Internat. J. Math. & Math. Sci.
VOL. 18 NO. 3 (1995) 463-468



# ON CERTAIN CLASSES OF MEROMORPHICALLY STARLIKE FUNCTIONS

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(Received May 11, 1993 and in revised form April 14, 1994)

**ABSTRACT.** The object of the present paper is to introduce a new class $\Sigma_n(\alpha)$ of meromorphic functions defined by a multiplier transformation and to investigate some properties for the class $\Sigma_n(\alpha)$. Further we study integrals of functions in $\Sigma_n(\alpha)$.

**KEY WORDS AND PHRASES.** Univalent functions, meromorphically starlike functions, integral operators.
**1991 AMS SUBJECT CLASSIFICATION CODE.** 30C45.

## 1. INTRODUCTION.

Let $\Sigma$ denote the class of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k (a_{-1} \neq 0)$$

which are regular in the punctured disk $D = \{z: 0 < \mid z \mid < 1\}$. For any integer $n$, let the operator $I^n$ operating on $f \in \Sigma$ be defined by

$$I^n f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} (k+2)^{-n} a_k z^k$$

Obviously, we have

$$I^n(I^m f(z)) = I^{n+m} f(z)$$

for all integers $m$ and $n$. For any nonpositive integer $n$, the operators $I^n$ are the differential operators studied by Uralegaddi and Somanatha [5]. Also the operators $I^n$ are closely related to the multiplier transformations introduced by Flett [2].

For any integer $n$, let $\Sigma_n(\alpha)$ denote the class of functions $f \in \Sigma$ satisfying the condition

$$Re\left\{\frac{I^{n-1}}{I^n f(z)} - 2\right\} < -\alpha \, (0 \le \alpha < 1, \, z \in U = \{z : |z| < 1\}).$$

In this paper, we prove that for the classes $\Sigma_n(\alpha)$ of functions in $\Sigma$, $\Sigma_n(\alpha) \subset \Sigma_{n+1}(\alpha)$ holds. Since $\Sigma_0(\alpha)$ equals $\Sigma^*(\alpha)$ (the class of meromorphically starlike functions of order $\alpha$), all members in $\Sigma_n(\alpha)$ are univalent for any nonpositive integer $n$. Further property preserving integrals are considered Our results generalize the some results of Bajpai [1], Goel and Sohi [3] and Uralegaddi and Somanatha [6].

## 2. MAIN RESULTS.

We begin with the statement of the following lemma due to Miller and Mocaun [4].

**LEMMA.** Let $\phi(u,v)$ be a complex valued function, $\phi: D \to C, D \subset C^2 \in$ ($C$ is the complex plane), and let $u = u_1 + iu_2, v = v_1 + iv_2$. Suppose that the function $\phi(u,v)$ satisfies the following condition

(i) $\phi(u,v)$ is continuous in $D$,

(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} > 0$;

(iii) $Re\{\phi(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{-(1+u_2^2)}{2}$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be regular in $U$ such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If

$$Re\{\phi(p(z), zp'(z))\} > 0 \qquad (z \in U),$$

then $Re\{p(z)\} > 0 \ (z \in U)$.

With the aid of above lemma, we drive

**THEOREM 1.** If $f \in \Sigma_n(\alpha)$, then $f \in \Sigma_{n+1}(\beta)$, where

$$\beta = \frac{5 + 2\alpha - \sqrt{(3-2\alpha)^2 + 8}}{4}. \tag{2.1}$$

**PROOF.** Define the function $p(z)$ by

$$\frac{I^n f(z)}{I^{n+1} f(z)} = \gamma + (1-\gamma)p(z), \tag{2.2}$$

where

$$\gamma = \frac{(3-2\alpha) + \sqrt{(3-2\alpha)^{2+8}}}{4} \ (\gamma > 1). \tag{2.3}$$

We see that $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is regular in $U$. Making use of the logarithmic differentiations of both sides in (2.2) and using the identity

$$z(I^n f(z))' = I^{n-1} f(z) - 2I^n f(z), \tag{2.4}$$

we obtain

$$\frac{I^{n-1} f(z)}{I^n f(z)} = \gamma + (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{\gamma + (1-\gamma)p(z)} \tag{2.5}$$

or

$$-Re\left\{\frac{I^{n-1} f(z)}{I^n f(z)} - 2 + \alpha\right\} = Re\left\{2 - (\alpha + \gamma - (1-\gamma p(z) - \frac{(1-\gamma)zp'(z)}{\gamma + (1-\gamma p(z)}\right\} > 0. \tag{2.6}$$

Let us define the function $\phi(u,v)$ by

$$\phi(u,v) = 2 - (\alpha + \gamma) - (1-\gamma)u - \frac{(1-\gamma)v}{\gamma + (1-\gamma)u}. \tag{2.7}$$

Then $\phi(u,v)$ satisfies

(i) $\phi(u,v)$ is continuous in $D = \left(C - \left\{\frac{\gamma}{\gamma-1}\right\}\right) \times C$;

(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} = 1 - \alpha > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{-(1+u_2^2)}{2}$,

$$\begin{aligned} Re\{\phi(iu_2, v_1)\} &= 2 - (\alpha + \gamma) - \frac{\gamma(1-\gamma)v_1}{\gamma^2 + (1-\gamma)^2 u_2^2} \\ &\leq 2 - (\alpha + \gamma) + \frac{\gamma(1-\gamma)v_2}{2(\gamma + (1-\gamma)^2 u_2^2)} \\ &\leq 0. \end{aligned}$$

Thus the function $\phi(u, v)$ satisfies the conditions in our Lemma This shows that $Re\{p(z)\} > 0$ $(z \in U)$, hence

$$Re\left\{\frac{I^n f(z)}{I^{n+1} f(z)}\right\} < \gamma \, (z \in U) \tag{2.8}$$

or

$$Re\left\{\frac{I^n f(z)}{I^{n+1} f(z)} - 2\right\} < -\beta \, (z \in U) \tag{2.9}$$

where $\beta$ is given by (2 1) Therefore we complete the proof of the theorem

Since $\beta - \alpha > 0$ in Theorem 1, we have

**COROLLARY 1.** $\Sigma_n(\alpha) \subset \Sigma_{n+1}(\alpha)$ for any integer $n$

**REMARK.** For nonpositive integers $n$, Corollary 1 is a similar result obtained by Uralegaddi and Somanatha [6]

Putting $n = 0$ and $\alpha = 0$ in Corollary 1, we obtain the following result of Bajpai [1]

**COROLLARY 2.** If $f(z) = \frac{a_{-1}}{z} + \Sigma_{k=0}^{\infty} a_k z^k$ $(a_{-1} \neq 0)$ is meromorphically starlike, then so is

$$F_1(z) = \frac{1}{x^2} \int_0^z t f(t) \, dt. \tag{2.10}$$

Next, we prove

**THEOREM 2.** Let $f \in \Sigma_n(\alpha)$ and let

$$F_c(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t) \, dt \;\; (c > 0) \tag{2.11}$$

Then $F_c \in \Sigma_n(\beta)$, where

$$\beta = \frac{(3 + 2\alpha c) - \sqrt{(5 - 2\alpha - 2c)^2 + 8(4c - 3 - 2\alpha c + 2\alpha)}}{4} \tag{2.12}$$

**PROOF.** Let $f \in \Sigma_n(\alpha)$. Then we have

$$Re\left\{\frac{I^{n-1} f(z)}{I^n f(z) - 2}\right\} < -\alpha. \tag{2.13}$$

From the definition of $F_c$, we obtain

$$z(I^n F_c(z))' = c I^n f(z) - (c+1) I^n F_c(z) \tag{2 14}$$

and also

$$z(I^nF_c(z))' = I^{n-1}F_c(z) - 2I^nF_c(z). \tag{2 15}$$

Using (2.14) and (2 15), the condition (2 13) may be written as

$$Re\left\{\frac{\frac{I^{n-2}F_c(z)}{I^{n-1}F_c(z)} + (c-1)}{1+(c-1)\frac{I^nF_c(z)}{I^{n-1}F_c(z)}} - 2\right\} < -\alpha. \tag{2.16}$$

Define the function $p(z)$ by

$$\frac{I^{n-1}F_c(z)}{I^nF_c(z)} = \gamma + (1-\gamma)p(z), \tag{2.17}$$

where

$$\gamma = \frac{(5-2\alpha-2c)+\sqrt{(5-2\alpha-2c)^2+8(4c-3-2\alpha c+2\alpha)}}{4} \quad (\gamma > 1). \tag{2.18}$$

Then $p(z) = 1 + p_1z + p_2z^2 + \cdots$ is regular in $U$. Differentiating (2.17) logarithmically and simplifying, we have

$$\frac{\frac{I^{n-2}F_c(z)}{I^{n-1}F_c(z)} + (c-1)}{1+(c-1)\frac{I^nF_c(z)}{I^{n-1}F_c(z)}} - 2 = -2+\gamma+(1-\gamma)p(z) + \frac{(1-\gamma zp'(z)}{(\gamma+c-1)+(1-\gamma p(z)}. \tag{2.19}$$

It follows from (2.19) that

$$\begin{aligned} &-Re\left\{\frac{\frac{I^{n-2}F_c(z)}{I^{n-1}F_c(z)} + (c-1)}{1+(c-1)\frac{I^nF_c(z)}{I^{n-1}F_c(z)}} - 2+\alpha\right\} \\ &= Re\left\{2-(\alpha+\gamma)-(1-\gamma)p(z) - \frac{(1-\gamma)zp'(z)}{(\gamma+c-1)+(1-\gamma)p(z)}\right\} \\ &> 0. \end{aligned} \tag{2.20}$$

If we define the function $\phi(u,v)$ by

$$\phi(u,v) = 2-(\alpha+\gamma)-(1-\gamma)u - \frac{(1-\gamma)v}{(\gamma+c-1)+(1-\gamma)u}, \tag{2.21}$$

then $\phi(u,v)$ satisfies

(i) $\phi(u,v)$ is continuous in $D = \left(C - \left\{\frac{\gamma+c-1}{\gamma-1}\right\}\right) \times C$;

(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} = 1-\alpha > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{-(1+u_2^2)}{2}$,

$$\begin{aligned} Re\{\phi(iu_2,v_1)\} &= 2-(\alpha+\gamma - \frac{(\gamma+c-1)(1-\gamma)v_1}{(\gamma+c-1)^2+(1-\gamma)^2u_2^2} \\ &\leq 2-(\alpha+\gamma) + \frac{(\gamma+c-1)(1-\gamma)(1+u_2^2)}{2\{(\gamma+c-1)^2+(1-\gamma)^{2u_2^2\}}} \\ &\leq 0. \end{aligned}$$

Since $\phi(u,v)$ satisfies the conditions in Lemma, we have that $Re\{p(z)\} > 0 (z \in U)$. This proves that

$$Re\left\{\frac{I^{n-1}F_c(z)}{I^n F_c(z)}\right\} < \gamma \ (z \in U) \tag{2.22}$$

or

$$Re\left\{\frac{I^{n-1}F_c(z)}{I^n F_c(z)} - 2\right\} < -\beta \ (z \in U), \tag{2.23}$$

where $\beta$ is given by (2 12). That is, $F_c(z) \in \Sigma_n(\beta)$.

Similarly, from Theorem 2, we have

**COROLLARY 3.** If $f \in \Sigma_n(\alpha)$, then the integral operator $F_c$ defined by (2.11) belongs to the class $\Sigma_n(\alpha)$.

Taking $n = 0$ and $\alpha = 0$ in Corollary 3, we obtained the following corresponding result of Goel and Sohi [3].

**COROLLARY 4.** If $f(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k$ is meromorphically starlike, then so is the integral operator $F_c$ defined by (2.11).

The following theorem gives us a characterization of the class $\Sigma_n(\alpha)$.

**THEOREM 3.** $f \in \Sigma_n(\alpha)$ if and only if the integral operator $F_1$ defined by (2.10) belongs to the class $\Sigma_{n-1}(\alpha)$.

**PROOF.** For $c = 1$, the identities (2.14) and (2.15) reduce to $I^n f(z) = I^{n-1}F_1(z)$ and hence $I^{n-1}f(z) = I^{n-2}F_1(z)$. Therefore

$$\frac{I^{n-1}f(z)}{I^n f(z)} = \frac{I^{n-2}F_1(z)}{I^{n-1}F_1(z)} \tag{2.24}$$

and the result follows.

**ACKNOWLEDGEMENT.** This paper was supported (in part) by Non-Directed Research Fund and the Research Institute Attached to University Program, Korea Research Foundation, 1993.

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