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CONSTRUCTIVE SOLUTION OF COUPLED SECOND ORDER DELAY DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT. In this paper, we study initial value problems for coupled second order delay differential equations with variable coefficients. By means of the application of the method of steps and the method of Frobenius, the exact solution of the problem is constructed. Then, in a bounded domain, a finite analytic solution with error bounds is provided. Given an admissible error ϵ, we give the number of terms to be taken in the infinite series exact solution so that the approximation error be smaller than ϵ in the bounded domain.

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1. INTRODUCTION.

In many fields of the contemporary science and technology systems with delaying links are often met and the dynamical processes in these are described by systems of delay differential equations, Bellman & Cooke [1], Driver [3], Marchuck [11], Okamoto & Hayashi [13]. The delay appears in complicated systems with logical and computing devices, where certain time for information processing is needed.

The theory of linear delay differential equations has been developed in the fundamental monographs Bellman & Cooke [1], Driver [3], Hale [10], Myshkis [12], Pinney [15]. Analytic solutions of some linear systems of delay differential equations have been investigated by Cherepennikov [2], Jódar & Martín [8], Jódar & Martín [9], Rodinov [16].

In this paper, we consider initial value problems for systems of second order delay differential equations of the form

$$\left.\begin{aligned} X''(t) + A(t)X'(t) + B(t)X(t) + B_1(t)X'(t-w) &= F(t), \ t > 0 \\ X(t) &= G(t), \ t \in [-w, 0], \ w > 0 \end{aligned}\right\} \qquad (1.1)$$

where $A(t)$ and $B(t)$ are analytic $\mathcal{C}^{r\times r}$ valued functions on the positive real line, $B_1(t)$ is a $\mathcal{C}^{r\times r}$ valued continuous function, the unknown $X(t)$ as well as $F(t)$ and $G(t)$ are $\mathcal{C}^r$ valued functions, with $F(t)$ continuous in $t > 0$ and $G(t)$ is a continuously differentiable function in $[-w, 0]$.

Problem (1.1) can be transformed into the equivalent extended first order system

$$Z(t) = \begin{bmatrix} X(t) \\ X'(t) \end{bmatrix}.$$

$$\left.\begin{array}{c} Z'(t) + \begin{bmatrix} 0 & 0 \\ 0 & B_1(t) \end{bmatrix} Z(t-w) + \begin{bmatrix} 0 & -I \\ B(t) & A(t) \end{bmatrix} + Z(t) = \begin{bmatrix} 0 \\ F(t) \end{bmatrix}, \ t > 0 \\ Z(t) = \begin{bmatrix} G(t) \\ G'(t) \end{bmatrix}, \ t \in [-w, 0] \end{array}\right\}$$

but this approach has some drawbacks, such as the increase of the computational cost and the lack of explicitness due to the relationship $X(t) = [I, 0]Z(t)$.

The aim of this paper is twofold. First of all we construct a series solution of problem (1.1) by means of a matrix method of Frobenius and the method of steps, but dealing directly with (1.1). Secondly we truncate the series solution and provide error bounds for the continuous finite approximate solution when $t \in [nw, (n+1)w]$ and n is a positive integer. For the constant coefficient case, systems of second order delay differential equations have been recently studied in Jódar & Martín [8] and Jódar & Martín [9] avoiding the transformation of the problem into an equivalent extended first order system.

This paper is organized as follows. In section 2 we construct a series solution of problem (1.1). Error analysis of the finite truncated series in terms of the data, for a given interval $[nw, (n+1)w]$, is studied in section 3.

If P is a matrix in $\mathcal{C}^{p\times q}$, we denote by $\|P\|$ the 2-norm of P defined in Golub & Van Loan [5, p.14]

2. A SERIES SOLUTION OF THE PROBLEM.

We begin this section considering the differential system

$$X''(t) + A(t)X'(t) + B(t)X(t) = 0. \tag{2.1}$$

Let us suppose that $A(t)$, $B(t)$ are $\mathcal{C}^{r\times r}$ valued analytic functions in $|t| < a$ with $0 < a \leq +\infty$ and

$$A(t) = \sum_{n\geq 0} A_n t^n, \ B(t) = \sum_{n\geq 0} B_n t^n, \ |t| < a, \tag{2.2}$$

where A_n, B_n are matrices in $\mathcal{C}^{r\times r}$. From the Cauchy inequalities, there exists a positive constant L, such that

$$\|A_n\|\rho^n \leq L, \ \|B_n\|\rho^n \leq L, \ 0 < \rho < a, \ n \geq 0. \tag{2.3}$$

Let us look for $\mathcal{C}^{r\times r}$ solutions of (2.1) of the form $X(t) = \sum_{n\geq 0} C_n t^n$, where C_n is a matrix in $\mathcal{C}^{r\times r}$ to be determined. Assuming the convergence of $X(t)$ and of its formal derivatives

$$X'(t) = \sum_{n\geq 0}(n+1)C_{n+1}t^n, \ X''(t) = \sum_{n\geq 0}(n+2)(n+1)C_{n+2}t^n$$

and substituting the expressions into (2.1), it follows that the coefficients C_n must satisfy

$$\sum_{n\geq 0}\left\{(n+2)(n+1)C_{n+2} + \left(\sum_{j=0}^{n}(j+1)A_{n-j}C_{j+1} + B_{n-j}C_j\right)\right\} t^n = 0.$$

Equating to zero the coefficient of each power t^n , one gets

$$(n+2)(n+1)C_{n+2} = -\sum_{j=0}^{n}\left((j+1)A_{n-j}C_{j+1} + B_{n-j}C_j\right), \ n \geq 0 \tag{2.4}$$

where C_0, C_1 are arbitrary matrices in $\mathcal{C}^{r\times r}$. Taking norms in (2.4) and using (2.3), it follows that

$$(n+2)(n+1)\|C_{n+2}\| \leq$$

$$\leq \sum_{j=0}^{n}\left((j+1)\|A_{n-j}\|\|C_{j+1}\| + \|B_{n-j}\|\|C_j\|\right) \leq \sum_{j=0}^{n}\left(\tfrac{(j+1)}{\rho^{n-j}}L\|C_{j+1}\| + \tfrac{L}{\rho^{n-j}}\|C_j\|\right) \leq$$

$$\leq \tfrac{L}{\rho^n}\sum_{j=0}^{n}\left((j+1)\|C_{j+1}\| + \|C_j\|\right)\rho^j + L\|C_{n+1}\|\rho.$$

We have added the last term for the sake of later convenience. Now, let us introduce the sequence of positive numbers $\{\gamma_n\}_{n\geq 0}$ defined by $\gamma_0 = \|C_0\|$, $\gamma_1 = \|C_1\|$, and for $n \geq 0$, γ_{n+2} is defined by the recurrent equation

$$(n+2)(n+1)\gamma_{n+2} = \frac{L}{\rho^n}\sum_{j=0}^{n}\left((j+1)\gamma_{j+1} + \gamma_j\right)\rho^j + L\gamma_{n+1}\rho,\ n \geq 0. \tag{2.5}$$

Hence,

$$\|C_n\| \leq \gamma_n,\ n \geq 0.$$

For $n \geq 1$, we may write (2.5) in the form

$$n(n+1)\gamma_{n+1} = \frac{L}{\rho^{n-1}}\sum_{j=0}^{n-1}\left((j+1)\gamma_{j+1} + \gamma_j\right)\rho^j + L\gamma_n\rho \tag{2.6}$$

and for $n \geq 2$,

$$\rho n(n+1)\gamma_{n+1} =$$

$$= \tfrac{L}{\rho^{n-2}}\sum_{j=0}^{n-2}\left((j+1)\gamma_{j+1} + \gamma_j\right)\rho^j + L\rho(n\gamma_n + \gamma_{n-1}) + L\gamma_n\rho^2 =$$

$$= n(n-1)\gamma_n + Ln\gamma_n\rho + L\gamma_n\rho^2,$$

by virtue of (2.6).
Hence,

$$\gamma_{n+1} = \frac{[n(n-1) + Ln\rho + L\rho^2]\gamma_n}{\rho(n+1)n},\ n \geq 2$$

and

$$\lim_{n\to+\infty}\frac{\gamma_{n+1}|t|^{n+1}}{\gamma_n|t|^n} = \lim_{n\to+\infty}\frac{[n(n-1) + Ln\rho + L\rho^2]\gamma_n}{\rho(n+1)n\gamma_n}|t| = \frac{|t|}{\rho},\ \rho \in (0,a).$$

Thus for any pair of starting matrices C_0 and C_1, the series $X(t) = \sum_{n\geq 0} C_n t^n$ with matrices C_n defined by (2.4) is absolutely convergent in $|t| < a$.

Let us denote by $X_1(t)$ the solution of (2.1) constructed by the above procedure with C_n defined by (2.4) with $C_0 = I$, $C_1 = 0$, and let $X_2(t)$ be defined in the same way with $C_0 = 0$, $C_1 = I$. Then, from Lemma 1 of Jódar & Legua [7] and the definition given in Jódar & Legua [7], the pair $\{X_1(t), X_2(t)\}$ is a fundamental set of solutions of (2.1) in $|t| < a$, and the set of all $\mathcal{C}^r$ solutions of (2.1) in $|t| < a$, is given by

$$X(t) = X_1(t)C + X_2(t)D,\ C, D \in \mathcal{C}^r.$$

Let $H(t)$ be a continuous $\mathcal{C}^r$ function in $|t| < a$, and let us consider the non-homogeneous problem

$$X''(t) + A(t)X'(t) + B(t)X(t) = H(t),\ |t| < a. \tag{2.7}$$

From Lemma 1 of Jódar & Legua [7], the $\mathcal{C}^{2r\times 2r}$ valued function $W(t)$ defined by

$$W(t) = \begin{bmatrix} X_1(t) & X_2(t) \\ X_1'(t) & X_2'(t) \end{bmatrix} \tag{2.8}$$

is invertible in $|t| < a$. Let us denote by

$$W^{-1}(t) = V(t) = \begin{bmatrix} V_{11}(t) & V_{12}(t) \\ V_{21}(t) & V_{22}(t) \end{bmatrix}, \quad V_{ij}(t) \in \mathcal{C}^{r\times r}, \ 1 \le i, j \le 2. \tag{2.9}$$

Let us look for a particular solution of (2.7) of the form

$$Z(t) = X_1(t)D_1(t) + X_2(t)D_2(t),$$

where $\{X_1(t), X_2(t)\}$ is the above fundamental set of solutions of (2.1) and $D_1(t)$, $D_2(t)$ are $\mathcal{C}^r$ valued functions satisfying

$$W(t)\begin{bmatrix} D_1'(t) \\ D_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ H(t) \end{bmatrix}, \quad |t| < a. \tag{2.10}$$

Note that (2.10) means that

$$X_1(t)D_1'(t) + X_2(t)D_2'(t) = 0,$$

$$X_1'(t)D_1'(t) + X_2'(t)D_2'(t) = H(t).$$

Hence $Z(t)$ satisfies

$$Z''(t) + A(t)Z'(t) + B(t)Z(t) =$$

$$= [X_1''(t) + A(t)X_1' + B(t)X_1(t)]D_1(t) +$$

$$+ [X_2''(t) + A(t)X_2' + B(t)X_2(t)]D_2(t) + H(t) = H(t),$$

$$Z(0) = D_1(0), \ Z'(0) = D_2(0).$$

A solution of (2.10) for $D_1(t)$, $D_2(t)$, satisfying $D_1(0) = D_2(0) = 0$ is given by

$$D_1(t) = \int_0^t V_{12}(x)H(x)dx, \ D_2(t) = \int_0^t V_{22}(x)H(x)dx.$$

Since (2.7) is linear, its general solution is given by

$$\begin{aligned} X(t) &= X_1(t)C_1 + X_2(t)C_2 + Z(t) = \\ &= X_1(t)\left[C_1 + \textstyle\int_0^t V_{12}(x)H(x)dx\right] + X_2(t)\left[C_2 + \textstyle\int_0^t V_{22}(x)H(x)dx\right] \end{aligned} \tag{2.11}$$

where C_1, C_2 are arbitrary vectors in $\mathcal{C}^r$. Given fixed initial conditions

$$X(0) = G_1, \ X_0' = G_2, \tag{2.12}$$

the unique solution of (2.7) satisfying (2.12) is given by (2.11), where the vectors C_1, C_2 must satisfy $G_1 = X(0) = C_1$ and $G_2 = X'(0) = C_2$. Thus the following result has been established.

THEOREM 1. Let us consider equation (2.7) where $A(t)$, $B(t)$ are analytic $\mathcal{C}^{r\times r}$ valued functions with power expansions defined by (2.2) and let $H(t)$ be a continuous vector function. Let $\{X_1(t), X_2(t)\}$ be the pair of $\mathcal{C}^{r\times r}$ analytic solutions of (2.1) satisfying $X_1(0) = I$, $X_1'(0) = 0$, $X_2(0) = 0$, $X_2'(0) = I$, and let $W(t)$ and $V(t)$ be the $\mathcal{C}^{2r\times 2r}$ valued functions defined by (2.8) and (2.9) respectively. Then, the unique solution of the initial value problem (2.7), (2.12), is given by (2.11) where $C_1 = G_1$, $C_2 = G_2$.

Let us consider the delay differential system (1.1) written in the form

$$\left.\begin{aligned} X''(t) + A(t)X'(t) + B(t)X(t) &= F(t) - B_1(t)X'(t-w),\ t > 0 \\ X(t) &= G(t),\ t \in [-w, 0] \end{aligned}\right\} \tag{2.13}$$

where $A(t)$, $B(t)$ are analytic $\mathcal{C}^{r\times r}$ valued functions on the real line, and note that for $t \in [0, w]$, the right-hand side of (2.13) is a known continuous function. Let $\{X_1(t), X_2(t)\}$ be the fundamental set of solutions of (2.1) on the real line provided by Theorem 1, let $W(t)$ and $V(t)$ be defined by Theorem 1 and let us introduce the matrix kernel $K : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{C}^{r\times r}$, by the expression

$$K(t, x) = X_1(t)V_{12}(x) + X_2(t)V_{22}(x), \tag{2.14}$$

and note that from the definition of $V(t)$ and $W(t)$ one gets $K(t, t) = 0$ for $t \in \mathcal{R}$.

Let $H(x)$ be a continuous $\mathcal{C}^r$ valued function and let $K(t, x)$ be defined by (2.14), then we introduce the operator $\mathcal{W}$ defined by

$$(\mathcal{W}H)(t) = \int_0^t K(t, x)H(x)dx,\ t \geq 0. \tag{2.15}$$

It is clear that $\mathcal{W}$ is a linear operator, i.e.,

$$\mathcal{W}(\alpha H + \beta J) = \alpha \mathcal{W}H + \beta \mathcal{W}J.$$

Here and below, when possible, we will drop the dependence on t for the sake of brevity.

From Theorem 1, the solution of (2.13) can be written in the form,

$$X = \mathcal{X} + \mathcal{W}(F - B_1X'_{-w}) = \mathcal{X} + \mathcal{W}F - \mathcal{W}(B_1X'_{-w}), \tag{2.16}$$

where

$$X'_{-w}(x) = X'(x - w)$$

and

$$\mathcal{X}(t) = X_1(t)G(0) + X_2(t)G'(0),\ t \geq 0. \tag{2.17}$$

Note that (2.16) is a feedback expression that provides the solution of (1.1) in an interval of length w, in terms of the solution in the previous intervals of length w. In order to find a closed form expression for the solution of (1.1) in any interval $[nw, (n+1)w]$, we introduce a recurrent sequence of integral operators. If H is a continuously differentiable function in $[-w, +\infty[$, we define for a positive integer k,

$$\mathcal{W}_1(B_1H) = \mathcal{W}(B_1H'), \tag{2.18}$$

and for $k > 1$,

$$\begin{aligned} (\mathcal{W}_k(B_1H))(t) &= \mathcal{W}(B_1(\mathcal{W}'_{k-1}(B_1H))_{-w})(t) = \\ &= \textstyle\int_0^t K(t, x)B_1(x)(\mathcal{W}'_{k-1}(B_1H))(x - w)dx. \end{aligned} \tag{2.19}$$

As above, $(\mathcal{W}'_{k-1}(B_1H))_{-w}(t) = \frac{\partial}{\partial t}(\mathcal{W}_{k-1}(B_1H))(t - w)$. Now, we prove that the solution of (1.1) can be written in the compact form

$$X(t) = G(t),\ t \in [-w, 0]$$

and for $t \in [nw, (n+1)w]$

$$\begin{aligned} X &= \mathcal{X} + \mathcal{W}F + \textstyle\sum_{k=1}^n (-1)^k \mathcal{W}_k(B_1(\mathcal{W}F)_{-w}) + \\ &+ \textstyle\sum_{k=1}^n (-1)^k \mathcal{W}_k(B_1\mathcal{X}_{-w}) + (-1)^{n+1}\mathcal{W}_{n+1}(B_1G_{-w}). \end{aligned} \tag{2.20}$$

Indeed, if $t \in [0, w]$, from (2.16) and (2.18) it follows that

$$X = \mathcal{X} + \mathcal{W}(F - B_1 G''_{-w}) = \mathcal{X} + \mathcal{W}F - \mathcal{W}_1(B_1 G_{-w}).$$

Thus (2.20) holds for $n = 0$. Let us suppose that (2.20) is true for $t \in [nw, (n+1)w]$ and let us take $t \in [(n+1)w, (n+2)w]$. From (2.16) and the induction hypothesis, we can write

$$X = \mathcal{X} + \mathcal{W}(F - B_1 X'_{-w}) =$$

$$= \mathcal{X} + \mathcal{W}\Big(F - B_1 \tfrac{\partial}{\partial t}\Big[\mathcal{X}_{-w} + (\mathcal{W}F)_{-w} + \textstyle\sum_{k=1}^{n}(-1)^k(\mathcal{W}_k(B_1(\mathcal{W}F)_{-w}))_{-w} +$$

$$+ \textstyle\sum_{k=1}^{n}(-1)^k(\mathcal{W}_k(B_1\mathcal{X}_{-w}))_{-w} + (-1)^{n+1}(\mathcal{W}_{n+1}(B_1 G_{-w}))_{-w}\Big]\Big) =$$

$$= \mathcal{X} + \mathcal{W}F - \mathcal{W}(B_1\mathcal{X}'_{-w}) - \mathcal{W}(B_1(\mathcal{W}F)'_{-w}) + \textstyle\sum_{k=1}^{n}(-1)^{k+1}\mathcal{W}(B_1(\mathcal{W}'_k(B_1(\mathcal{W}F)_{-w}))_{-w}) +$$

$$+ \textstyle\sum_{k=1}^{n}(-1)^{k+1}\mathcal{W}(B_1(\mathcal{W}'_k(B_1\mathcal{X}_{-w}))_{-w}) + (-1)^{n+2}\mathcal{W}(B_1(\mathcal{W}'_{n+1}(B_1 G_{-w}))_{-w}) =$$

$$\begin{aligned} &= \mathcal{X} + \mathcal{W}F + \textstyle\sum_{k=1}^{n+1}(-1)^k\mathcal{W}_k(B_1(\mathcal{W}F)_{-w}) + \\ &+ \textstyle\sum_{k=1}^{n+1}(-1)^k\mathcal{W}_k(B_1\mathcal{X}_{-w}) + (-1)^{n+2}\mathcal{W}_{n+2}(B_1 G_{-w}). \end{aligned} \tag{2.21}$$

Note that (2.21) coincides with (2.20) replacing n by $n+1$. Thus the following result has been proved.

THEOREM 2. Let us consider the problem (1.1) under the hypothesis of Theorem 1. Let $\{\mathcal{W}_k\}_{k\geq 1}$ be the sequence of operators defined by (2.18) and (2.19) where $B_1(t)$ is a $\mathcal{C}^{r\times r}$ valued continuous function and let $K(t,x)$ be defined by (2.14). If $\mathcal{X}(t)$ is defined by (2.17), then the exact solution of (1.1) in the interval $[nw, (n+1)w]$, for $n \geq 0$, is given by (2.20).

REMARK. It is easy to show that the integral operators $\mathcal{W}_k$ defined by (2.18), (2.19) can be written in terms of the data in the form

$$\begin{aligned} &(\mathcal{W}_k(B_1 H))(t) = \\ &= \textstyle\int_0^t \int_0^{t_n - w} \cdots \int_0^{t_{p+1}-w} \Big[K(t,t_n)B_1(t_n)\frac{\partial}{\partial t_n}K(t_n - w, t_{n-1})\cdots \\ &\cdots B_1(t_{p+1})\frac{\partial}{\partial t_{p+1}}K(t_{p+1} - w, t_p)B_1(t_p)H(t_p)\Big]\, dt_p dt_{p+1}\cdots dt_n, \end{aligned} \tag{2.22}$$

where $p = n - k + 1$.

From a computational point of view, the solution provided by Theorem 2 has the drawback that the expression of $K(t,x)$ and $\mathcal{X}(t)$ is given in terms of infinite series involving $X_1(t)$, $X_2(t)$ and the block entries of $V(t)$ defined by Theorem 1. In the sequel we construct finite approximate solutions of (1.1) obtained by truncation of the quoted infinite series.

3. FINITE ANALYTIC APPROXIMATE SOLUTIONS AND ERROR BOUNDS.

Let $X_1(t)$ and $X_2(t)$ be the pair of $\mathcal{C}^{r\times r}$ analytic series solutions of (2.1) given by Theorem 1, let $\imath \geq 1$ and let

$$X_1(t) = \sum_{n=0}^{+\infty} C_n t^n,\ X_2(t) = \sum_{n=0}^{+\infty} D_n t^n,\ 0 < t < +\infty,$$

$$X_{1\imath}(t) = \sum_{n=0}^{\imath} C_n t^n,\ X_{2\imath}(t) = \sum_{n=0}^{\imath} D_n t^n,$$

$$X_i(t) = X_{1i}(t)G(0) + X_{2i}(t)G'(0), \tag{3.1}$$

$$K_i(t,x) = X_{1i}(t)V_{12}^i(x) + X_{2i}(t)V_{22}^i(t), \tag{3.2}$$

where $V_{12}^i(x)$ and $V_{22}^i(x)$ denote the block entries of the inverse of the i-th partial sum of $W(x)$. (See below (3.9)-(3.12)).

Note that from Lemma 1 of Jódar & Legua [7], the matrix function $W(t)$ defined by (2.8) satisfies

$$W'(t) = C(t)W(t),\ W(0) = I_{2r},\ C(t) = \begin{bmatrix} 0 & I \\ -B(t) & -A(t) \end{bmatrix}.$$

Let S_q be the positive constant

$$S_q = exp\left(\int_0^{(q+1)w} \|C(x)\|dx\right).$$

From Flett [4, p.114], it follows that

$$\|W(t)\| \le S_q,\ \|W^{-1}(t)\| = \|V(t)\| \le S_q,\ t \in [0,(q+1)w] \tag{3.3}$$

and

$$max\{\|X_1(t)\|, \|X_1'(t)\|, \|X_2(t)\|, \|X_2'(t)\|, \|V_{12}(t)\|, \|V_{22}(t)\|\} \le S_q, \quad t \in [0,(q+1)w]. \tag{3.4}$$

From the Cauchy inequalities, it follows that

$$\|C_n\| \le S_q[z(q+1)w]^{-1},\ \|D_n\| \le S_q[z(q+1)w]^{-1},\ n \ge 1,\ t \in [0,(q+1)w]$$

z and r be the positive constants defined by

$$r = z(q+1)w,\ z > 1.$$

Then, for $0 \le t \le (q+1)w$, it follows that

$$\begin{aligned} \|X_1(t) - X_{1i}(t)\| &= \|\textstyle\sum_{n=i+1}^{+\infty} C_n t^n\| \le \sum_{n=i+1}^{+\infty} \|C_n\| |t|^n \le \\ &\le S_q \textstyle\sum_{n=i+1}^{+\infty} |t|^n [z(q+1)w]^{-n} = S_q \frac{[(q+1)w]^{i+1}}{[r-(q+1)w]r^i} = \frac{S_q}{z^i(z-1)} = E_{iq}. \end{aligned} \tag{3.5}$$

Analogously,

$$\|X_2(t) - X_{2i}(t)\| \le \frac{S_q}{z^i(z-1)} = E_{iq}. \tag{3.6}$$

$$\|X_1'(t) - X_{1i}'(t)\| = \|\textstyle\sum_{n=i}^{+\infty} nC_n t^{n-1}\| \le \sum_{n=i}^{+\infty} n\|C_n\| |t|^{n-1} \le$$

$$\le \textstyle\sum_{n=i}^{+\infty} n\frac{S_q}{r^n}|t|^{n-1} = \frac{S_q}{r}\sum_{n=i}^{+\infty} n\left(\frac{|t|}{r}\right)^{n-1} \le \frac{S_q}{r}\sum_{n=i}^{+\infty} n\left(\frac{(q+1)w}{r}\right)^{n-1} \le$$

$$\le \frac{S_q}{r}\left[\frac{iz}{(z-1)z^{i-1}} + \frac{z^2}{(z-1)^2 z^i}\right] = \frac{S_q(z-1)(i+z-1)}{(q+1)wz^{i-1}} = D_{iq},$$

$$t \in [0,(q+1)w],$$

$$\|X_2'(t) - X_{2i}'(t)\| \le D_{iq},\ t \in [0,(q+1)w],$$

where

$$D_{iq} = E_{iq}\left[\frac{z(z-1)^2(i+z-1)}{(q+1)w}\right] = \frac{S_q(z-1)(i+z-1)}{(q+1)wz^{i-1}}. \tag{3.7}$$

From (2.17), (3.1), (3.4), (3.5) and (3.6), it follows that

$$\|X(t) - X_i(t)\| \leq (\|G(0)\| + \|G'(0)\|)E_{iq},\ t \in [0,(q+1)w].$$

$$\|X(t)\| \leq (\|G(0)\| + \|G'(0)\|)S_q,\ t \in [0,(q+1)w].$$

Since $\{E_{iq}\}$ and $\{D_{iq}\}$ defined by (3.5) and (3.7) respectively, converge to zero as $i \to +\infty$, let us choose i_0 as the first positive integer satisfying

$$E_{iq} + D_{iq} < (2S_q)^{-1},\ i \geq i_0. \tag{3.8}$$

If we denote by $W_i(t)$ the i-th partial sum of $W(t)$, from the perturbation Lemma Ortega [14, p.32] and the inequality

$$\|W(t) - W_i(t)\| \leq 2(E_{iq} + D_{iq}) < S_q^{-1} < \|W^{-1}(t)\|^{-1}, \tag{3.9}$$

it follows that $W_i(t)$ is invertible and from the Banach Lemma Ortega [14, p.32]

$$\|W^{-1}(t)\| \leq S_q[1 - 2S_q(E_{iq} + D_{iq})] = M_{iq}, \tag{3.10}$$

$$\|W^{-1}(t) - W_i^{-1}(t)\| \leq 2S_q(E_{iq} + D_{iq})M_{iq},\ t \in [0,(q+1)w], \tag{3.11}$$

where

$$W_i^{-1}(t) = \begin{bmatrix} V_{11}^i(t) & V_{12}^i(t) \\ V_{21}^i(t) & V_{22}^i(t) \end{bmatrix}. \tag{3.12}$$

Thus the approximate kernel $K_i(t,x)$ given by (3.2) for $i \geq i_0$, is well defined for $0 \leq t \leq (q+1)w$, in the sense that $W_i(t)$ is invertible for $i \geq i_0$. Then, we can define

$$(\mathcal{W}^i H)(t) = \int_0^t K_i(t,x)H(x)dx,\ t \geq 0,\ i \geq i_0, \tag{3.13}$$

$$\mathcal{W}_{1i}(B_1 H) = \mathcal{W}^i(B_1 H'),$$

$$(\mathcal{W}_{ki}(B_1 H))(t) = \mathcal{W}^i(B_1(\mathcal{W}'_{k-1,i}(B_1 H))_{-w})(t) =$$

$$= \textstyle\int_0^t K_i(t,x)B_1(x)(\mathcal{W}'_{k-1,i}(B_1 H))(x-w)dx,\ k > 1.$$

In accordance with (2.22) we can write for $k = n - p + 1$

$$(\mathcal{W}_{k,i}(B_1 H))(t) =$$

$$= \textstyle\int_0^t \int_0^{t_n - w} \cdots \int_0^{t_{p+1}-w} \left[K_i(t,t_n)B_1(t_n)\frac{\partial}{\partial t_n}K_i(t_n - w, t_{n-1}) \cdots \right. \tag{3.14}$$

$$\left. \cdots B_1(t_{p+1})\frac{\partial}{\partial t_{p+1}}K_i(t_{p+1} - w, t_p)B_1(t_p)H(t_p)\right] dt_p dt_{p+1} \cdots dt_n.$$

Note that from (3.3), (3.9) and the triangular inequality, it follows that

$$\|W_i(t)\| \leq \|W(t)\| + \|W(t) - W_i(t)\| \leq S_q + S_q^{-1},$$
$$t \in [0,(q+1)w],\ i \geq i_0. \tag{3.15}$$

From (3.15) and taking into account that

$$W_i(t) = \begin{bmatrix} X_{1i}(t) & X_{2i}(t) \\ X'_{1i}(t) & X'_{2i}(t) \end{bmatrix}$$

if $i \geq i_0$, $0 \leq t \leq (q+1)w$, it follows that

$$max\{\|X_{1i}(t)\|, \|X'_{1i}(t)\|, \|X_{2i}(t)\|, \|X'_{2i}(t)\|\} \leq S_q + S_q^{-1}. \tag{3.16}$$

Otherwise, note that from the definition of $K(t,x)$ and $K_i(t,x)$ given by (2.11) and (3.2) respectively, and from (3.4) and (3.16), it follows that

$$\begin{aligned}
&\|X_i(t)\| \leq (\|G(0)\| + \|G'(0)\|)[S_q + S_q^{-1}],\\
&\|K(t,x) - K_i(t,x)\| \leq \gamma_{iq} = 4S_q^2(E_{iq} + D_{iq})M_{iq} + 2E_{iq}M_{iq},\\
&\|K(t,x)\| \leq 2S_q^2,\ \|K_i(t,x)\| \leq 2M_{iq}[S_q + S_q^{-1}],\\
&\|\tfrac{\partial}{\partial t}K(t,x)\| \leq 2S_q^2,\ \|\tfrac{\partial}{\partial t}K_i(t,x)\| \leq 2M_{iq}(S_q + S_q^{-1}),\\
&\|\tfrac{\partial}{\partial t}K(t,x) - \tfrac{\partial}{\partial t}K_i(t,x)\| \leq \beta_{iq} = 4S_q^2(E_{iq} + D_{iq})M_{iq} + 2D_{iq}M_{iq},\\
&t \in [0,(q+1)w],\ x \in [0,(q+1)w],\ i \geq i_0.
\end{aligned} \tag{3.17}$$

Let us introduce the constants g, f_q , b_q defined by

$$max\{\|G(t)\|,\ -w \leq t \leq 0\} = g,\ max\{\|F(t)\|,\ 0 \leq t \leq (q+1)w\} = f_q,$$

$$max\{\|B_1(t)\|,\ 0 \leq t \leq (q+1)w\} = b_q.$$

From Gradshteyn [6, p.620], it follows that

$$\int_0^{t-w}\int_0^{t_n-w}\cdots\int_0^{t_{p+1}-w} dt_p dt_{p+1}\cdots dt_n = \frac{(t-w)[t-(n-p+2)w]^{n-p}}{(n-p+1)!}. \tag{3.18}$$

In particular, for $t = (n+3)w$ we have

$$\int_0^{(n+2)w}\int_0^{t_n-w}\cdots\int_0^{t_{p+1}-w} dt_p dt_{p+1}\cdots dt_n = \frac{(n+2)(p+1)w^{n-p+1}}{(n-p+1)!} = I_{np}. \tag{3.19}$$

From (2.22) and (3.14) if $k = n-p+1$, $t = t_{n+1}$ and $p \in \mathcal{N}$, we can write

$$\begin{aligned}
&(\mathcal{W}_k(B_1H))(t) - (\mathcal{W}_{ki}(B_1T))(t) =\\
&= \textstyle\int_0^t \int_0^{t_n-w}\cdots\int_0^{t_{p+1}-w} \Big(\{[K(t,t_n) - K_i(t,t_n)]\,B_1(t_n)\\
&\tfrac{\partial}{\partial t_n}K_i(t_n-w,t_{n-1})\cdots B_1(t_{p+1})\tfrac{\partial}{\partial t_{p+1}}K_i(t_{p+1}-w,t_p)B_1(t_p)T(t_p)\Big\} +\\
&+\textstyle\sum_{j=p}^n \Big\{K(t,t_n)B_1(t_n)\tfrac{\partial}{\partial t_n}K(t_n-w,t_{n-1})\cdots\\
&\cdots B_1(t_j)\left[\tfrac{\partial}{\partial t_j}K(t_j-w,t_{j-1}) - \tfrac{\partial}{\partial t_j}K_i(t_j-w,t_{j-1})\right]B_1(t_{j-1})\cdots\\
&\cdots B_1(t_{p+1})\tfrac{\partial}{\partial t_{p+1}}K_i(t_{p+1}-w,t_p)B_1(t_p)T(t_p)\Big\} +\\
&+\Big\{K(t,t_n)B_1(t_n)\tfrac{\partial}{\partial t_n}K(t_n-w,t_{n-1})\cdots B_1(t_{p+1})\tfrac{\partial}{\partial t_{p+1}}K(t_{p+1}-w,t_p)\\
&B_1(t_p)\,[H(t_p) - T(t_p)]\}\Big)\,dt_p dt_{p+1}\cdots dt_n.
\end{aligned} \tag{3.20}$$

If we denote by α_1 and α_2 the constants

$$\alpha_1 = max\{\|T(t_p)\|,\ pw \leq t_p \leq (p+1)w\},$$

$$\alpha_2 = max\{\|H(t_p) - T(t_p)\|,\ pw \leq t_p \leq (p+1)w\},$$

from (3.18)-(3.20) it follows that

$$\begin{gathered} \|\mathcal{W}_k(B_1 H) - \mathcal{W}_{k,\imath}(B_1 T)\| \leq \\ \leq 2^{n-p+1} I_{np} \left(\prod_{h=p}^{n} b_h\right) \left[\alpha_1 \gamma_{\imath,n+1} \left(\prod_{h=p}^{n} (S_h + S_h^{-1}) M_{\imath h}\right) + \right. \\ \left. + \alpha_2 \left(\prod_{h=p}^{n} S_h^2\right) + \alpha_1 \sum_{j=p}^{n} \beta_{\imath j} \left(\prod_{h=j+1}^{n} S_h^2\right)\right) \left(\prod_{h=p}^{j} (S_h + S_h^{-1}) M_{\imath h}\right)\right]. \end{gathered} \tag{3.21}$$

If $p = 0$ and $H = T = G_{-w}$, the expression (3.21) takes the form

$$\|\mathcal{W}_{n+1}(B_1 G_{-w}) - \mathcal{W}_{n+1,\imath}(B_1 G_{-w})\| \leq \rho_{n,\imath}$$

where

$$\begin{gathered} \rho_{n,\imath} = 2^{n+1} I_{n0}\, g \left(\prod_{h=0}^{n} b_h\right) \left[\gamma_{\imath,n+1} \left(\prod_{h=0}^{n} (S_h + S_h^{-1}) M_{\imath h}\right) + \right. \\ \left. + \sum_{j=0}^{n} \beta_{\imath j} \left(\prod_{h=j+1}^{n} S_h^2\right)\right) \left(\prod_{h=0}^{j} (S_h + S_h^{-1}) M_{\imath h}\right)\right]. \end{gathered}$$

If we consider $H = \mathcal{X}_{-w}$, $T = \mathcal{X}^{\imath}_{-w}$, from (2.17), (3.1) and (3.5) for $t \in [pw, (p+1)w]$, it follows that

$$\|\mathcal{X}_{-w} - \mathcal{X}_{\imath,-w}\| \leq (\|G(0)\| + \|G'(0)\|) E_{\imath,p-1},$$

and from (3.21) we can write

$$\|\mathcal{W}_k(B_1 \mathcal{X}_{-w}) - \mathcal{W}_{k,\imath}(B_1 \mathcal{X}_{\imath,-w})\| \leq Y_{n,\imath}$$

where

$$\begin{gathered} Y_{n,\imath} = 2^{n-p} I_{np} (\|G(0)\| + \|G'(0)\|) \left(\prod_{h=p}^{n} b_h\right) \\ \left[(S_p + S_p^{-1}) \gamma_{\imath,n+1} \left(\prod_{h=p}^{n} (S_h + S_h^{-1}) M_{\imath h}\right) + E_{\imath,p-1} \left(\prod_{h=p}^{n} S_h^2\right) + \right. \\ \left. + (S_p + S_p^{-1}) \sum_{j=p}^{n} \beta_{\imath j} \left(\prod_{h=j+1}^{n} S_h^2\right)\right) \left(\prod_{h=p}^{j} (S_h + S_h^{-1}) M_{\imath h}\right)\right]. \end{gathered}$$

Taking $H = (\mathcal{W}F)_{-w}$, $T = (\mathcal{W}^{\imath}F)_{-w}$ and using that $t_p \in [pw, (p+1)w]$, from (3.13) and (3.17), it follows that

$$\begin{gathered} \|(\mathcal{W}F)_{-w} - (\mathcal{W}^{\imath}F)_{-w}\| \leq \\ \leq \int_0^{t-w} \|K(t-w,x) - K_\imath(t-w,x)\| \|F(x)\| dx \leq \\ \leq \int_0^{t-w} \gamma_{\imath,p-1} f_{p-1} dx \leq pw \gamma_{\imath,p-1} f_{p-1}, \end{gathered}$$

and from (3.21) and (3.17),

$$\begin{gathered} \|(\mathcal{W}^{\imath}F)_{-w}\| \leq \int_0^{t-w} \|K_\imath(t-w,x)\| \|F(x)\| dx \leq \\ \leq pw (S_{p-1} + S_{p-1}^{-1}) M_{\imath,p-1} f_{p-1}. \end{gathered} \tag{3.22}$$

From (3.21) and (3.22) we can write

$$\|\mathcal{W}_n(B_1(\mathcal{W}F)_{-w}) - \mathcal{W}_{n,\imath}(B_1(\mathcal{W}^{\imath}F)_{-w})\| \leq U_{n,\imath},$$

where

$$U_{n,i} = 2^{n-p+1} l_{np} p w f_{p-1} \left(\prod_{h=p}^{n} b_h\right)$$

$$\Big[(S_{p-1} + S_{p-1}^{-1}) M_{i,p-1} \gamma_{i,n+1} \left(\prod_{h=p}^{n} (S_h + S_h^{-1}) M_{ih}\right) + \gamma_{i,p-1} \left(\prod_{h=p}^{n} S_h^2\right) +$$

$$+ (S_{p-1} + S_{p-1}^{-1}) M_{i,p-1} \sum_{j=p}^{n} \beta_{ij} \left(\prod_{h=j+1}^{n} S_h^2\right)\right) \left(\prod_{h=p}^{j} (S_h + S_h^{-1}) M_{ih}\right)\Big].$$

If we denote by $X_i(t)$ for $i \geq i_0$ the approximate solution of (1.1) defined in $[nw, (n+1)w]$ by

$$\begin{aligned} X_i &= X_i' + W^i F + \sum_{k=1}^{n} (-1)^k W_{ki} (B_1 (W^i F)_{-w}) + \\ &+ \sum_{k=1}^{n} (-1)^k W_{ki} (B_1 (X_i')_{-w}) + (-1)^{n+1} W_{n+1,i} (B_1 G_{-w}), \end{aligned} \tag{3.23}$$

from (2.20), (3.23) and the previous comments of Section 3, it follows that the error $X(t) - X_i(t)$ of the approximation $X_i(t)$ with respect to the exact solution $X(t)$ of (1.1), for $i \geq i_0$ and $t \in [nw, (n+1)w]$ is bounded above by

$$\begin{aligned} &\|X(t) - X_i(t)\| \leq \\ &\leq (\|G(0)\| + \|G''(0)\|) E_{i,n} + (n+1) w f_n \gamma_{i,n} + \rho_{i,n} + \sum_{k=1}^{n} (U_{k,i} + Y_{k,i}). \end{aligned} \tag{3.24}$$

Thus, for a fixed interval $[n_0 w, (n_0+1)w]$ and an admissible error ϵ, to construct a finite analytic approximate solution whose error be smaller than ϵ in $[n_0 w, (n_0+1)w]$ it is sufficient to take $i \geq i_0$ such that

$$(\|G(0)\| + \|G'(0)\|) E_{i,n} + (n+1) w f_n \gamma_{i,n} + \rho_{i,n} + \sum_{k=1}^{n} (U_{k,i} + Y_{k,i}) < \epsilon. \tag{3.25}$$

Hence the following result has been proved.

THEOREM 3. Let us consider the problem (1.1) under the hypotheses of Theorem 2 and let us use the previous notation. If i_0 is the first positive integer i satisfying (3.8) and $X_i(t)$ is the function defined by (3.23) for $i \geq i_0$, then $X_i(t)$ converges to the exact solution $X(t)$ of (1.1) as $i \to +\infty$, for any $t > 0$. If n_0 is a positive integer, then the error of the approximate solution $X_i(t)$ with respect to the exact solution $X(t)$ satisfies (3.24) for $t \in [n_0 w, (n_0+1)w]$ and $i \geq i_0$. Given an admissible error $\epsilon > 0$, taking $i \geq i_0$ satisfying (3.25), one gets an approximation whose error is bounded above by ϵ for $t \in [n_0 w, (n_0+1)w]$.

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