w = 2n_1 n_2 w_{xy} - n_1^2 w_{yy} - n_2^2 w_{xx} \\
B_2 w = \frac{\partial}{\partial \tau}\left[(n_1^2 - n_2^2) w_{xy} + n_1 n_2 (w_{yy} - w_{xx})\right]
\end{array}
\right\}
$$

Also, $F(w)$ satisfies the system of equations

$$
(1.2)\qquad
\left.
\begin{array}{l}
\Delta^2 F = -[w,w] \\
F = \frac{\partial}{\partial \nu} F = 0 \quad \text{on } \Sigma = \Gamma \times (0,\infty)
\end{array}
\right\}
$$

where

$$
[\phi, \psi] = \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 \psi}{\partial x^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 \psi}{\partial x \partial y}.
$$

The well-posedness and regularity of such a system is both a delicate and interesting problem. Such results are important in solving the problem of stabilization for system (1.1). Usual PDE techniques require the existence and uniqueness of "smooth" solutions to justify computations used in determining the stability and controllability of dynamical models. The stabilization of thin plates (and particularly the von Kármán system) is of current interest in the literature (see ([1], [2], [3], [4], [5])). The von Kármán nonlinearity poses many difficulties in obtaining the well-posedness and regularity results we seek. Difficulties also arise from the higher order boundary conditions on $\Sigma$. To handle these difficulties we adapt abstract results proven in [6] to our more difficult boundary conditions.

This paper will proceed as follows. In Section 2 we state the main results of our paper. After this we state the appropriate abstract results from [6] which will be useful in the proofs of our results. In Section 3 we prove the results stated in Section 2.

**2. STATEMENT OF RESULTS.** Before stating the results we intend to prove, we define the meaning of "weak solutions" through a variational equality. Let

$$a(w,v) = \int_\Omega (\Delta w \Delta v + (1-\mu)(2w_{xy}v_{xy} - w_{xx}vyy - w_{yy}v_{xx}))d\Omega.$$

We define the spaces

$$H^2_{\Gamma_0}(\Omega) = \left\{ w \in H^2(\Omega) : w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma_0 \right\}$$

with norm

$$\|w\|^2_{H^2_{\Gamma_0}(\Omega)} = a(w,w)$$

and

$$H^1_{\Gamma_0}(\Omega) = \left\{ w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_0 \right\}$$

with norm

$$\|w\|^2_{H^1_{\Gamma_0}(\Omega)} = \int_\Omega (w^2 + \gamma^2 |\nabla w|^2) d\Omega.$$

We define the solution space $\mathcal{H} = H^2_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$.

DEFINITION 2.1. *A function pair $(w, w_t) \in C((0,T);\mathcal{H})$ is said to be a weak solution to system (1.1) if $(w(\cdot,0), w_t(\cdot,0)) = (w_0, w_1)$ and $w$ satisfies the variational equation*

$$\begin{aligned} 0 &= (w_{tt},\varphi) + \gamma^2(\nabla w_{tt}, \nabla\varphi) + a(w,w) - ([w,F],\varphi) \\ &\quad + < w_t - \tfrac{\partial^2 w_t}{\partial \tau^2}, \varphi > + < \tfrac{\partial w_t}{\partial \nu}, \tfrac{\partial \varphi}{\partial \nu} > \qquad \forall \varphi \in H^2_{\Gamma_0}(\Omega), \end{aligned} \tag{2.1}$$

*where here and throughout the paper $(\cdot,\cdot)$ denotes either the $L^2(\Omega)$-inner product or the duality pairing between $H^2_{\Gamma_0}(\Omega)$ and $[H^2_{\Gamma_0}(\Omega)]'$, as is appropriate by context, and $< \cdot,\cdot >$ represents the $L^2(\Gamma)$-inner product. We note that (2.1) holds in $H^{-1}[0,T]$.*

THEOREM 2.1. *Given initial data $(w_0, w_1) \in \mathcal{H}$, there exists a unique weak solution to system (1.1), $(w, w_t) \in C([0,T),\mathcal{H})$ for any $T > 0$.*

THEOREM 2.2. (Regularity): *Assume in addition to Theorem 2.1 that the initial data satisfy*

$$\begin{aligned} &\text{(i)} \qquad w_0 \in H^3(\Omega); \quad w_1 \in H^2_{\Gamma_0}(\Omega); \\ &\text{(ii)} \qquad \left.\begin{aligned} \Delta w_0 + (1-\mu)B_1 w_0 &= -\tfrac{\partial}{\partial \nu} w_1 \\ \tfrac{\partial \Delta w_0}{\partial \nu} + (1-\mu)B_2 w_0 &= w_1 - \tfrac{\partial^2}{\partial \tau^2} w_1 \end{aligned}\right\} \quad \textit{on } \Gamma_1. \end{aligned} \tag{2.2}$$

*Then the unique solution to (1.1) has the regularity*

(i) $(w, w_t) \in C((0,T);(H^3(\Omega) \cap H^2_{\Gamma_0}(\Omega)) \times H^2_{\Gamma_0}(\Omega))$;
(ii) $w_{tt} \in C((0,T); H^1_{\Gamma_0}(\Omega))$
(iii) *equation (2.1) is satisfied for every $t \in [0,T)$.*

THEOREM 2.3. (Strong Regularity): *In addition to Theorems 2.1 and 2.2 we assume that*

$$\begin{aligned} &\text{(i)} \qquad w_0 \in H^4(\Omega); \quad w_1 \in H^3(\Omega) \cap H^2_{\Gamma_0}(\Omega). \\ &\text{(ii)} \qquad \left.\begin{aligned} \Delta w_0 + (1-\mu)B_1 w_1 &= -\tfrac{\partial w_{tt}(0)}{\partial \nu} \\ \tfrac{\partial \Delta w_1}{\partial \nu} + (1-\mu)B_2 w_1 &= w_{tt}(0) - \tfrac{\partial^2 w_{tt}(0)}{\partial \tau^2} \end{aligned}\right\} \quad \textit{on } \Gamma_1, \end{aligned} \tag{2.3}$$

*where $w_{tt}(0)$ is derived from the equation (1.1). Then the unique solution guaranteed by Theorem 2.1 has the following regularity properties:*

(i) $(w, w_t) \in C((0,T); (H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega)) \times (H^3(\Omega) \cap H^2_{\Gamma_0}(\Omega)))$;
(ii) $w_{tt} \in C((0,T); H^2_{\Gamma_0}(\Omega))$;
(iii) $w_{ttt} \in C((0,T); H^1_{\Gamma_0}(\Omega))$.

*Moreover, equation (1.1) holds in the $L^2$-sense for each $t \in [0,T]$.*

The proofs of Theorems 2.1–2.3 will be based primarily on the work of Favini and Lasiecka [6]. That paper deals with abstract problems of the form

$$\text{(2.4)} \qquad \begin{cases} Mw_{tt}(t) + \mathcal{A}w(t) + \mathcal{A}GG^*\mathcal{A}w_t(t) + \mathcal{A}Gf(w)(t) = \mathcal{F}(w)(t) \\ w(t=0) = w_0; \quad w_t(t=0) = w_1, \end{cases}$$

which will be described in detail shortly. Our intention in this paper is to recast system (1.1) in the abstract framework of (2.4). We will then show that the results of [6] may be applied directly to or may be adapted for our system. For the purpose of self-containment, we now state the necessary background and results from [6] which will be useful in this present context.

Let $\mathcal{A}$ be a closed, positive self-adjoint operator on a Hilbert space $H$ with $\mathcal{D}(\mathcal{A}) \subset H$. Let $V$ be another (appropriately chosen) Hilbert space such that

$$\mathcal{D}(\mathcal{A}^{1/2}) \subset V \subset H \subset V' \subset [\mathcal{D}(\mathcal{A}^{1/2})]'.$$

We assume that $M : V \to V'$ is both bounded and boundedly invertible so that the restriction $\widetilde{M} \equiv M|_H$ with domain $\mathcal{D}(\widetilde{M}) = \{u \in V : Mu \in H\}$ gives that $V = \mathcal{D}(\widetilde{M}^{1/2})$.

The operator $G$ is defined on another Hilbert space, $U$. It is assumed that $G : U \to H$ is a bounded linear operator such that $G^*\mathcal{A} \in \mathcal{L}(\mathcal{D}(\mathcal{A}^{1/2}); H)$.

Finally, the nonlinear term $\mathcal{F} : \mathcal{D}(\mathcal{A}^{1/2}) \to V'$ is assumed to be Frechét differentiable with derivative, denoted $D\mathcal{F}$, satisfying

$$\|D\mathcal{F}(u)h\|_{V'} \le C(\|u\|_{\mathcal{D}(\mathcal{A}^{1/2})})\|h\|_{\mathcal{D}(\mathcal{A}^{1/2})}.$$

We note that for our purposes, $f \equiv 0$.

We now state the results from [6] which form the framework for Theorem 2.1–2.3.

**Theorem 2.4.** (F-L Theorem 2.1): *For each initial data $(w_0, w_1) \in \mathcal{D}(\mathcal{A}^{1/2}) \times V$, there exists $T_0 > 0$ such that there exists a unique weak solution $(w(t), w_t(t))$ to (2.4).*

**Theorem 2.5.** (F-L Theorem 2.4): *In addition to the hypotheses of Theorem 2.4 we assume that for all $\tilde{w} = (w, w_t) \in C(0, T_0; \mathcal{D}(\mathcal{A}^{1/2}) \times V)$ and such that $G^*\mathcal{A}w_t \in L^2(0, T_0; U)$ the following inequality holds for all $t \in [0, T_0)$:*

$$\text{(2.5)} \qquad \begin{aligned} \int_0^t (\mathcal{F}(w(\tau)), w_t(\tau))d\tau &\le C_1 \int_0^t (\|w(\tau)\|^2_{\mathcal{D}(\mathcal{A}^{1/2})} + \|w_t(\tau)\|^2_V)d\tau \\ &+ C_2(\|(w_0, w_1)\|_{\mathcal{D}(\mathcal{A}^{1/2})\times V}) \equiv C_0. \end{aligned}$$

*Then the weak solution $(w(t), w_t(t))$ is global for any $T > 0$.*

**Theorem 2.6.** (F-L Theorem 2.2): *Assume that the initial data $(w_0, w_1)$ satisfy*

$$\text{(2.6)} \qquad \begin{aligned} &\text{(i)} \quad w_1 \in \mathcal{D}(\mathcal{A}^{1/2}) \\ &\text{(ii)} \quad \mathcal{A}(w_0 + \beta GG^*\mathcal{A}w_1) \in V'. \end{aligned}$$

*Moreover, assume that*

$$\text{(2.7)} \qquad \|\mathcal{A}^{-1/2}D\mathcal{F}(w)h\|_H \le C(\|w\|_{\mathcal{D}(\mathcal{A}^{1/2})})\|h\|_V.$$

*Then the solution satisfies (2.4) in the sense of the $[\mathcal{D}(\mathcal{A}^{1/2})]'$ topology for each $T \in (0,T)$. $(w(0), w_t(0)) = (w_0, w_1)$ and has the following regularity:*

$$(w, w_t) \in C(0,T; \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2})),$$
$$w_{tt} \in C(0,T;V).$$

By showing that system (1.1) can be formulated in the framework of the abstract equation (2.4) while satisfying the hypotheses of Theorems 2.5–2.7, we will have proven Theorems 2.1 and 2.2. For the additional regularity given in Theorem 2.3, we will need an additional proof which does not follow directly from results of [6].

**3. PROOFS OF THEOREMS 2.1–2.3.** Let $V = H^1_{\Gamma_0}(\Omega)$, $H = L^2(\Omega)$ and $U = (L^2(\Gamma_1))^3$. We define $\mathcal{A}$ on $H^2_{\Gamma_0}(\Omega)$ by

$$\begin{aligned} \mathcal{A}w &\equiv \Delta^2 w \text{ with domain} \\ D(\mathcal{A}) &= \Big\{ w \in H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega) : \Delta w + (1-\mu)B_1 w = 0 \\ &\qquad \text{and } \frac{\partial}{\partial\nu}\Delta w + (1-\mu)B_2 w = 0 \text{ on } \Gamma_1 \Big\}, \end{aligned} \tag{3.1}$$

which is well-defined, positive and self-adjoint. By the results of Grisvard [7], we see that $\mathcal{D}(\mathcal{A}^{1/2}) = H^2_{\Gamma_0}(\Omega)$. We also define the Green maps, $G_1 : H^s(\Gamma) \to H^{5/2+s}(\Omega)$, $G_2 : H^s(\Gamma) \to H^{7/2+s}(\Omega)$ and $G_3 : H^s(\Gamma) \to H^{5/2+s}(\Omega)$ by

$$G_1 h = v \iff \begin{cases} \Delta^2 v = 0 & \text{in } Q \\ v = \frac{\partial}{\partial\nu} v = 0 & \text{on } \Sigma_0 \\ \left.\begin{array}{l} \Delta v + (1-\mu)B_1 v = h \\ \frac{\partial}{\partial\nu}\Delta v + (1-\mu)B_2 v = 0 \end{array}\right\} & \text{on } \Sigma_1, \end{cases} \tag{3.2}$$

$$G_2 h = v \iff \begin{cases} \Delta^2 v = 0 & \text{in } Q \\ v = \frac{\partial}{\partial\nu} v = 0 & \text{on } \Sigma_0 \\ \left.\begin{array}{l} \Delta v + (1-\mu)B_1 v = 0 \\ \frac{\partial}{\partial\nu}\Delta v + (1-\mu)B_2 v = h \end{array}\right\} & \text{on } \Sigma_1, \end{cases} \tag{3.3}$$

and

$$G_3 h = G_2 \frac{\partial h}{\partial\tau}. \tag{3.4}$$

A straightforward computation shows that for $w \in \mathcal{D}(\mathcal{A})$,

$$\begin{aligned} G_1^* \mathcal{A} w &= \frac{\partial w}{\partial\nu}\Big|_{\Gamma_1} \\ G_2^* \mathcal{A} w &= -w|_{\Gamma_1}. \\ G_3^* \mathcal{A} w &= \frac{\partial w}{\partial\tau}\Big|_{\Gamma_1}. \end{aligned} \tag{3.5}$$

Let $\bar{u} \in [L^2(\Gamma)]^3$. Define $G\bar{u} = -G_1 u_1 - G_2 u_2 - G_3(\frac{\partial u_3}{\partial\tau})$. Then $G : [L^2(\Gamma)]^3 \to L^2(\Omega)$ is bounded and $G^*\mathcal{A} \in \mathcal{L}(H^2_{\Gamma_0}(\Omega); [L^2(\Gamma)]^3)$.

We now introduce the operator $M : \mathcal{D}(M) \subset H^2(\Omega) \to L^2(\Omega)$.

$$Mw = (I - \gamma^2\Delta)w + \gamma^2 \mathcal{A} G_2 \frac{\partial w}{\partial\nu}.$$

We observe that for $v, w \in H^1_{\Gamma_0}(\Omega)$,

$$\begin{aligned} (Mw, v) &= (w, v) + \gamma^2(\nabla w, \nabla v) \\ &\quad + \gamma^2 \langle \frac{\partial w}{\partial\nu}, G_2^* \mathcal{A} v \rangle - \gamma^2 \langle \frac{\partial w}{\partial\nu}, v \rangle \\ &= (v, w) + \gamma^2(\nabla v, \nabla w), \end{aligned} \tag{3.6}$$

where we have interpreted the $\gamma^2 \mathcal{A} G_2 \frac{\partial w}{\partial \nu}$ term in the sense of duality. Using (3.5), we see that $M : H^1_{\Gamma_0}(\Omega) \to [H^1_{\Gamma_0}(\Omega)]'$ is an isomorphism (by the Lax-Milgram Theorem).

By a straightforward computation, we see that

$$(3.7) \qquad (\mathcal{A}w, \varphi) = a(w, \varphi)$$

and that

$$(3.8) \qquad < G^* \mathcal{A} w_t, G^* \mathcal{A} \varphi > = \langle w_t - \frac{\partial^2 w_t}{\partial \tau^2}, \varphi \rangle + \langle \frac{\partial w_t}{\partial \nu}, \frac{\partial \varphi}{\partial \nu} \rangle.$$

Defining $\mathcal{F}(w) = [w, F(w)]$ and using (3.6) - (3.8), we can now rewrite system (1.1) in the form of (2.4).

To see that the von Kármán nonlinearity is Frechét differentiable, we define the operator

$$(3.9) \qquad A_0 w = \Delta^2 w \text{ with } \mathcal{D}(A_0) = H^4(\Omega) \cap H^2_0(\Omega).$$

Then $F(w) = -A_0^{-1}[w, w]$ so that $\mathcal{F}(w) = -[w, A_0^{-1}[w, w]]$. By straightforward (but somewhat lengthy) computations we see that

$$(3.10) \qquad D\mathcal{F}(w)h = [h, A_0^{-1}[w, w]] + 2[w, A_0^{-1}[w, h]].$$

To prove that $\|D\mathcal{F}(w)h\|_{[H^1_{\Gamma_0}(\Omega)]'} \le C(\|w\|_{H^2_{\Gamma_0}(\Omega)})\|h\|_{H^2_{\Gamma_0}(\Omega)}$, we use the following lemma, which is proved in [3].

LEMMA 3.1. *The mapping* $(u, v, w) \to [u, A_0^{-1}[v, w]]$ *is continuous from* $[H^2(\Omega)]^3 \to H^{-\varepsilon}(\Omega)$ *for* $0 < \varepsilon < 1/2$.

Consequently, we have

$$\|D\mathcal{F}(w)h\|_{[H^1_{\Gamma_0}(\Omega)]'} \le \|D\mathcal{F}(w)h\|_{H^{-\epsilon}(\Omega)} \le C\|w\|^2_{H^2_{\Gamma_0}(\Omega)}\|h\|_{H^2_{\Gamma_0}(\Omega)}.$$

*Remark.* An interesting estimate which arises in the proof of Lemma 3.1 is

$$(3.11) \qquad \|A_0^{-1}[w, v]\|_{H^{3-\epsilon}(\Omega)} \le C\|w\|_{H^2(\Omega)}\|v\|_{H^2(\Omega)}.$$

This will be useful to us later in the proof.

PROOF OF THEOREM 2.1. To complete the proof, it suffices to show that (2.5) holds. Let $(w, w_t) \in C([0,T]; H^2_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega))$. Then

$$\begin{aligned}
\int_0^t \int_\Omega [w, F(w)] w_t d\Omega dt &= \int_0^t \int_\Omega [w, w_t] F(w) d\Omega dt \\
&= \int_0^t \int_\Omega \frac{1}{2} \left( \frac{d}{dt}[w, w] \right) F(w) d\Omega dt \\
&= -\frac{1}{2} \int_0^t \int_\Omega \frac{d}{dt} (\Delta^2 F(w)) F(w) d\Omega dt \\
&= -\frac{1}{4} \int_0^t \int_\Omega \frac{d}{dt} (\Delta F(w))^2 d\Omega dt \\
&\le \frac{1}{4} \int_\Omega (\Delta F(w_0))^2 d\Omega = C\|F(w_0)\|^2_{H^2(\Omega)} \\
&\le C\|w_0\|^2_{H^2(\Omega)},
\end{aligned}$$

where the last inequality holds by (3.11). Hence, (2.5) holds with $C_1 \equiv 0$.

PROOF OF THEOREM 2.2. It suffices to verify (2.6) and (2.7) and to apply Theorem 2.6. We note that (2.6)(i) is satisfied by hypothesis (2.2)(i) in Theorem 2.2. As for (2.6)(ii), we see that in p.d.e. form this is equivalent to

$$\begin{aligned}
&\Delta^2 w_0 \in [H^1_{\Gamma_0}(\Omega)]' \\
&w_0 = \tfrac{\partial w_0}{\partial \nu} = 0 \qquad \text{on } \Gamma_0 \\
&\left.\begin{aligned}
\Delta w_0 + (1-\mu) B_1 w_0 &= -\tfrac{\partial w_1}{\partial \nu} \\
\tfrac{\partial \Delta w_0}{\partial \nu} + (1-\mu) B_2 w_0 &= w_1 - \tfrac{\partial^2 w_1}{\partial \tau^2}
\end{aligned}\right\} \quad \text{on } \Gamma_1.
\end{aligned}$$

But then if $w_0 \in H^3(\Omega) \cap H^2_{\Gamma_0}(\Omega)$ and $(w_0, w_1)$ satisfy the compatibility relation (2.2)(ii), we see that (2.6) must hold also.

We now prove (2.7). We need to show that for $w \in H^2_{\Gamma_0}(\Omega)$, $h \in H^1_{\Gamma_0}(\Omega)$ and $\varphi \in H^2_{\Gamma_0}(\Omega)$ that

$$|(D\mathcal{F}(w)h, \varphi)| \leq C(\|w\|_{H^2_{\Gamma_0}(\Omega)})\|h\|_{H^1_{\Gamma_0}(\Omega)}\|\varphi\|_{H^2_{\Gamma_0}(\Omega)}.$$

Recalling (3.9)–(3.10), we compute

$$\begin{aligned}(3.12)\qquad |([h, A_0^{-1}[w, w]], \varphi)| &= |([\varphi, F(w)], h)| \\ &\leq \|h\|_{H^1_{\Gamma_0}(\Omega)}\|[\varphi, F(w)]\|_{[H^1_{\Gamma_0}(\Omega)]'} \\ &\leq \|h\|_{H^1_{\Gamma_0}(\Omega)}\|[\varphi, A_0^{-1}[w, w]]\|_{H^{-\varepsilon}(\Omega)} \\ &\leq C(\|w\|^2_{H^2_{\Gamma_0}(\Omega)})\|h\|_{H^1_{\Gamma_0}(\Omega)}\|\varphi\|_{H^2_{\Gamma_0}(\Omega)},\end{aligned}$$

where we have used Lemma 3.1.

We now compute

$$\begin{aligned}|([w, A_0^{-1}[w, h]], \varphi)| &= |([w, \varphi], A_0^{-1}[w, h])| \\ (3.13)\qquad &\leq \|[w, \varphi]\|_{H^{-1-\epsilon}(\Omega)}\|A_0^{-1}[w, h]\|_{H_0^{1+\epsilon}(\Omega)} \\ &\leq C\|w\|_{H^2_{\Gamma_0}(\Omega)}\|\varphi\|_{H^2(\Omega)}\|A_0^{-(\frac{3}{4})+\varepsilon/4}[w, h]\|_{L^2(\Omega)},\end{aligned}$$

where we have again used the results of Grisvard [7] to give us $\mathcal{D}(A_0^{\frac{1+\epsilon}{4}}) \sim H_0^{1+\varepsilon}(\Omega)$.

We now examine the term $\|A_0^{-3/4+\varepsilon/4}[w, h]\|_{L^2(\Omega)}$. Let $\psi \in \mathcal{D}(A_0^{3/4-\varepsilon/4})$ so that (again by Grisvard's results) we have $\psi \in H^{3-\varepsilon}(\Omega) \cap H_0^2(\Omega)$. Then

$$(3.14)\qquad |([w, h], \psi)| = |([w, \psi], h)| \leq C\|w\|_{H^2(\Omega)}\left(\int_\Omega(\psi_{yy}^2 + \psi_{xx}^2 + \psi_{xy}^2)h^2 d\Omega\right)^{1/2}$$

But then since $h \in H^1(\Omega) \subset L^q(\Omega)$, $1 \leq q < \infty$, and by Hölder's inequality, we have, for example,

$$\begin{aligned}\int_\Omega \psi_{yy}^2 h^2 d\Omega &\leq \left(\int_\Omega \psi_{yy}^{2p} d\Omega\right)^{1/p}\left(\int_\Omega h^{2q} d\Omega\right)^{1/q} \\ &= \|\psi_{yy}\|^2_{L^{2p}(\Omega)}\|h\|^2_{L^{2q}(\Omega)} \\ &\leq \tilde{C}\|\psi_{yy}\|^2_{L^{2+\epsilon_0}(\Omega)}\|h\|^2_{H^1(\Omega)}.\end{aligned}$$

Using the Sobolev imbeddings (see [8], Theorem 7.58 p. 218), this implies

$$\|\psi_{yy}h\|^2_{L^2(\Omega)} \leq C\|\psi\|^2_{H^{2+\epsilon_1}(\Omega)}\|h\|^2_{H^1(\Omega)},$$

where $\varepsilon_1 = \frac{\varepsilon_0}{2+\varepsilon_0}$. Substituting back into (3.14), we obtain

$$\begin{aligned}(3.15)\qquad |([w, \psi], h)| &\leq \tilde{C}\|w\|_{H^2(\Omega)}\|\psi\|_{H^{2+\epsilon}(\Omega)}\|h\|_{H^1(\Omega)} \\ &\leq C\|w\|_{H^2(\Omega)}\|\psi\|_{H^{3-\epsilon}(\Omega)}\|h\|_{H^1(\Omega)}.\end{aligned}$$

Putting (3.13)–(3.15) together implies

$$(3.16)\qquad |([w, A_0^{-1}[w, h]], \varphi)| \leq C\|w\|^2_{H^2_{\Gamma_0}(\Omega)}\|h\|^2_{H^1_{\Gamma_0}(\Omega)}\|\varphi\|_{H^2_{\Gamma_0}(\Omega)}.$$

Then taking (3.12) with (3.16) gives us the estimate in (2.7). Applying Theorem 2.6, we have the result.

PROOF OF THEOREM 2.3. Here we would like to use the following strong regularity result from [6].

THEOREM 3.2. ([6] Theorem 2.3 – Regularity Revisited): *In addition to the assumptions of the previous theorem (our Theorem 2.6) assume that $\mathcal{F}$ is twice Frechét differentiable $\mathcal{D}(\mathcal{A}^{1/2}) \to V'$. Moreover, assume*

$$(3.17)\qquad \widetilde{M}^{-1} \in \mathcal{L}(H; \mathcal{D}(\mathcal{A}^{1/2}))$$

(3.18) $$\mathcal{F}(w_0) \in H;$$

*and*

(3.19) $$\begin{cases} \text{(i)} & w_0 + \beta GG^*\mathcal{A}w_1 \in \mathcal{D}(\mathring{\mathcal{A}}) \\ \text{(ii)} & \mathcal{A}(w_1 + \beta GG^*\mathcal{A}\{-\mathcal{M}^{-1}[\mathring{\mathcal{A}}(w_0 + \beta GG^*\mathcal{A}w_1) - \mathcal{F}(w_0)]\}) \in V'. \end{cases}$$

*Then,*

(3.20) $$(w_{tt}, w_{ttt}) \in C([0,T]; \mathcal{D}(\mathcal{A}^{1/2}) \times V);$$

(3.21) $$\begin{cases} \mathring{\mathcal{A}}(w + \beta GG^*\mathcal{A}w_t) - \mathcal{F}(w) \in C([0,T]; H), \\ \mathring{\mathcal{A}}(w_t + \beta GG^*\mathcal{A}w_{tt}) - D\mathcal{F}(w)w_t \in C([0,T]; V'), \end{cases}$$

*and the equation*

(3.22) $$Mw_{tt} + \mathcal{A}(w(t) + \beta GG^*\mathcal{A}w_t(t)) - \mathcal{F}(w(t)) = 0$$

*holds for all $t \geq 0$ on $H$.*

Unfortunately, system (1.1) fails to satisfy hypothesis (3.17), since for general $L^2$-functions, $\widetilde{\mathcal{M}}^{-1}$ cannot recover both boundary conditions on $\Gamma_0$. However, to follow the proof of the theorem given in [6], we need only

(3.23) $$M^{-1}\mathcal{A}(w_0 + \beta GG^*\mathcal{A}w_1) + M^{-1}\mathcal{F}(w_0) \in \mathcal{D}(\mathcal{A}^{1/2}),$$

which, in terms of system (1.1) requires $w_{tt}(0) \in \mathcal{D}(\mathcal{A}^{1/2})$. By virtue of hypothesis on $w_0$, $w \in \mathcal{D}(\mathcal{A}^{1/2})$, it suffices that $\widetilde{M}^{-1}$: $L^2(\Omega) \to H^2(\Omega)$. But this follows directly from the definition of $\widetilde{M}$. Consequently, system (1.1) satisfies the weaker, but sufficient, hypothesis (3.23). We now show that under the hypotheses of Theorem 2.3, we may apply the modified version of Theorem 3.1 to system (1.1).

By straightforward computations one can see that the von Kármán nonlinearity is twice Frechét differentiable with

$$\begin{aligned} D^2\mathcal{F}(w)(h,v) &= [-2A_0^{-1}[w,h],v] \\ &+ [-2A_0^{-1}[v,h],w] + [h,-2A_0^{-1}[w,v]]. \end{aligned}$$

By Lemma 3.1 we see that for $w, h, v \in H^2_{\Gamma_0}(\Omega)$ with $\varepsilon < 1/2$,

$$\begin{aligned} \|D^2\mathcal{F}(w)(h,v)\|_{[H^1_{\Gamma_0}(\Omega)]'} &\leq \|D^2\mathcal{F}(w)(h,v)\|_{H^{-\epsilon}(\Omega)} \\ &\leq C\|w\|_{H^2_{\Gamma_0}(\Omega)}\|h\|_{H^2_{\Gamma_0}(\Omega)}\|v\|_{H^2_{\Gamma_0}(\Omega)}. \end{aligned}$$

By hypothesis (2.3)(i), we see that $\mathcal{F}(w_0) \in L^2(\Omega)$ is trivially satisfied.

In terms of the p.d.e., (3.19)(i) is equivalent to (2.3)(i) with (2.2)(ii). We also observe that by (2.4)

$$-\mathcal{M}^{-1}[\mathcal{A}(w_0 + \beta GG^*\mathcal{A}w_1) - \mathcal{F}(w_0)] = w_{tt}(0),$$

So that the p.d.e. equivalent of (3.19)(ii) is

$$\begin{aligned} &\Delta^2 w_1 \in [H^1_{\Gamma_0}(\Omega)]' \\ &w_1 = \tfrac{\partial w_1}{\partial\nu} = 0 \qquad \text{on } \Gamma_0 \\ &\left.\begin{aligned} \Delta w_1 + (1-\mu)B_1 w_1 &= -\tfrac{\partial}{\partial\nu}w_{tt}(0) \\ \tfrac{\partial\Delta w_1}{\partial\nu} + (1-\mu)B_2 w_1 &= w_{tt} - \tfrac{\partial^2}{\partial\tau^2}w_{tt}(0) \end{aligned}\right\} \text{ on } \Gamma_1. \end{aligned}$$

But these are precisely satisfied by hypothesis (2.3)

Applying the results of Theorem 3.1, we obtain the regularity results of Theorem 2.3.

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