ON A FAMILY OF WIENER TYPE SPACES

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ABSTRACT. Research on Wiener type spaces was initiated by N.Wiener in [15]. A number of authors worked on these spaces or some special cases of these spaces. A kind of generalization of the Wiener's definition was given by H.Feichtinger in [2] as a Banach spaces of functions (or measures, distributions) on locally compact groups that are defined by means of the global behaviour of certain local properties of their elements. In the present paper we discussed Wiener type spaces using the spaces $A_{w,\omega}^{p,q}(G)$ and $F_{w,\omega}^{p,q}(G)$ (c.f. [8]) as a local component, and $L_\nu^r(G)$ as a global component, where w and ν are Beurling weights on G and ω is a Beurling weight on $\hat{G}$(c.f. [13]).

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NOTATIONS. Troughout this paper G denotes a locally compact abelian group (non-compact, non-discrete) with the dual group $\hat{G}$. Certain well known terms and their definitions may be found, e.g., in [2], [3], [7],[13], [14], [9], [8]. In the present paper we have used two kinds of Fourier Transforms: The classical Fourier transform($\hat{}$)(c.f. e.g[13])and a generalized Fourier transform F (c.f. [4]) discussed below.

Let Q be a fixed open and relatively compact subset of G. We define

$$S_0(G) = \{f \mid f = \Sigma L_{y_n} f_n, y_n \in G, f_n \in A_Q, n \geq 1 \text{ and } \Sigma \|f_n\|_A < \infty\}. \tag{1}$$

where

$$A_Q = \{h \in A(G) \mid \text{supp}(h) \subset Q\}$$

and L_y denotes the translation operator. Any Representation of f in the form (1) is called an admissible representation. Endowed with the norm

$$\|f\|_{S_0} = \inf\{\|f_n\|_A, f = \Sigma L_{y_n} f_n \text{ admissible}\},$$

$S_o(G)$ is the smallest strongly character invariant Segal algebra on G. It is well-known that the Fourier trensforms induce isomorphisms between the spaces $S_o(G)$ and $S_o(\hat{G})$. The generalized Fourier transform then is defined by

$$<\hat{\sigma},f>=<\sigma,\hat{f}> \quad \text{for } f\epsilon S_o(\hat{G}), \ \sigma\epsilon S_o'(G).$$

It is easy to see that above mentioned generalized Fourier transform coincides with the Fourier transform in the sense of tempered distributions for the special case of $G=\mathbb{R}^m$.

Throughout this work, we also will use Beurling weights, i.e. real-valued, measurable and locally bounded functions w on a locally compact abelian group G which satisfy $1\leq w(x)$, $w(x+y)\leq w(x)w(y)$ for x,yϵG. For $1\leq p<\infty$, we set

$$L_w^p(G)=\{f\mid f.w\epsilon L^p(G)\}.$$

Under the norm $\|f\|_{p,w}=\|f.w\|_p$, this is a Banach space. When p=1, $L_w^1(G)$ becomes an algebra under convolution, called Beurling algebra, c.f. [13].

In this paper another two important tools are the spaces $A_{w,\omega}^{p,q}(G)$ and $F_{w,\omega}^{p,q}(G)$ with the norms $\|\cdot\|_{w,\omega}^{p,q}$ and $\|\cdot\|_F$ respectively [8]. These spaces and the norms are defined as follows:

$$A_{w,\omega}^{p,q}(G)=\{f\mid f\epsilon L_w^p(G) \quad \text{and} \quad \hat{f}\epsilon L_\omega^q(\hat{G})\},$$

$$\|f\|_{w,\omega}^{p,q}=\|f\|_{p,w}+\|\hat{f}\|_{q,\omega}$$

and

$$F_{w,\omega}^{p,q}(\hat{G})=(A_{w,\omega}^{p,q}(G))^\wedge, \quad \|\hat{f}\|_F=\|f\|_{w,\omega}^{p,q}$$

where F is the generalized Fourier transform, w and ω are Beurling weights on G and $\hat{G}$ respectively.

The main parts of this work deal with certain Wiener-type spaces in the sense [2]: the definition is the following: Let B be a Banach space. Assume that there exists a homogeneous Banach space $(A,\|\cdot\|_A)$, continuously embedded into $(C_b(G),\|\cdot\|_\infty)$, which is a Banach algebra under pointwise multiplication and is stable under complex conjugation, such that $(B,\|\cdot\|_B)$ is continuously embedded into $A_c'(G)=(A(G)\cap C_c(G))'$ and also is a Banach module under pointwise multiplication. Here $C_c(G)$ is the space of continuous functions with compact support, $A_c(G)=A(G)\cap C_c(G)$ is given the (locally convex) inductive limit topology of its subspaces $(A_K(G),\|\cdot\|_A)$, K$\subset$G compact, and $A_c'(G)$ is the topological dual of $A_c(G)$. Let now B_{loc} be the space of all $f\epsilon A_c'(G)$ such that $hf\epsilon B$ for $h\epsilon A_c(G)$; this is a locally convex vector space whose topology is defined by the seminorms $f\rightarrow\|hf\|_B$, $h\epsilon A_c(G)$. Fix an open, relatively compact set Q$\subset$G and define, for $f\epsilon B$ and xϵG, the "restriction norm $\|f\|_{B(x+Q)}$ of f over x+Q" to be

$$\inf\{\|g\|_B \,|\, g\in B,\ hf\cdot hg \text{ for } h\in A_{x+Q}(G)\}.$$

For $f\in B_{loc}$, set $F_f(x)\colon \|f\|_{B(x+Q)}$. If now C is a solid, translation in-variant BF-space on G, one defines the Wiener-type space $W(B,C)$ by

$$W(B,C)=\{f\in B_{loc}\,|\,F_f\in C\} \text{ and } \|f\|_{W(B,C)}\colon\|F_f\|_F\,).$$

Lastly recall that a Banach convolution triple (BCT) is a triple (B^1,B^2,B^3) of BF-spaces such that convolution, given by

$$f^1\ast f^2(x)=\int_G f^1(x-y)f^2(y)\,dy$$

for $f^i\in B^i\cap C_c(G)$ (i=1,2), extends to a continuous bilinear map $B^1\times B^2\to B^3$ (of norm 1).

1. THE WIENER-TYPE SPACES $W(A^{p,q}_{w,\omega}(G),L^r)$

The construction of the Wiener-type spaces mentioned in the section title requires some preliminary considerations, notably Theorem 1.5 below. First of all we introduce the Banach spaces.

$$A^u(G)=F(L^1_u(\hat{G}))$$

where u is an arbitrary weight function on $\hat{G}$ with the norm $\|\hat{g}\|_u=\|g\|_{1,u}$ and F is the classical Fourier transform. We set $A^u_c(G)=A^u(G)\cap C_c(G)$ and equip it with the inductive limit topology τ_u of the subspaces $A_K(G)=A^u(G)\cap C_K(G)$, $K\subset G$ compact, equipped with their $\|\cdot\|_u$-norms. Since it is obvious that τ_u then is finer than the norm-topology of $A_c(G)$, it is Hausdorff and hence the dual space $A^u_c(G)'=(A^u_c(G),\tau_u)'$ separates the points of $A^u_c(G)$. Note also that the subspaces $A^u_K(G)$ are closed in $A^u(G)$: Indeed, if $(h_n)\subset A^u_K(G)$ converges to $h\in A^u(G)$ with respect to $\|\cdot\|_u$, the sequence also converges uniformly to h and so supp(h)$\subset$K. The same reasoning shows that for $K\subset L$, $A^u_K(G)$ is $\|\cdot\|_u$-closed in $A^u_L(G)$. Consequently, if G is σ-compact, then $(A^u_c(G),\tau_u)$ also is complete since then it is a strict inductive limit of Banach spaces.

LEMMA 1.1. If $w_1<w_2$ and w_2 satisfies Beurling Domar condition (shortly (BD) i.e $\sum_{n>1}\dfrac{\log w_2(t^n)}{n^2}<\infty$, $t\in\hat{G}$), then $A^{w_2}_c(G)\subset A^{w_1}_c(G)$ and $A^{w_2}_c(G)$ is everywhere dense in $A^{w_1}_c(G)$.

PROOF. Since $w_1<w_2$ then the inclusion $C_c(G)\subset L^1_{w_2}(G)\subset L^1_{w_1}(G)$ is satisfied and $A^{w_2}(G)$ is everywhere dense in A^{w_1} with respect to the norm $\|\cdot\|_{w_1}$. Also since w_2 satisfies (BD) then there is an approximate identity $(e_\alpha)_{\alpha\in I}\subset A^K_{w_2}(G)$, where $A^K_{w_2}=\{f\,|\,f\in L^1_{w_2}(G),\text{supp}\hat{f} \text{ compact}\}$. Take any $\hat{g}\in A^{w_1}_c(G)$ and $\varepsilon>0$. There exists a function $\hat{f}\in A^{w_2}(G)$ such that

$$\|\hat{f}-\hat{g}\|_{w_1}<\frac{\varepsilon}{2}\ ,$$

and $\hat{f}.\hat{e}_\alpha \in A_c^{w_2}(G)$ for all $\alpha \in I$. It is easily seen that $(\hat{f}.\hat{e}_\alpha)_{\alpha \in I}$ conver-
ges to $\hat{g}$ in $A_c^{w_1}(G)$. This proves our lemma.

COROLLARY 1.2. Let w_1 and w_2 be satisfied the hypothesis in lem-
ma 1.1. We endow the space $A_c^{w_1}(G)$ with the induced inductive limit to-
pology by $A^{w_1}(G)$. $A_c^{w_2}(G)$ is also everywhere dense in $A_c^{w_1}(G)$ with respect
to induced inductive limit topology because it is strictly finer than
the original topology on $A_c^{w_1}(G)$ which is induced topology by $A^{w_1}(G)$.

COROLLARY 1.3. Again let w_1, w_2 be satisfied the hypothesis in lem-
ma 1.1. Using the inclusion $A_c^{w_2}(G) \subset A_c^{w_1}(G)$ and the Corollary 1.2. one
obtains that $A_c^{w_1}(G)'$ is continuously embedded into $A_c^{w_2}(G)'$. Specially
if $w_1 \sim w_2$ then $(A_c^{w_1}(G))' = (A_c^{w_2}(G))'$.

PROPOSITION 1.4. If $1 \leq p \leq \infty$ and the weight function ω on G satis-
fies (BD) condition then $A_{w,\omega}^{p,q}(G)$ is continuously embedded into $\sigma(A_c^\omega(G)',$
$A_c^\omega(G))$.

PROOF. It is known that $A_{w,\omega}^{p,q}(G)$ is continuously embedded into
$\sigma(A_c'(G), A_c(G))$ [8]. If one uses the above embedding and the Corollary
1.3., easily proves the Proposition.

In order to obtain all the properties of $A^u(G)$, etc., required
for the construction of Wiener-type spaces in the sense e.g. of Feich-
tinger, cf.[2], we assume henceforth that the weight function u on $\hat{G}$
satisfies (BD) and symmetric and $\omega \leq u$, where ω is the second weight func-
tion in $A_{w,\omega}^{p,q}(G)$ (For example one can take $u = \omega(x) + \omega(-x)$).

First of all, $A^u(G)$ now satisfies the requirements of [2]: It is
clear that $A^u(G)$ is continuously embedded into $C_b(G)$; in fact, the em-
bedding map has norm ≤ 1. Moreover, since u satisfies (BD), $A^u(G)$ is
known to be a Wiener space, cf.[13], hence is regular. It is a Banach
algebra under pointwise multiplication and also is translation-invari-
ant: $L_u^1(\hat{G})$ is character-invariant and that the maps M_x are isometries;
moreover, for $g \in L^1(G)$, $x \to M_x g$ is continuous [8]. By taking Fourier
transforms, we now conclude that $A^u(G)$ is translation-invariant, trans-
lation maps are isometries and that $x \to L_x h$ is continuous from G into
$A^u(G)$ for each $h \in A^u(G)$. In other words: $A^u(G)$ is homogeneous Banach
space. Lastly, the symmetry of u implies that $A^u(G)$ is closed under
complex conjugation.

Secondly $A_{w,\omega}^{p,q}(G)$ is a pointwise Banach module over $A^\omega(G)$, [8]. Since $\omega \leq u$
then $A_{w,\omega}^{p,q}(G)$ is also a Banach module over $A^u(G)$. Consequently one shows
that $A_{w,\omega}^{p,q}(G)$ is continuously embedded into $(A_c^u(G)', \sigma)$ by the Proposi-
tion 1.4.

With this, Feichtinger's general hypotheses are satisfied and the
construction of the Wiener-type spaces $W(A_{w,\omega}^{p,q}(G), C)$, C a solid BF-space
proceeds in the standard manner.

Using the arguments in Theorem 1. in [2] and doing some small
modification, the proof of the following theorem is completed.

THEOREM 1.5. (i) The Wiener type space $W(A_{W,\omega}^{p,q}(G), L_\nu^r(G))$ is a Banach space under the norm

$$\| f \| = \| F \|_{r,\nu},$$

where $f \epsilon W(A_{W,\omega}^{p,q}(G), L_\nu^r(G))$ and

$$F(z) = F_f(z) = (\| f \|_{W,\omega}^{p,q} | z+\Omega), \quad z \epsilon G \text{ and}$$

$\Omega \subset G$ is any open subset of G with compact closure. It is also continuously embedded into $(A_{W,\omega}^{p,q}(G))_{loc}$

(ii) The set

$$\Lambda_o = \{ f \epsilon A_{W,\omega}^{p,q}(G) | \text{suppf is compact} \}$$

is continuously embedded into $W(A_{W,\omega}^{p,q}(G), L_\nu^r(G))$.

(iii) $W(A_{W,\omega}^{p,q}(G), L_\nu^r(G))$ is left(right) invariant, and

$$\| | L_x | \| \leq \| | L_x | \|_{W,\omega}^{p,q} \cdot \| | L_x | \|_{r,\nu},$$

where $\| | \cdot | \|$, $\| | \cdot | \|_{W,\omega}^{p,q}$ and $\| | \cdot | \|_{r,\nu}$ are operator norms on $W(A_{W,\omega}^{p,q}(G), L_\nu^r(G))$, $A_{W,\omega}^{p,q}(G)$ and $L_\nu^r(G)$ respectively.

(iv) The translation is strongly continuous in the Wiener type spaces $W(A_{W,\omega}^{p,q}(G), L_\nu^r(G))$.

(v) $W(A_{W,\omega}^{p,q}(G), L_\nu^r(G))$ is a Banach module over $W(A(G), L^\infty(G))$ with respect to the pointwise multiplication.

PROPOSITION 1.6. Let $p \geq 1$. Then $W(A_{W,\omega}^{p,q}(G), L_W^p(G))$ is a Banach module over $W(A_{W,\omega}^{1,q}(G), L_W^1(G))$ with respect to convolution.

PROOF. It is easy to show that every locally compact Abelian group is a $\mathbb{N}$-group (i.e. a group having a compact neighbourhood of identity is invariant under inner automorphisms). It is known that $A_{W,\omega}^{p,q}(G)$ is a left(right) convolution module over $A_{W,\omega}^{1,q}(G)$ [8]. Hence since

$$(A_{W,\omega}^{1,q}(G), A_{W,\omega}^{p,q}(G), A_{W,\omega}^{p,q}(G)) \text{ and } (L_W^1(G), L_W^p(G), L_W^p(G))$$

are a Banach convolution triples on G. Then

$$(W(A_{W,\omega}^{1,q}(G), L_W^1(G)), W(A_{W,\omega}^{p,q}(G), L_W^p(G)), W(A_{W,\omega}^{p,q}(G), L_W^p(G)))$$

is a Banach convolution triples on G and the inequality

$$\| f * g | W(A_{W,\omega}^{p,q}(G), L_W^p(G)) \| \leq$$

$$\leq \| f | W(A_{W,\omega}^{1,q}(G), L_W^p(G)) \| \cdot \| g | W(A_{W,\omega}^{p,q}(G), L_W^p(G)) \|$$

is satisfied by the Theorem 3. in [2] for all $f \epsilon W(A_{W,\omega}^{1,q})(G), L_W^p(G))$ and $g \epsilon W(A_{W,\omega}^{p,q}(G), L_W^p(G))$. One can easily show the algebraic conditions which are needed to be a module.

PROPOSITION 1.7. $W(A^{p,q}_{w,\omega}(G), L^r_\nu(G))$ is a BF-space on G.

PROOF. By the Theorem 1.5. (i), $W(A^{p,q}_{w,\omega}(G), L^r_\nu(G))$ is continuously embedded into $(A^{p,q}_{w,\omega}(G))_{loc}$. That means given any $h \epsilon A_c$ (thus a seminorm $P_h(f) = \|h \cdot f\|^{p,q}_{w,\omega}$ on $(A^{p,q}_{w,\omega})_{loc})$ there exists a constant number $D_h > 0$ such that

$$\|hf\|^{p,q}_{w,\omega} \leq D_h \| f | W(A^{p,q}_{w,\omega}(G), L^r_\nu(G))\| . \tag{1}$$

for all f $W(A^{p,q}_{w,\omega}(G), L^p_\nu(G))$. Hence one can write

$$\|h \cdot f\|_p \leq D_h \cdot \| f | W(A^{p,q}_{w,\omega}(G), L^r_\nu(G))\| . \tag{2}$$

Also $A^\omega(G)$ is a regular Banach algebra with respect to pointwise multiplication, then one may choice a function $h \epsilon A^\omega_c(G) = A^\omega(G) \cap C_c(G)$ satisfying $0 \leq h < 1$ and $h(x) = 1$ for all $x \epsilon K$. We let supph K_o. Then $\chi_K(x) \leq h(x)$, hence $\chi_K(x) |f(x)| \leq h(x) |f(x)|$ for all $x \epsilon G$. Since $L^p \hookrightarrow L^1_{loc}$, then there exists $D_{K_o} > 0$ such that

$$\int_{K_o} |h(x) f(x)| dx \leq D_{K_o} \cdot \|h \cdot f\|_p . \tag{3}$$

Also one has

$$\int_K |f(x)| dx \leq \int_{K_o} |f(x) h(x)| dx . \tag{4}$$

The proof is completed combining the formülas (2), (3) and (4).

PROPOSITION 1.8. The Wiener type space $W(A^{p,q}_{w,\omega}(G), L^r_\nu(G))$ is a Banach convolution module (left and right because G is an Abelian group) over some Beurling algebra $L^1_{w_o}(G)$.

PROOF. We proved in Proposition 1.7 that $W(A^{p,q}_{w,\omega}(G), L^r_\nu(G))$ is a BF-space, thus $W(A^{p,q}_{w,\omega}(G), L^r(G)) \hookrightarrow L^1_{loc}(G)$. By the Theorem 1.5., this space is left invariant and translation operator in $W(A^{p,q}_{w,\omega}(G), L^r_\nu(G)$ is continuous. Now a simple application of vector valued integral shows that $W(A^{p,q}_{w,\omega}(G), L^r_\nu(G))$ is a Banach module over $L^1_{w_o}(G)$, where

$$W_o(x) = \max(1, \||L_x| W(A^{p,q}_{w,\omega}(G), L^r_\nu(G))\||) .$$

COROLLARY 1.9. $W(A^{p,q}_{w,\omega}(G), L^r_\nu(G))$ is a left(right) Banach module over $L^1_v(G)$ if $v(x)$ is a weight satisfying $v(x) > w_o(x)$ for all $x \epsilon G$, where $w_o(x)$ is defined as in Proposition 1.8.

REMARK. By the Theorem 1.5 (iii), one writes

$$\||L_x\|| \leq \||L_x\||^{p,q}_{w,\omega} \cdot \||L_x\||_{r,\nu} , \tag{1}$$

where $\|| \cdot \||$, $\|| \cdot \||^{p,q}_{w,\omega}$ and $\|| \cdot \||_{r,\nu}$ are operator norms on $W(A^{p,q}_{w,\omega}(G), L^r_\nu(G))$, $A^{p,q}_{w,\omega}(G)$ and $L^r_\nu(G)$ respectively. It is also known that $\||L_x\||_{r,\nu} \leq \nu(x)$ and $\||L_x\||^{p,q}_{w,\omega} \leq W(x)$ [8]. Then we have

$$\||L_x|\| \leq W(x) \cdot \nu(x),$$

for all $x \in G$. Since W, ν are weight functions, then the function $W.\nu$ is locally bounded. Using the inequality (2) it is easily seen that $x \to \||L_x|\|$ is also locally bounded.

Given a weighted space $L_w^p(G)$ the associated weighted sequence space is denoted by ℓ_w^p and defined

$$\ell_w^r = \{ (a_i)_{i \in I} \in \ell^r \mid (a_i W(i))_{i \in I} \in \ell^r \},$$

where the discrete weight W given by $W(i) = W(x_i)$ for $i \in I$. It is known that ℓ_w^r is a Banach space with respect to the norm

$$\|z\|_{r,w} = (\sum_{i \in I} |a_i W(i)|^r)^{\frac{1}{r}},$$

where $z = (a_i)_{i \in I}$.

LEMMA 1.10. For any $z \in \ell_w^r$, $z \neq 0$ the function $z \to \|L_p z\|_{r,w}$ is equivalent to the weight function w, i.e there is a constant $c > 0$ such that one has

$$c^{-1} W(p) \leq \|L_p z\|_{r,w} \leq c. W(p).$$

PROOF. Result can be obtained by a slight modifications of the proof of Lemma 2.2. in [7].

It is also easy to prove the following lemma using the closed graph theorem.

LEMMA 1.11. If $\ell_{w_1}^{r_1} \subset \ell_{w_2}^{r_2}$ then there is some constant $c > 0$ such that

$$\|z\|_{r_2,w_2} \leq C. \|z\|_{r_1,w_1}$$

for all $z = (a_i)_{i \in I} \in \ell_{w_1}^{r_1}$

If we use the Lemma 1.10 and Lemma 1.11 easily prove the following lemma:

LEMMA 1.12. $\ell_{w_1}^r \subset \ell_{w_2}^r$ if and only if $w_2 < w_1$.

THEOREM 1.13. Let U_1 and U_2 be the weight functions in construction of Wiener type spaces $W(A_{w_1,\omega_1}^{p,q}(G), L_{\nu_1}^{r_1}(G))$ and $W(A_{w_2,\omega_2}^{p,q}(G), L_{\nu_2}^{r_2}(G))$ respectively. Also assume that w_1, w_2, ν_1, ν_2 weights on G; ω_1, ω_2 weights on $\hat{G}$ and $1 \leq p,q,r_1,r_2 < \infty$. If $U_1 \sim U_2$ $r_2 \leq r_1$, $\nu_1 < \nu_2$ and

$$A_{w_2,\omega_2}^{p,q}(G) \subset A_{w_1 \omega_1}^{p,q}(G)$$

then

$$W(A_{w_2,\omega_2}^{p,q}(G), L_{\nu_2}^{r_2}(G)) \hookrightarrow W(A_{w_1,\omega_1}^{p,q}(G), L_{\nu_1}^{r_1}(G)). \quad (1)$$

PROOF. Since $\Lambda^{p,q}_{w_2,\omega_2}(G) \subset \Lambda^{p,q}_{w_1,\omega_1}(G)$ then there exists $c>0$ such that

$$\| f \|^{p,q}_{w_1,\omega_1} \leq C. \| f \|^{p,q}_{w_2,\omega_2} \qquad (2)$$

for all $f \in \Lambda^{p,q}_{w_2,\omega_2}(G)$ and $w_1 < w_2$, $\omega_1 < \omega_2$ [8]. Also since $U_1 \sim U_2$ then $\Lambda^{u_1}_c(G) = \Lambda^{u_2}_c(G)$ and $(\Lambda^{u_1}_c(G))' = (\Lambda^{u_2}_c(G))'$ by the Lemma 1.1. Hence a simple calculation shows that

$$(\Lambda^{p,q}_{w_2,\omega_2}(G))_{loc} \hookrightarrow (\Lambda^{p,q}_{w_1,\omega_1}(G))_{loc}. \qquad (3)$$

Now using the definition of Wiener-type space, (2) and (3) we have

$$W(\Lambda^{p,q}_{w_2,\omega_2}(G), L^{r_2}_{\nu_2}(G)) \hookrightarrow W(\Lambda^{p,q}_{w_1,\omega_1}(G), L^{r_2}_{\nu_2}(G)).$$

Also because the Proposition 3.7. in [6] we write

$$W(\Lambda^{p,q}_{w_1,\omega_1}(G), L^{r_2}_{\nu_2}(G)) \hookrightarrow W(\Lambda^{p,q}_{w_1,\omega_1}(G), L^{r_1}_{\nu_1}(G)) \qquad (4)$$

if and only if

$$(L^{r_2}_{\nu_2}(G))_d \subset (L^{r_1}_{\nu_1}(G))_d, \qquad (5)$$

where $(L^{r_2}_{\nu_2}(G))_d$ and $(L^{r_1}_{\nu_1}(G))_d$ are the discretes of the spaces $L^{r_2}_{\nu_2}(G)$ and $L^{r_1}_{\nu_1}(G)$ respectively. Since $r_2 \leq r_1$ and $\nu_1 < \nu_2$ then

$\ell^{r_2}_{\nu_2} \subset \ell^{r_1}_{\nu_1}$. This completes the proof.

It is known that $\Lambda^{p,q}_{w_2,\omega_2}(G) \subset \Lambda^{p,q}_{w_1,\omega_1}(G)$ if and only if $w_1 < w_2$ and $\omega_1 < \omega_2$ [8]. If one uses Theorem 1.13 and Lemma 1.12 easily proves the following corollary.

COROLLARY 1.14. Let $u_1 \sim u_2$, $w_1 < w_2$ and $\omega_1 < \omega_2$. Then

$$W(\Lambda^{p,q}_{w_2,\omega_2}(G), L^{r}_{\nu_2}(G)) \hookrightarrow W(\Lambda^{p,q}_{w_1,\omega_1}(G), L^{r}_{\nu_1}(G))$$

if and only if $\nu_1 < \nu_2$.

COROLLARY 1.15. If $w_1 \sim w_2$, $\omega_1 \sim \omega_2$, $\nu_1 \sim \nu_2$ $u_1 \sim u_2$ and $r_1 = r_2$ then

$$W(\Lambda^{p,q}_{w_2,\omega_2}(G), L^{r_2}_{\nu_2}(G)) = W(\Lambda^{p,q}_{w_1,\omega_1}(G), L^{r_1}_{\nu_1}(G))$$

2. THE WIENER-TYPE SPACES $W(F_{w,\omega}^{p,q}(G), L_\nu^r(\hat{G}))$

Let u be a weight function on G. Proceeding as we did in Section 1, we set $A_c(G) = A(G) \cap C_c(\hat{G})$, $A_c^u(\hat{G}) = A^u(\hat{G}) \cap C_c(\hat{G})$ equipped with their natural inductive limit topologies, thus the topology c.q of $A_c^u(\hat{G})$ is the inductive limit τ_A of subspaces $A_K^u(G) = A^u(\hat{G}) \cap K_K(\hat{G})$, $K \subset G$ compact, and their $\|\cdot\|_A$-topologies. Recall that τ_A is (strictly) finer than the topology τ_c induced by the usual inductive limit topology of $C_c(\hat{G})$.

We assume that the weight function u satisfies (BD) and symmetric and the first weight w in $A_{w,\omega}^{p,q}(G)$ satisfies the condition $w < u$.

Since W is symmetric then $F_{w,\omega}^{p,q}(\hat{G}) = A_{\omega,w}^{q,p}(\hat{G})$, (see [8]).

Now all conditions are satisfied required for the construction of Wiener-type space $W(F_{w,\omega}^{p,q}(\hat{G}), L_\nu^r(\hat{G})) = W(A_{\omega,w}^{q,p}(\hat{G}), L_\nu^r(\hat{G}))$. If one uses the properties of the space $A_{\omega,w}^{q,p}(\hat{G})$ obtains all properties of the Wiener-type space $W(F_{w,\omega}^{p,q}(\hat{G}), L_\nu^r(\hat{G}))$ like to that of $W(A_{w,\omega}^{p,q}(G), L_\nu^r(G))$ in Section 1.

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