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A NOTE ON FINITE CODIMENSIONAL LINEAR ISOMETRIES OF *C(X)* INTO *C(Y)*

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ABSTRACT. Let (X,Y) be a pair of compact Hausdorff spaces. It is shown that a certain property of the class of continuous maps of Y onto X is equivalent to the non-existence of linear isometry of $C(X)$ into $C(Y)$ whose range has finite codimension > 0.

KEY WORDS AND PHRASES. Compact Hausdorff space, $C(X)$, linear isometry, finite codimension
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1. INTRODUCTION

In [1], A. Gutek, D. Hart, J. Jamison and M. Rajagopalan proved that there are no isometric shift operators on $C([a,b])$, a result first proved in the real scalars by Holub [3]. Here $[a,b]$ is any closed interval in the real line and $C([a,b])$ is the Banach space of all continuous complex-valued functions on $[a,b]$. By observing carefully the proof given in [1], one can note that $C([a,b])$ does not admit an isometric shift operator because the space $[a,b]$ has the property that the set

$$\{(x,y) \in [a,b] \times [a,b] : \phi(x) = \phi(y), x \neq y\}$$

is infinite for every continuous map ϕ of $[a,b]$ onto itself which is not injective.

The purpose of the note is to prove the following theorem which is based on the above idea:

THEOREM. *Let (X,Y) be a pair of compact Hausdorff spaces. Then the following two conditions are equivalent:*

(i) If there is a continuous map ϕ of Y onto X which is not injective, then the set

$$\{(y_1,y_2) \in Y \times Y : \phi(y_1) = \phi(y_2), y_1 \neq y_2\}$$

is infinite.

(ii) If there is a linear isometry of $C(X)$ into $C(Y)$ which has a finite codimension, then it is surjective.

Since both $([0,1],[0,1])$ and (T^1,T^1) satisfy the condition (i), where T^1 is the unit circle in the complex plane, we get from this

COROLLARY 1. *The only possible codimension of linear isometries $C([0,1]) \to C([0,1])$ and $C(T^1) \to C(T^1)$ are zero or infinite.*

Moreover, if V is the canonical linear map of $C(T^1)$ into $C([0,1])$ defined by

$$(Vf)(t) = f(e^{2\pi it}) \quad (f \in C(T^1), 0 \leq t \leq 1),$$

then V is an isometry and the range of V is the set of all $g \in C([0,1])$ such that $g(0) = g(1)$ Hence V has codimension 1, and if there is a finite codimensional linear isometry of $C([0,1])$ into $C(T^1)$, say T, then VT is a linear isometry of $C([0,1])$ into itself such that $VT(C([0,1])) \subsetneq C([0,1])$ and codim$(T)+1$ From Corollary 1 it follows that VT must be surjective, a contradiction hence we have also proved

COROLLARY 2. *There is no finite codimensional linear isometry of $C([0,1])$ into $C(T^1)$*

2. LEMMAS

In order to prove the main theorem, we have to prepare some lemmas

LEMMA 1. *Let X be a compact Hausdorff space, M a subspace of $C(X)$ whose codimension is $n < +\infty$, and K a closed boundary of X with respect to M (i.e., for any $f \in M$ there exists a point x in K with $|f(x)| = \|f\|_{\infty}$, the supremum norm of f on X). Then the set $X \backslash K$ has at most n points.*

PROOF. Assume that $X\backslash K$ has at least $n+1$ points, say $x_1, ..., x_{n+1}$ For each $1 \leq i \leq n+1$, choose a function f_i in $C(X)$ such that $f_i(x_i) = 1$ and $f_i(x) = 0$ for $x \in K \cup \{x_1, ..., x_{n+1}\}\backslash\{x_i\}$ since K is closed In this case, $\{f_1 + M, .., f_{n+1} + M\}$ is linearly independent in $C(X)/M$ since if

$$c_1(f_1 + M) + ... + c_{n+1}(f_{n+1} + M) = 0$$

for some complex numbers $c_1, ..., c_{n+1}$ there exists a function $g \in M$ such that $c_1 f_1 + ... + c_{n+1} f_{n+1} + g = 0$ and (since K is a boundary of X with respect to M) a point x_0 in K such that $\|g\|_{\infty} = |g(x_0)|$ Then

$$\|g\|_{\infty} = |c_1 f_1(x_0) + \cdots + c_{n+1} f_{n+1}(x_0)| = 0,$$

implying $c_1 = 0, ..., c_{n+1} = 0$ since $\{f_1, .., f_{n+1}\}$ is linearly independent, and it follows that codim$(M) \geq n+1$

LEMMA 2. *Let X and Y be compact Hausdorff spaces and ϕ a continuous map of Y onto X. If g is a function in $C(Y)$ such that $g(y_1) = g(y_2)$ for all pairs $(y_1, y_2) \in Y \times Y$ satisfying $\phi(y_1) = \phi(y_2)$, then there is a function f in $C(X)$ such that $f(\phi(y)) = g(y)$ for all $y \in Y$.*

PROOF. Let g be a function in $C(Y)$ such that $g(y_1) = g(y_2)$ for all pairs $(y_1, y_2) \in Y \times Y$ satisfying $\phi(y_1) = \phi(y_2)$ Let Y/ϕ be the quotient space of Y defined by ϕ, π_ϕ the canonical map of Y onto Y/ϕ, and τ the canonical map of Y/ϕ onto X Then the complex-valued function $\tilde{g}$ on Y/ϕ defined by $\tilde{g}(\tilde{y}) = g(y)$ for each $\tilde{y} \in Y/\phi$ is continuous, so setting $f = \tilde{g} \circ \tau^{-1}$ it is easy to see that f is a function with the desired properties

Finally, we will need the following result whose proof is straightforward

LEMMA 3. *Let X be a compact Hausdorff space, K a compact subset of X, and A_K the Banach subspace of $C(X)$ consisting of all $f \in C(X)$ which are constant on K. Then the Banach space $C(X)/A_K$ is isomorphic to a quotient space of $C(K)$.*

3. PROOF OF THEOREM

(i) $\Rightarrow$ (ii) Let T be a linear isometry of $C(X)$ into $C(Y)$ which has a finite codimension By the decomposition theorem of Holsztynski [2], there exists a closed boundary K of Y with respect to $T(C(X))$, a continuous map h of K onto X, and a continuous unimodular function u on Y such that

$$(Tf)(y) = u(y)f(h(y))$$

for all $f \in C(X)$ and $y \in K$ Since T has a finite codimension, it follows from Lemma 1 that K is a closed subset of Y whose complement is a finite set. Then h has a continuous extension to Y, say $\tilde{h}$. We claim that the map $\tilde{h}$ is injective Assume the contrary Then by the condition (i) there is a mutually different sequence $\{\alpha_1, \beta_1, \alpha_2, \beta_2, ...\}$ in Y such that $\tilde{h}(\alpha_n) = \tilde{h}(\beta_n)$ for all positive integers n, and where we can assume without loss of generality that $\{\alpha_1, \beta_1, \alpha_2, \beta_2, ...\} \subset K$. Let n be any positive integer, and for each $1 \leq i \leq n$ choose a function g_i in $C(Y)$ such that $g_i(\alpha_i) = 1$ and $g_i(y) = 0$ for all $y \in Y \backslash U_i$, where U_i is a sufficiently small neighborhood of α_i. In this case $\{g_1 + T(C(X)), ...g_n + T(C(X))\}$ is linearly independent in $C(Y)/T(C(X))$, since if

$$c_1(g_1 + T(C(X))) + ... + c_n(g_n + T(C(X))) = 0$$

for some complex numbers $c_1, ..., c_n$ there exists $f \in C(X)$ such that $c_1 g_1 + ... + c_n g_n = Tf$, implying

$$\begin{aligned} c_i &= c_1 g_1(\alpha_i) + ... + c_n g_n(\alpha_i) \\ &= (Tf)(\alpha_i) \\ &= u(\alpha_i) f(h(\alpha_i)) \\ &= u(\alpha_i) f(h(\beta_i)) \\ &= \frac{u(\alpha_i)}{u(\beta_i)} (Tf)(\beta_i) \\ &= \frac{u(\alpha_i)}{u(\beta_i)} \{c_1 g_1(\beta_1) + ... + c_n g_n(\beta_i)\} \\ &= 0 \end{aligned}$$

for each $i = 1, ..., n$. It follows that T has an infinite codimension since n is arbitrary, a contradiction. Consequently, $\tilde{h}$ must be injective, $K = Y$, and h is a homeomorphism of Y onto X. If for any $g \in C(Y)$, we set

$$f(x) = \frac{1}{u(h^{-1}(x))} g(h^{-1}(x))$$

for each $x \in X$, then we obtain that $f \in C(X)$ and $Tf = g$, so that T is surjective.

(ii) $\Rightarrow$ (i). Let ϕ be a continuous map of Y onto X which is not injective. Then we have to show that the set

$$\{(y_1, y_2) \in Y \times Y : \phi(y_1) = \phi(y_2), y_1 \neq y_2\}$$

is infinite under the condition (ii). If not, then all $\phi^{-1}(x)(x \in X)$ are non-empty finite sets, and also $\{x \in X : \text{card}(\phi^{-1}(x)) \geq 2\}$ is a non-empty finite set, say $\{x_1, ..., x_n\}$, where "card" denotes the cardinal number. Set

$$(T_\phi f)(y) = f(\phi(y))$$

for each $f \in C(X)$ and $y \in Y$. Then T_ϕ is a linear isometry of $C(X)$ into $C(Y)$ and since ϕ is not injective, it follows that T_ϕ is not surjective. Put

$$A_i = \{g \in C(Y) : g \text{ is constant on } \phi^{-1}(x_i)\} \quad (i = 1, ..., n)$$

and

$$A = \left\{ g \in C(Y) : g \text{ is constant on } \bigcup_{i=1}^{n} \phi^{-1}(x_i) \right\}.$$

Then $A \subseteq \bigcap_{i=1}^{n} A_i$, and hence $C(Y)/\bigcap_{i=1}^{n} A_i$ is isomorphic to $(C(Y)/A)/I$, where $I = \{g + A \in C(Y)/A : g \in \bigcap_{i=1}^{n} A_i\}$. On the other hand, $T_\phi(C(X)) = \bigcap_{i=1}^{n} A_i$ since the inclusion

$T_\phi(C(X)) \subseteq \bigcap_{i=1}^{n} A_i$ is trivial, and the reverse inclusion follows immediately from Lemma 2 Also by Lemma 3, $C(Y)/A$ is isomorphic to a quotient of $C(Y_0)$, where $Y_0 = \bigcup_{i=1}^{n} \phi^{-1}(x_i)$ Consequently,

$$\begin{aligned} \operatorname{codim}(T_\phi) &= \dim(C(Y)/T_\phi(C(X))) \\ &\leq \dim(C(Y)/A) \\ &\leq \dim(C(Y_0)) \\ &\leq \sum_{i=1}^{n} \operatorname{card}(\phi^{-1}(x_i)) \\ &< +\infty . \end{aligned}$$

Hence T_ϕ has a finite codimension, and so must be surjective by the condition (ii) But this is a contradiction, so the implication is proved

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