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EXISTENCE OF PERIODIC SOLUTIONS FOR NONLINEAR LIENARD SYSTEMS

WAN SE KIM
Department of Mathematics
Dong-A University
Pusan 604 - 714
Republic of Korea

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ABSTRACT. We prove the existence and multiplicity of periodic solutions for nonlinear Lienard System of the type

$$x''(t)+\frac{d}{dt}[\nabla F(x(t))]+g(x(t))+h(t,x(t))=e(t)$$

under various conditions upon the functions *g*, *h* and *e*.

KEY WORDS AND PHRASES: Nonlinear Lienard system, multiplicity of periodic solution.
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1. INTRODUCTION

Let R^n be n-dimensional Euclidean space. We define $\|x\| = [\Sigma_{i=1}^n |x_i|^2]^{1/2}$ for $x=(x_1,x_2,\ldots,x_n)\in R^n$. By $L^2([0,2\pi],R^n)$ we denote the space of all measurable functions $x:[0,2\pi]\to R^n$ for which $\|x(t)\|^2$ is integrable. The norm is given by

$$\|x\|_{L^2}=\left[\sum_{i=1}^n\|x_i\|_{L^2}^2\right]^{1/2}.$$

By $C^k([0.2\pi],R^n)$ we denote the Banach space of 2π-periodic continuous functions $x:[0,2\pi]\to R^n$ whose derivatives up to order k are continuous. The norm is given by

$$\|x\|_{C^k}=\sum_{i=0}^k\|x^{(i)}\|_\infty$$

where $\|y\|_\infty=\sup_{t\in[0,2\pi]}\|y(t)\|$ which is a norm in $C([0,2\pi],R^n)$. We use the symbol $(\cdot,\cdot)$ for the Euclidean inner product in the space R^n. For x, y in $C([0,2\pi],R^n)$ we define the L^2-inner product as follows

$$\langle x,y\rangle=\int_0^{2\pi}(x(t),y(t))dt\,.$$

The mean value $\bar{x}$ of x and the function of mean value zero are defined by $\bar{x}=\frac{1}{2\pi}\int_0^{2\pi}x(t)dt$ and $\tilde{x}(t)=x(t)-\bar{x}$, respectively.

We define inequalities in R^n componentwise, i.e. $x,y\in R^n$, $x\le y$ if and only if $x_i\le y_i$ for $i=1,2,\ldots,n$, and $x<y$ if and only if $x_i<y_i$ for $i=1,2,\ldots,n$. In this work, we will study the existence of periodic solutions and multiple periodic solutions for the problem

$$(E)\qquad x''(t)+\frac{d}{dt}[\nabla F(x(t))]+g(x)+h(t,x)=e(t)$$

$$(B)\qquad x(0)-x(2\pi)=x'(0)-x'(2\pi)=0$$

where $F: R^n \to R$ is a C^2-function, $g: R^n \to R^n$ is continuous, $h: [0,2\pi]\times R^n \to R$ is continuous in both variables and 2π-periodic in t, and $e: [0,2\pi] \to R$ is in $L^2([0,2\pi],R^n)$. We assume that $g(x) = (g_1(x_1), g_2(x_2), \ldots, g_n(x_n))$ for all $x = (x_1, x_2, \ldots, x_n) \in R^n$ and $h(t,x) = (h_1(t,x), h_2(t,x), \ldots, h_n(t,x))$ for all $(t,x) \in [0,2\pi]\times R^n$.

Moreover, we assume the following:

(H_1) h is bounded; i.e., for each $i = 1,2,3\ldots,n$, there exists $K_i > 0$ such that

$$|h_i(t,x)| \le K_i$$

for all $(t,x) \in [0,2\pi]\times R^n$.

(H_2) for each $i = 1,2,\ldots,n$,

$$\frac{d}{dt}\frac{\partial F(x)}{\partial x_i} = \frac{\partial^2 F(x)}{\partial x_i^2} x_i{}'$$

and there exists $C_i > 0$ such that

$$\left|\frac{\partial^2 F(x)}{\partial x_i^2}\right| \ge C_i$$

for all $x = (x_1, x_2, \ldots, x_n) \in R^n$.

The purpose of this work is to give existence and multiplicity results for periodic solutions of coupled Lienard system in R^n. This paper was motivated by the results in [1] and so our results in this work extend some results in [1]. To prove our results we adapt Mawhin's continuation theorem in [2], and we give appropriate region for the system's multiplicity by finding an a'priori bound.

2. A'priori Bound

To prove our assertion, we consider the following homotopy:

$$(E_\lambda) \qquad x''(t) + \lambda\frac{d}{dt}[\nabla F(x(t))] + \lambda g(x) + \lambda h(t,x) = \lambda e(t).$$

Let $\lambda \in (0,1)$ and let $x(t)$ be a possible solution of the problem $(E_\lambda)(B)$. Taking L^2-inner product by $x'(t)$ on both sides of (E_λ), we have

$$\lambda\sum_{i=1}^{n}\int_0^{2\pi}\frac{\partial^2 F(x(t))}{\partial x_i^2}[x_i'(t)]^2 dt + \lambda\sum_{i=1}^{n}\int_0^{2\pi} g_i(x_i(t))x_i'(t)dt$$

$$+\lambda\sum_{i=1}^{n}\int_0^{2\pi} h_i(t,x(t))x_i'(t)dt = \lambda\sum_{i=1}^{n}\int_0^{2\pi} e_i(t)x_i'(t)dt.$$

By the continuity of $\frac{\partial^2 F(x)}{\partial x_i^2}$, (H_2) and the periodicity of $x_i(t)$ in t, we have

$$\sum_{i=1}^{n} C_i \int_0^{2\pi}[x_i'(t)]^2 dt \le \left|\sum_{i=1}^{n}\int_0^{2\pi}\frac{\partial^2 F(x)}{\partial x_i^2}[x_i'(t)]^2 dt\right|$$

$$\le \sum_{i=1}^{n}\sqrt{2\pi}\left[\sum_{i=1}^{n}K_i^2\right]^{1/2}\left[\int_0^{2\pi}|x_i'(t)|^2 dt\right]^{1/2} + \left[\sum_{i=1}^{n}\int_0^{2\pi}|\bar{e}_i(t)|^2 dt\right]^{1/2}\left[\sum_{i=1}^{n}\int_0^{2\pi}[x_i'(t)]^2\right]^{1/2}.$$

Hence

$$\|x'\|_{L^2} \le \left(\frac{1}{\min_{1\le i\le n} C_i}\right)\left[\sqrt{2\pi}\left[\sum_{i=1}^{n}K_i^2\right]^{1/2} + \|\bar{e}\|_{L^2}\right] = M_0.$$

By the Sobolev inequality, we have

$$\|\tilde{x}\|_\infty \le \sqrt{\frac{\pi}{6}}M_0 = M_1.$$

Suppose there exist $a=(a_1,a_2,\ldots,a_n), b=(b_1,b_2,\ldots,b_n)$ in R^2 such that $a\le b$; if $x(t)$ is a solution of $(E_\lambda)(B)$ such that $a\le\bar{x}\le b$ and $\|\tilde{x}\|_\infty\le M_1$, then

$$\|x\|_\infty\le\left[\sum_{i=1}^n[\max(|a_i|,|b_i|)]^2\right]^{1/2}+M_1 .$$

Taking L^2-inner product by $x''(t)$ on both sides of (E_λ), we have

$$\sum_{i=1}^n\int_0^{2\pi}[x_i''(t)]^2dt+\lambda\sum_{i=1}^n\int_0^{2\pi}\frac{\partial^2F(x)}{\partial x_i^2}x_i'(t)x_i''(t)dt$$

$$+\lambda\sum_{i=1}^n\int_0^{2\pi}g_i(x_i(t))x_i''(t)dt+\lambda\sum_{i=1}^n\int_0^{2\pi}h_i(t,x(t))x_i''(t)dt$$

$$=\lambda\sum_{i=1}^n\int_0^{2\pi}\tilde{e}_i(t)x_i''(t)dt .$$

Since F is a C^2-function, for each $i=1,2,\ldots,n$, there exists $i>0$ such that

$$\left|\frac{\partial^2F(x)}{\partial x_i^2}\right|\le D_i ,$$

and also since g is continuous, for each $i=1,2,\ldots,n$, there exists $L_i>0$ such that

$$|g_i(x_i)|\le L_i .$$

Hence

$$\sum_{i=1}^n\int_0^{2\pi}[x_i''(t)]^2dt\le\left(\max_{1\le i\le n}D_i\right)\left[\sum_{i=1}^n\int_0^{2\pi}|x_i'(t)|^2dt\right]^{1/2}\left[\sum_{i=1}^n\int_0^{2\pi}|x_i''(t)|^2dt\right]^{1/2}$$

$$+\sqrt{2\pi}\left[\sum_{i=1}^nL_i^2\right]^{1/2}+\left[\sum_{i=1}^nK_i^2\right]^{1/2}\left[\sum_{i=1}^n\int_0^{2\pi}|x_i''(t)|^2\right]^{1/2}$$

$$+\left[\sum_{i=1}^n\int_0^{2\pi}|\tilde{e}_i(t)|^2dt\right]^{1/2}\left[\sum_{i=1}^n\int_0^{2\pi}\left|x_i''(t)\right|^2dt\right]^{1/2} .$$

and thus we have

$$\|x''\|_{L^2}\le\left(\max_{1\le i\le n}D_i\right)M_0+\sqrt{2\pi}\left[\sum_{i=1}^nL_i^2\right]^{1/2}+\left[\sum_{i=1}^nK_i^2\right]^{1/2}+\|\tilde{e}\|_{L^2}\equiv M_2 .$$

By the Sobolev inequality

$$\|x'\|_\infty\le\sqrt{\frac{\pi}{6}}M_2$$

for every solution of the problem $(E_\lambda)(B)$ where M_2 depends on a, b, M_0 and M_1.

3. OPERATOR FORMULATION

Define

$$L:D(L)\subseteq C^1([0,2\pi],R^n)\to L^2([0,2\pi],R^n)$$

by

$$(x_1(t),x_2(t),\ldots,x_n(t))\to(x_1''(t),x_2''(t),\ldots,x_n''(t))$$

where $D(L)=C^2([0,2\pi],R^n)$. Then $KerL=R^2$ and

$$ImL = \left\{ e \in L^2([0,2\pi],R^n) \mid \int_0^{2\pi} e(t)dt = 0 \right\}.$$

Consider two continuous projections

$$P: C^1([0,2\pi],R^n) \to C^1([0,2\pi],R^n)$$

such that

$$ImP = KerL$$

and

$$Q: L^2([0,2\pi],R^n) \to L^2([0,2\pi],R^n)$$

defined by

$$(Qe)(t) = \frac{1}{2\pi} \int_0^{2\pi} e(t)dt .$$

Then

$$KerQ = ImL, C([0,2\pi],R^n) = KerL \oplus KerP$$

and $L^2([0,2\pi],R^n) = ImL \oplus ImQ$ as a topological sum. Since

$$dim[L^2([0,2\pi],R^n)/ImL] = dim[ImQ] = dim[KerL] = n ,$$

L is a Fredholm mapping of index zero and hence there exists an isomorphism $J: ImQ \to KerL$. The operator L is not bijective but the restriction of L on $DomL \cap KerP$ is one-to-one and onto ImL, so it has its algebraic right inverse K_R and, as well known, it is compact. Define

$$N: C^1([0,2\pi],R^n) \to L^2([0,2\pi],R^n)$$

by

$$x(t) \to -\frac{d}{dt}[\nabla F(x(t))] - g(x(t)) - h(t,x(t)) + e(t)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$. Then N is continuous and maps bounded sets into bounded sets. Let G be any open bounded subset of $C^1([0,2\pi],R^n)$, then $QN: G \to L^2([0,2\pi],R^n)$ is bounded and $K_R(I-Q): \overline{G} \to L^2([0,2\pi],R^n)$ is compact and continuous. Hence N is L-compact on G. Now we see $x \in D(L)$ is a solution to the problem $(E_\lambda)(B)$ if and only if

$$Lx = \lambda Nx .$$

4. MAIN RESULTS

THEOREM 4.1. Besides conditions on F, g, e, and $(H_1), (H_2)$, we assume

(H_3) there exists $r = (r_1, r_2, \dots, r_n)$, $s = (s_1, s_2, \dots, s_n)$, $A = (A_1, A_n, \dots, A_n)$ and $B = (B_1, B_2, \dots, B_n)$ in R^n such that $r < s$ and $A \le B$

$$\frac{1}{2\pi} \int_0^{2\pi} g(r + \tilde{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \overline{x} + \tilde{x}(t))dt \le A$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} g(s + \tilde{x}(t))dt + \frac{1}{2\pi} \int_0^{2\pi} h(t, \overline{x} + \tilde{x}(t))dt \ge B$$

for every $\overline{x} \in R^n$ such that

$$\|\overline{x}\| \le [\sum_{i=1}^{n} [\max(|r_i|, |s_i|)^2]^{1/2} ,$$

and for every $\tilde{x} \in C^1([0,2\pi],R^n)$ having mean value zero, satisfying the boundary condition (B) and such that

$$\|\tilde{x}\|_\infty \le \sqrt{\frac{\pi}{6}}\left(\frac{1}{\min_{1\le i\le n} C_i}\right)\left[\sqrt{2\pi}\left[\sum_{i=1}^{n} K_i^2\right]^{1/2} + \|\tilde{e}\|_{L^2}\right].$$

Then $(E)(B)$ has at least one solution if

$$A < \frac{1}{2\pi}\int_0^{2\pi} e(t)dt < B .$$

PROOF. We construct a bounded open set Ω in $C^1(([0,2\pi]),R^n)$ to apply Mawhin's continuation theorem in [2]. Using a'priori estimate, we have

$$\|x'\|_{L^2} \le \left(\frac{1}{\min_{1\le i\le n} C_i}\right)\left[\sqrt{2\pi}\left[\sum_{i=1}^{n} K_i^2\right]^{1/2} + \|\tilde{e}\|_{L^2}\right] = M_0$$

for any solution $x(t)$ of $(E_\lambda)(B)$, $\lambda \in (0,1)$. Hence $\|\tilde{x}\|_\infty \le \sqrt{\frac{\pi}{6}}M_0 = M_1$. Define a bounded set Ω^0 by

$$\Omega^0 = \{x \in C^1([0,2\pi],R^n) | r \le \bar{x} \le s, \|\tilde{x}\|_\infty \le M_1\} .$$

Then, for any solution $x(t)$ of $(E_\lambda)(B)$ lying in Ω^0, we have

$$\|x\|_\infty \le \left[\sum_{i=1}^{n}[\max(|r_i|,|s_i|)]^2\right]^{1/2} + M_1$$

and

$$\|x''\|_{L^2} \le \left(\max_{1\le i\le n} D_i\right)M_0 + \sqrt{2\pi}\left[\sum_{i=1}^{n} L_i^2\right]^{1/2} + \left[\sum_{i=1}^{n} K_i^2\right]^{1/2} + \|\tilde{e}\|_{L^2} = M_2 ,$$

where L_i depends on r, s and M_1. Thus $\|x'\|_\infty \le \sqrt{\frac{\pi}{6}}M_2$. Define a bounded open set Ω by

$$\Omega = \left\{x \in C^1([0,2\pi],R^n) \,|\, r < \bar{x} < s, \|\tilde{x}\|_\infty < 2M_1, \|x'\|_\infty < \sqrt{\frac{2\pi}{6}}M_2\right\}.$$

Let $(x,\lambda) \in [D(L)\cap\partial\Omega]\times(0,1)$ and if (x,λ) is any solution to $Lx = \lambda Nx$, then (x,λ) is a solution to the problem $(E_\lambda)(B)$,

$$\|\bar{x}\| \le \left[\sum_{i=1}^{n}[\max(|r_i|,|s_i|)]^2\right]^{1/2}, \quad \|\tilde{x}\| \le M_1$$

and there exists some $i \in \{1,2,\dots,n\}$ such that $\bar{x}_i = r_i$ or s_i. Take L^2-inner product with $e_i = (0,0,\dots,0,1,0,\dots,0)$ on both sides of (E_λ), we have

$$\lambda\int_0^{2\pi} g_i(x_i(t))dt + \lambda\int_0^{2\pi} h_i(t,x(t))dt = \lambda\int_0^{2\pi} e_i(t)dt ,$$

or

$$\int_0^{2\pi} g_i(x_i(t))dt + \int_0^{2\pi} h_i(t,x(t))dt - \int_0^{2\pi} e_i(t)dt = 0$$

if $\bar{x}_i = r_i$, then, by assumption

$$\int_0^{2\pi} g_i(r_i+\tilde{x}_i(t))dt + \int_0^{2\pi} h_i(t,\bar{x}_1+\tilde{x}_1(t),\dots,r_i+\tilde{x}_i(t),\dots,\bar{x}_n+\tilde{x}_n(t))dt - \int_0^{2\pi} e_i(t)dt < 0 .$$

If $\bar{x}_i = s_i$, then again by assumption,

$$\int_0^{2\pi} g_i(s_i+\tilde{x}_i(t))dt + \int_0^{2\pi} h_i(t,\bar{x}_1+\tilde{x}_1(t),\ldots,s_i+\tilde{x}_i(t),\ldots,\bar{x}_n+\tilde{x}_n(t))dt - \int_0^{2\pi} e_i(t)dt < 0 .$$

Thus, for each $\lambda \in (0,1)$, for every solution of

$$Lx = \lambda Nx$$

is such that $x \notin \partial\Omega$.

Next, we will show that $QNx \neq 0$ for each $x \in KerL \cap \partial\Omega$ and $d_B[JQN, \Omega\cap KerL, 0] \neq 0$ where d_B is the Brouwer topological degree. Since $J: ImQ \to KerL$ is an isomorphism and $dim[ImQ] = dim[KerL] = n$, we may take J to be the identity on R^n and hence

$$(JQN)(x)(t) = -\frac{1}{2\pi}\int_0^{2\pi} g(x(t))dt - \frac{1}{2\pi}\int_0^{2\pi} h(t,x(t))dt + \frac{1}{2\pi}\int_0^{2\pi} e(t)dt$$

with, for $i = 1,2,\ldots,n$,

$$(JQN)_i(x)(t) = -\frac{1}{2\pi}\int_0^{2\pi} g_i(x_i(t))dt - \frac{1}{2\pi}\int_0^{2\pi} h_i(t,x(t))dt + \frac{1}{2\pi}\int_0^{2\pi} e_i(t)dt$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$.

Let $x \in KerL \cap \partial\Omega$, then $x = \bar{x}$ is constant in R^n,

$$\|\bar{x}\| \le \left[\sum_{i=1}^{n} [\max(|r_i|,|s_i|)]^2\right]^{1/2},$$

and there exists $i \in \{1,2,\ldots,n\}$ such that $x_i = \bar{x}_i = r_i$ or s_i. In a similar manner we have $(QN)_i(x) \neq 0$.

Thus $QNx \neq 0$ for each $x \in KerL \cap \partial\Omega$. It is easy to see that $P \equiv \overline{\Omega \cap KerL} = \Pi_{i=1}^n [r_i, s_i]$. Let $P_i = \{x \in P \mid x_i = r_i\}$, $P_i' = \{x \in P \mid x_i = s_i\}$ and $x \in P_i, x' \in P_i', i = 1,2,\ldots,n$.

Then $x = \bar{x}, x' = \bar{x}'$ are constant with

$$\|\bar{x}\|, \text{ and } \|\bar{x}'\| \le \left[\sum_{i=1}^{n} [\max(|r_i|,|s_i|)]^2\right]^{1/2},$$

and $x_i = \bar{x}_i = r_i, x_i' = \bar{x}_i' = s_i$. Hence

$$(JQN)_i(x) = -\frac{1}{2\pi}\int_0^{2\pi} g_i(r_i)dt - \frac{1}{2\pi}\int_0^{2\pi} h_i(t,x_i,\ldots,r_i,\ldots,x_n)dt + \frac{1}{2\pi}\int_0^{2\pi} e_i(t)dt > 0$$

and

$$(JQN)_i(x') = -\frac{1}{2\pi}\int_0^{2\pi} g_i(s_i)dt - \frac{1}{2\pi}\int_0^{2\pi} h_i(t,x_i',\ldots,s_i,\ldots,x_n')dt + \frac{1}{2\pi}\int_0^{2\pi} e_i(t)dt < 0 .$$

Thus $(JQN)_i(x)(JQN)_i(x') < 0$ for $i = 1,2,\ldots,n$. Therefore, by the generalized intermediate value theorem, $d_B[JQN, \Omega\cap KerL, 0] \neq 0$. Hence, by Mawhin's continuation theorem, the problem $(E)(B)$ has at least one solution in $D(L) \cap \bar{\Omega}$.

THEOREM 4.2. Besides conditions on F, g, e, and (H_1) and (H_2), we assume

(H_4) there exists $q = (q_1, q_2, \ldots, q_n)$, $r = (r_1, r_2, \ldots, r_n)$, $s = (s_1, s_2, \ldots, s_n)$, $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ in R^n such that $q < r < s$ and $A \le B$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} g(q+\tilde{x}(t))dt + \frac{1}{2\pi}\int_0^{2\pi} h(t,\bar{x}+\tilde{x}(t))dt \ge B ,$$

$$\frac{1}{2\pi}\int_0^{2\pi} g(r+\tilde{x}(t))dt + \frac{1}{2\pi}\int_0^{2\pi} h(t,\bar{x}+\tilde{x}(t))dt \le A ,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} g(s+\tilde{x}(t))dt + \frac{1}{2\pi}\int_0^{2\pi} h(t,\bar{x}+\tilde{x}(t))dt \geq B$$

for every $\bar{x} \in R^n$ such that

$$\|\bar{x}\| \leq \left[\sum_{i=1}^{n} \max(|q_i|,|r_i|,|s_i|)^2\right]^{1/2}$$

and for every $\tilde{x} \in C^1([0,2\pi],R^n)$ having mean value zero, satisfying the boundary condition (B) such that

$$\|\tilde{x}\|_\infty \leq \sqrt{\frac{\pi}{6}}\left(\frac{1}{\min_{1\leq i\leq n} C_i}\right)\left[\sqrt{2\pi}\left[\sum_{i=1}^{n} K_i^2\right]^{1/2} + \|\tilde{e}\|_{L^2}\right]$$

Then $(E)(B)$ has at least 2^n solutions if

$$A < 1/2\pi \int_0^{2\pi} e(t)dt < B \ .$$

PROOF. We construct 2^n bounded open sets in $C^1([0,2\pi],R^n)$ to apply Mawhin's continuation theorem in [3]. Using a'priori estimate, we have

$$\|x'\|_{L^2} \leq \left(\frac{1}{\min_{i\leq i\leq n} C_i}\right)\left[\sqrt{2\pi}\left[\sum_{i=1}^{n} K_i^2\right]^{1/2} + \|\tilde{e}\|_L^2\right] = M_0$$

for any solution $x(t)$ of $(E_\lambda)(B), \lambda \in (0,1)$. Hence $\|\tilde{x}\|_\infty \leq \sqrt{\frac{\pi}{6}}M_0 = M_1$. Let I, J be two disjoint subsets of $\{1,2,\ldots,n\}$ such that $I \cup J = \{1,2,\ldots,n\}$ and define Ω_{IJ}^0 by $\Omega_{IJ}^0 = \{x \in C^1([0,2\pi],R^n) \mid q_i \leq \bar{x}_i \leq r_i$ for $i \in I, r_j \leq \bar{x}_j \leq s_j$ for $j \in J, \|\tilde{x}\|_\infty \leq M_1\}$; then the number of such sets is 2^n and for any solution, $x(t)$ of $(E_\lambda)(B)$ lying in Ω_{IJ}^0, we have

$$\|x\|_\infty \leq \left[\sum_{i\in I}[\max(|q_i|,|r_i|)]^2 + \sum_{j\in J}[\max(|r_j|,|s_j|)]^2\right]^{1/2} + M_1$$

and

$$\|x''\|_{L^2} \leq \left(\max_{1\leq i\leq n} D_i\right)M_0 + \sqrt{2\pi}\left[\sum_{i=1}^{n} L_i^2\right]^{1/2} + \left[\sum_{i=1}^{n} K_i^2\right]^{1/2} + \|\tilde{e}\|_{L^2} = M_2$$

where L_i depends on q,r,s and M_1. Thus $\|x'\|_\infty \leq \sqrt{\frac{\pi}{6}}M_2$. Define a bounded open set Ω_{IJ} by

$$\Omega_{IJ} = \{x \in C^1([0,2\pi],R^n) \mid q_i < \bar{x}_i < r_i \quad \text{for} \quad i \in I \, , r_j < \bar{x}_j < s_j$$

$$\text{for} \quad j \in J, \|\tilde{x}\|_\infty < 2M_1, \|x''\|_\infty < \sqrt{\frac{2\pi}{3}}M_2 \, .$$

Let $(x,\lambda) \in [D(L) \cap \partial\Omega_{IJ}] \times (0,1)$ and if (x,λ) is any solution to

$$Lx = \lambda Nx \ ,$$

then (x,λ) is a solution to the problem $(E_\lambda)(B)$,

$$\|\bar{x}\| \leq \left[\sum_{i\in I}[\max(|q_i|,|r_i|)]^2 + \sum_{j\in J}[\max(|r_j|,|s_j|)]^2\right]^{1/2}, \|\tilde{x}\| \leq M_1$$

and there exists some $i \in \{1,2,\ldots,n\}$, such that $\bar{x}_i = q_i, r_i$ or s_i. By (H_4) and assumption we can see for each $\lambda \in (0,1)$, for every solution of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega_{IJ}$. And similarly, we can also see $QNx \neq 0$ for each $x \in KerL \cap \partial\Omega_{IJ}$. It is easy to see $P = \Omega_{IJ} \cap KerL = \Pi_{i\in I}[q_i,r_i] \times \Pi_{j\in J}[r_j,s_j]$. Let

$$P_i = \{x \in p \mid x_i = q_i\} \quad \text{if} \quad i \in I,$$

$$P_j = \{x \in p \mid x_j = r_j\} \quad \text{if} \quad j \in J,$$

$$P_i' = \{x \in p \mid x_i = r_i\} \quad \text{if} \quad i \in I,$$

$$P_j' = \{x \in p \mid x_i = s_j\} \quad \text{if} \quad j \in I,$$

and let $x \in P_i$, $x' \in P_i'$ with $i \in I \cup J$. Then, for $i \in I$, we have $x_i = q_i$, $x_i = r_i$. Hence $(JQN)_i(x)(JQN)_i(x') < 0$ for $i \in I$. For $j \in J$, we have $x_j = r_j, x_i' = s_j$. Thus $(JQN)_j(x)(JQN)_j(x') < 0$ for $j \in J$. Therefore, we have $d_B[JQN, \Omega_{IJ} \cap KerL, 0] \neq 0$. Thus, by Mawhin's continuation theorem, the problem $(E_\lambda)(B)$ has at least one solution in $D(L) \cap \overline{\Omega}_{IJ}$. Thus $(E_\lambda)(B)$ has at least 2^n solutions.

Corollary 4.3. Besides the conditions on F, g and e, and (H_1) and (H_2), we assume

(H_5) there exists $T = (T_1, T_2, \ldots, T_n) > 0$ in R^n such that

$$g(T+x) = g(x) \quad \text{and} \quad h(t, T+x) = h(t,x)$$

for all $(t,x) \in [0, 2\pi] \times R^n$.

(H_6) there exists $r = (r_1, r_2, \ldots, r_n)$, $s = (s_1, s_2, \ldots, s_n)$, $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ in R^n such that $0 < s - r < T$, $r < s$, $A \leq B$

$$\frac{1}{2\pi}\int_0^{2\pi} g(r + \tilde{x}(t))dt + \frac{1}{2\pi}\int_0^{2\pi} h(t, \overline{x} + \tilde{x}(t))dt \leq A,$$

$$\frac{1}{2\pi}\int_0^{2\pi} g(s + \tilde{x}(t))dt + \frac{1}{2\pi}\int_0^{2\pi} h(t, \overline{x} + \tilde{x}(t))dt \geq B$$

for every $\overline{x} \in R^n$ such that

$$\|\overline{x}\| \left[\sum_{i=1}^{n} [\max(|s_i - T_i|, |r_i|, |s_i|)]^2\right]^{1/2}$$

and for every $\tilde{x} \in C^1([0.2\pi], R^n)$ having mean value zero, satisfying the boundary condition (B) and such that

$$\|\tilde{x}\|_\infty \leq \sqrt{\frac{\pi}{6}}\left(\frac{1}{\min_{1 \leq i \leq n} C_i}\right)\left[\sqrt{2\pi}\left[\sum_{i=1}^{n} K_i^2\right]^{1/2} + \|\tilde{e}\|_{L^2}\right].$$

Then $(E)(B)$ has at least 2^n solutions if

$$A < \frac{1}{2\pi}\int_0^{2\pi} e(t)dt < B.$$

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REFERENCES

[1] MAWHIN, J. and WILLEM, M. Multiple solutions of the periodic boundary value problem for some forced pendulum-type equations, J. Diff. Eq., 52, 2 (1984), 264-287.

[2] GAINES, R. E. and MAWHIN, J. Coincidence degree and nonlinear differential equations, Springer-Verlag, New York, 1977.

[3] DRABEK, P. Remarks on multiple periodic solutions of nonlinear ordinary differential equations, Comment. Math. Univ. Carolinae 211 (1980), 155-160.

[4] DRABEK, P. Periodic solutions for systems of forced coupled pendulum-like equations, J. Diff. Eq., 70, 3 (1987), 390-401.

[5] ZANOLIN, B. Remarks on multiple periodic solutions for nonlinear ordinary differential systems of Lienard type, Boll. U.M.I. (6) I-B (1982), 683-698.