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ON THE DIAPHONY OF ONE CLASS OF ONE-DIMENSIONAL SEQUENCES

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ABSTRACT In the present paper, we consider a problem of distribution of sequences in the interval $[0,1)$, the so-called 'P_r-sequences' We obtain the best possible order $O(N^{-1}(logN)^{1/2})$ for the diaphony of such P_r-sequences For the symmetric sequences obtained by symmetrization of P_r-sequences, we get also the best possible order $O(N^{-1}(logN)^{1/2})$ of the quadratic discrepancy

KEY WORDS AND PHRASES Distribution of sequences, quadratic discrepancy and Pr-sequences
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1 INTRODUCTION

1 1 Let $\sigma = (x_n)_{n=0}^{\infty}$ be an infinite sequence in the unit interval $E = [0,1)$ For every real number $x \in E$ and every positive integer N we denote $A_N(\sigma, x)$ the number of terms x_n, $0 \le n \le N-1$, which are less than x

The sequence σ is called uniformly distributed in E if for every real number $x \in E$ we have

$$lim_{N\to\infty} A_N(\sigma; x)N^{-1} = x.$$

The systematic study of the theory of uniformly distributed sequences was initiated by Weyl [1]

A classical measure for the irregularity of the distribution of a sequence σ in E is its quadratic discrepancy $T_N(\sigma)$, which is defined for every positive integer N as

$$T_N(\sigma) = (\int_0^1 \mid A_N(\sigma; x)/N - x \mid^2 dx)^{1/2}.$$

The irregularity of distribution with respect to the quadratic discrepancy was first studied by Roth [2]

In 1976, Zinterhof (see [3,4]) proposed a new measure for distribution, which he named diaphony The diaphony $F_N(\sigma)$ of σ is defined for every positive integer N as

$$F_N(\sigma) = (2\sum_{h=1}^{\infty} h^{-2} \mid N^{-1} S_N(\sigma; h) \mid^2)^{1/2}$$

where

$$S_N(\sigma; h) = \sum_{n=0}^{N-1} exp(2\pi i h x_n)$$

signify trigonometric sum of σ

We note that the diaphony of σ can be written in the form

$$F_N(\sigma) = (N^{-2} \sum_{n,k=0}^{N-1} g(x_n - x_k))^{1/2}$$

where

$$g(x) = \pi^2(2x^2 - 2x + 1/3)$$

It is well known (see [5], p 115, [4]) that both equalities

$$lim_{N\to\infty} T_N(\sigma) = 0 \text{ and } lim_{N\to\infty} F_N(\sigma) = 0$$

are equivalent to the definition that the sequence σ is uniformly distributed in E

1 2 Using the well-known theorem of Roth [2] it can be proved (see Neiderreiter [7], p 158; Proinov [8]) that for any infinite sequence σ in E, the estimate

$$T_N(\sigma) > 214^{-1}N^{-1}(logN)^{1/2} \tag{1.1}$$

holds for infinitely many integers N The exactness of the order of magnitude of this estimate was proved by Proinov ([9], [10], [11])

Proinov [8] proved that for any sequence σ in E the estimate

$$F_N(\sigma) > 68^{-1}N^{-1}(logN)^{1/2} \tag{1.2}$$

holds for infinitely many N.

From (1.1) and (1.2) becomes clearly that the best possible order of diaphony and quadratic discrepancy of every sequence σ in E is $O(N^{-1}(logN)^{1/2})$.

2. A SEQUENCE OF r-ADIC RATIONAL TYPE.

2.1 CONSTRUCTION OF SEQUENCE OF r-ADIC RATIONAL TYPE

In this part we generalize Sobol's ([12], [5], p 117, [13], p. 23) construction of sequences of binary rational type

Let $r \geq 2$ is fixed integer. We consider the infinite matrix

$$(v_{s,j}) = \begin{pmatrix} v_{11} & v_{21} & --- \\ v_{12} & v_{22} & --- \\ --- & --- & --- \end{pmatrix} \tag{2.1}$$

where for every $s, j = 1, 2, \cdots, v_{s,j} \in \{0, 1, \cdots, r-1\}$. We suppose that in every column, the quantity of $v_{s,j}$, which are different from zero is a positive integer number, i.e., $v_{s,j} = 0$ for j sufficiently big Such matrix we shall call guiding matrix

To every column of the matrix (2 1) corresponds a r-adic rational numbers

$$V_s = 0,\ v_{s,1}\ v_{s,2} \cdots v_{s,j} \cdots (s = 1, 2 \cdots) \tag{2.2}$$

The numbers determined in (2.2) are called guiding numbers.

We signify $N_0 = N \cup \{0\}$, with N the set of natural integers.

A sequence of r-adic rational type (or RP-sequence) is a sequence $(\varphi(i))_{i=0}^{\infty}$, which is generated by the guiding matrix $(v_{s,j})$ in the following way: If in the r-adic number system

$$i = e_m e_{m-1} \cdots e_1$$

then in the r-adic number system

$$\varphi(i) = 0, W_1^* W_2^* \cdots {}^* W_m$$

where for $j = 1, 2, \cdots, m$

$$W_j = e_j V_j = \underbrace{V_j^* V_j^* \cdots {}^* V_j}_{e_j - terms}, \tag{2.3}$$

and * is the operation of the digit-by-digit addition modulo r of elements of $Z_r = \{0, 1, \cdots, r-1\}$.

A RP-sequence $(\varphi(i))_{i=0}^{\infty}$, which is generated by the guiding matrix $(v_{s,j})$ can be also constructed by following the three mentioned below rules:

(1) $\varphi(0) = 0$.

(2) If $i = r^s (s \in N_0)$, then $\varphi(i) = V_{s+1}$.

(3) If $r^s < i < r^{s+1}$, then $\varphi(i) = e_{s+1}\ \varphi(r^s)^* \varphi(i - e_{s+1}\ r^s)$, where e_{s+1} is higher significant digit in r-adic development of i and $e_{s+1}\varphi(r^s) = \underbrace{V_{s+1}^* V_{s+1}^* \cdots {}^* V_{s+1}}_{e_{s+1} - terms}$.

Obviously the operation * has commutative and associative property.

We shall prove that the two definitions of the PR-sequences are equivalent.

Let us suppose that the first definition is valid for RP-sequence.

(1) If $i = 0$, then obviously $\varphi(i) = 0$.

(2) If $i = r^s (s \in N_0)$, then $\varphi(i) = V_{s+1}$.

(3) Let us assume that $r^s < i < r^{s+1}$ and $i = (e_{s+1}\ e_s \cdots e_1)_r$. Since the operator * is commutative and associative we have

$$\varphi(i) = 0, ((e_1\ V_1)^* \cdots {}^*(e_s\ V_s))^*(e_{s+1}\ V_{s+1}).$$

Since $V_{s+1} = \varphi(r^s)$ and $i - e_{s+1}\ r^s = (e_s\ e_{s-1} \cdots e_1)_r$, then $\varphi(i - e_{s+1}\ r^s) = 0, (e_1\ V_1)^* \cdots {}^*(e_s\ V_s)$. Finally $\varphi(i) = e_{s+1}\varphi(r^s)^*\varphi(i - e_{s+1}\ r^s)$. The three rules in the second definition for RP-sequence are proved.

Reversely, let the second definition for PR-sequence is valid and i is given positive integer. Then there exists uniquely positive integer s that $r^s \leq i < r^{s+1}$. We shall prove definition 1 by induction on s. If $s = 0$, then $1 \leq i < r$ and

$$\varphi(i) = i\varphi(r^0)^*\varphi(0) = 0,\ iV_1.$$

We make inductive supposition that for some $s \in N$ and every integer i, $r^{s-1} \leq i < r^s$ definition 1 holds. Let us assume that $r^s \leq i < r^{s+1}$ and $i = (e_{s+1}e_s \cdots e_1)_r$. From rule 3 we have

$$\varphi(i) = e_{s+1}\ \varphi(r^s)^*\varphi(i - e_{s+1}\ r^s).$$

If we denote $j = i - e_{s+1}\ r^s$, then $j = (e_s\ e_{s-1} \cdots e_1)_r$ and $r^{s-1} \leq j < r^s$. Then by inductive supposition

$$\varphi(j) = 0, (e_1V_1)^* \cdots {}^*(e_sV_s).$$

By rule 2, $\varphi(r^s) = V_{s+1}$ and we have

$$\varphi(i) = 0, (e_1V_1)^* \cdots {}^*(e_sV_s)^*(e_{s+1}V_{s+1}).$$

Definition 1 holds for every positive integer s.

In the following lemma we give a property of the functions φ.

LEMMA 2.1. Let $(v_{s,j})$ is an arbitrary guiding matrix, and $(\varphi(i))_{i=0}^{\infty}$ is RP-sequence, which is generated by $(v_{s,j})$. Let ν, m, n be integer numbers such that $\nu \in N_0$, $0 \leq n < r^\nu$ and $m \equiv 0(mod\ r^\nu)$. Then we have

$$\varphi(m + n) = \varphi(m)^*\varphi(n).$$

The proof of the lemma is obvious.

For every integer $a \in Z_r$ we define $\overline{a}$ the only integer, which is a solution of the equation

$$a + \overline{a} \equiv 0(mod\ r).$$

If $\alpha = 0, \alpha_1\alpha_2 \cdots \alpha_t$, where, for $\tau = 1, 2, \cdots, t$ $\alpha_\tau \in Z_r$, then we define $\overline{\alpha} = 0, \overline{\alpha}_1\overline{\alpha}_2 \cdots \overline{\alpha}_t$.

2.2. SEQUENCES OF r-ADIC RATIONAL TYPE, WHICH ARE P_r-SEQUENCES.

The theory of the P_r-sequences was first studied by Faure ([14];[15]) and generalized by Neiderreiter ([16];[17]).

A r-adic elementary interval is an interval

$$l_{m,j} = [(j - 1)/r^m, j/r^m),$$

in which $1 \leq j \leq r^m$, for any integer m.

Let $N = r^m$. We shall call the net

$$X = (x_0, x_1, \cdots, x_{N-1})$$

be a net of type P_r^m (or P_r^m-type), if every r-adic elementary interval $l_{m,j}$, having length $1/\ N$ contain one point of the net X.

A r-adic section of the sequence $X = (x_i)_{i=0}^{\infty}$ is a set of terms x_i, with numbers i, satisfying the inequalities

$$kr^s \leq i < (k + 1)r^s,$$

for every integers k and s, such that $k = 0, 1, \cdots; s = 1, 2, \cdots$.

The sequence $(x_i)_{i=0}^{\infty}$ is called a sequence of type P_r (or P_r-sequence) if every r-adic section is a P_r^m-net.

THEOREM 2.1. Let in the guiding matrix $(v_{s,j})$ every $v_{s,s} = 1$ and for $j > s$ every $v_{s,j} = 0$, i.e.,

$$(v_{s,j}) = \begin{pmatrix} 1 & v_{21} & v_{31} & --- & v_{j1} & -- \\ 0 & 1 & v_{32} & --- & v_{j2} & -- \\ 0 & 0 & 1 & --- & v_{j3} & -- \\ --- & --- & --- & --- & --- & --- \\ 0 & 0 & 0 & --- & 1 & -- \\ --- & --- & --- & --- & --- & --- \end{pmatrix}.$$

Then the corresponding RP-sequence is P_r-sequence

PROOF. We choose arbitrary r-adic section of the RP-sequence $(\varphi(i))_{i=0}^{\infty}$, the length of which is r^m. We write the numbers i, belonging to this section in the r-adic number system:

$$i = c_\mu c_{\mu-1} \cdots c_{m+1} e_m e_{m-1} \cdots e_1, \tag{2.4}$$

where c_k are fixed and e_k are arbitrary r-adic numbers

We choose now an arbitrary r-adic interval l, with length $|\,l\,| = r^{-m}$. In the r-adic system this interval is determined by the inequality

$$0, a_1 a_2 \cdots a_m \leq x < 0, a_1 a_2 \cdots a_m + 0, \underbrace{0 \cdots 0}_{m - zeros} 1,$$

where $a_1, \cdots, a_m$ are r-adic numbers

We shall prove, that for every choice of the numbers c_k and a_k among the numbers i, in the form (2.4) there exists exactly one i, for which $\varphi(i) \in l$.

. In the r-adic number system we write

$$\varphi(i) = 0, g_{i,1} g_{i,2} \cdots g_{i,j} \cdots.$$

From (2.3) we have

$$g_{i,j} = e_1 v_{1,j}^* \cdots {}^* e_m v_{m,j}^* c_{m+1} v_{m+1,j}^* \cdots {}^* c_\mu v_{\mu,j},$$

where the sense of $e_k v_{k,j}$ is the same as in (2.3).

The condition $\varphi(i) \in l$ is equivalent to the following conditions

$$g_{i,j} = a_j, \text{ for } 1 \leq j \leq m.$$

We get that for each j, $1 \leq j \leq m$

$$g_{i,j} = (e_1 v_{1,j}^* \cdots {}^* e_m v_{m,j})^* (c_{m+1} v_{m+1,j}^* \cdots {}^* c_\mu v_{\mu,j}),$$

from which we get

$$e_1 v_{1,j}^* \cdots {}^* e_m v_{m,j} = a_j^* \overline{(c_{m+1} v_{m+1,j}^* \cdots {}^* c_\mu v_{\mu,j})} \quad (1 \leq j \leq m) \tag{2.5}$$

Let us call f_j the right-side of (2.5) for $1 \leq j \leq m$. Having in mind that for $s = 1, 2, \cdots$, $v_{s,s} = 1$ and in case $j > s$, $v_{s,j} = 0$, the system (2.5) become

$$e_j v_{j,j}^* e_{j+1} v_{j+1,j}^* \cdots {}^* e_m v_{m,j} = f_j (1 \leq j \leq m).$$

In this system the unknowns $e_1, e_2, \cdots, e_m$ are successively so determined that it has only one solution.

The theorem is proved.

In the following lemma we shall show some property of P_r-sequences.

LEMMA 2.2. Let $N = r^\nu$ where $\nu \in N_0$. For every guiding matrix $(v_{s,j})$ in which $v_{s,s} = 1$ and $v_{s,j} = 0$ for $j > s (s = 1, 2, \cdots)$ and for the RP-sequence $(\varphi(i))_{i=0}^{\infty}$, which is product of $(v_{s,j})$ we have

$$\{\varphi(i) : 0 \leq i < r^\nu\} = \{j/N : 0 \leq j < N\} \tag{2.6}$$

PROOF. We shall make the proof by induction on ν. If $\nu = 0$ and $\nu = 1$, then we make directly examination.

We make inductive supposition, that for some $\nu \in N$ the equality (2.6) is true and for $j = 0, 1, \cdots, r-1$ we consider the multitudes $A_j = \{\varphi(i) : jr^\nu \leq i < (j+1) r^\nu\}$. Then obviously

$$A = \bigcup_{j=0}^{r-1} A_j \tag{2.7}$$

where $A = \{\varphi(i) : 0 \leq i < r^{\nu+1}\}$.

We consider that $j = 0$. By the inductive supposition

$$A_0 = \{\varphi(i) : 0 \leq i < r^\nu\} = \{m / r^{\nu+1} : 0 \leq m < r^{\nu+1}, m \equiv 0 \ (mod\ r)\} \tag{2.8}$$

Let us now consider that $1 \leq j \leq r-1$. We shall prove the following equality

$$A_j = \{m/r^{\nu+1}: 0 \leq m < r^{\nu+1}, m \equiv j \ (mod\ r)\}. \tag{2.9}$$

Let $j, 1 \leq j \leq r-1$ is fixed integer and consider that $jr^\nu \leq i < (j+1)r^\nu$. Let us represent i in the form $i = jr^\nu + k$, where $0 \leq k < r^\nu$.

Then by Lemma 2 1 we have

$$\varphi(i) = \varphi(jr^\nu)^* \varphi(k) \tag{2.10}$$

It is obvious that

$$\varphi(jr^\nu) = \underbrace{V^*_{\nu+1} V^*_{\nu+1} \cdots {}^* V_{\nu+1}}_{j-terms} \tag{2.11}$$

Let us put

$$\varphi(jr^\nu) = 0, w_{\nu+1,1} w_{\nu+1,2} \cdots w_{\nu+1,\nu+1}.$$

From (2 11) is clear, that $w_{\nu+1,\nu+1} = j$. Let k has r-adic development $k = k_\nu k_{\nu-1} \cdots k_1$. Then

$$\varphi(k) = 0, (k_1 V_1)^* \cdots {}^*(k_\nu V_\nu)$$

$$\varphi(k) = 0, a_1 a_2 \cdots a_\nu, \text{ where } a_s \in \{0, 1, \cdots, r-1\}, s = 1, 2, \cdots, \nu) \tag{2.12}$$

From (2.10), (2.11) and (2.12) we get

$$\varphi(i) = 0, (a_1^* w_{\nu+1,1}) \cdots (a_\nu^* w_{\nu+1,\nu}) j = 0, b_1 b_2 \cdots b_\nu j \tag{2.13}$$

When $0 \leq k < r^\nu$, then $0 \leq (b_1 b_2 \cdots b_\nu)_r < r^\nu$ and from (2.13) we get that for $1 \leq j \leq r-1$

$$A_j = \{\varphi(i): jr^\nu \leq i < (j+1)r^\nu\} = \{m/r^{\nu+1}: 0 \leq m < r^{\nu+1}, m \equiv j \ (mod\ r)\}$$

The inequalities (2.9) are proved.

By induction on ν the lemma is proved.

LEMMA 2.3 Let $(\varphi(i))_{i=0}^{\infty}$ be a P_r-sequence. Then for every $\nu \in N_0$ holds the equality

$$\begin{aligned}&\{\varphi(m+j): m \equiv 0(mod\ r^\nu), 0 \leq j < r^\nu\} \\ &\quad = \{\varphi(m) + \varphi(j)(mod\ 1): m \equiv 0(mod\ r^\nu), 0 \leq j < r^\nu\}\end{aligned} \tag{2.14}$$

PROOF. Let us consider that $m = kr^\nu$, for some positive integer k. The equality (2.14) is equivalent to the equality

$$\begin{aligned}&\{\varphi(m+j): m \equiv 0(mod\ r^\nu), 0 \leq j < r^\nu\} \\ &\quad = \bigcup_{l=0}^{r^{\nu-1}-1} \{\varphi(m+j): m \equiv 0(mod\ r^\nu), lr \leq j < (l+1)r\}\end{aligned} \tag{2.15}$$

First, we shall prove that for every fixed l, $0 \leq l \leq r^{\nu-1}$ exists uniquely l' $0 \leq l' < r^{\nu-1}$, such that

$$\begin{aligned}&\{\varphi(m+j): m = kr^\nu, lr \leq j < (l+1)r\} \\ &\quad = \{\varphi(m) + \varphi(j)(mod\ 1): m = kr^\nu, l'r \leq j < (l'+1)r\}.\end{aligned} \tag{2.16}$$

Let $k = (k_n k_{n-1} \cdots k_1)_r$. Then we have

$$\varphi(m) = 0, g_1^m g_2^m \cdots g_{n+\nu}^m,$$

where for $1 \leq i \leq n+\nu$ $g_i^m = \sum_{h=1}^{n} k_h v_{h+\nu,i} \ (mod\ r)$.

Let $0 \leq l < r^{\nu-1}$ be fixed integer and $l = (l_{\nu-1} \cdots l_1)_r$. Then $lr = (l_{\nu-1} \cdots l_1\ 0)_r$.

When j is such integer that $lr \leq j < (l+1)r$, we have $j = (l_{\nu-1} \cdots l_1 l_0)_r$, where $l_{\nu-1}, \cdots, l_1$ are fixed integers and l_0 takes r different values in the set $\{0, 1, ..., r-1\}$. Let $\varphi(j) = 0,\ a_1 \cdots a_\nu$, where

$$a_1 = l_0 + \sum_{h=1}^{\nu-1} e_h\ v_{h+1,1}\ (mod\ r)$$

$$a_2 = l_1 + \sum_{h=2}^{\nu-1} e_h\ v_{h+1,2}\ (mod\ r)$$

— — — — — — — — — — — — — — — — —

$$a_{\nu-1} = l_{\nu-2} + l_{\nu-1}\ v_{\nu,\nu-1} (mod\ r)$$

$$a_\nu = 1_{\nu-1}.$$

It is obvious that and a_1 takes r different values in the set $\{0, 1, ..., r-1\}$.

From the Lemma 2.1 we have

$$\varphi(m+j) = (0, g_1^m g_2^m \cdots g_\nu^m g_{\nu+1}^m \cdots g_{n+\nu}^m) * (0, a_1 a_2 \cdots a_\nu)$$
$$= 0, b_1 b_2 \cdots b_\nu g_{\nu+1}^m \cdots g_{n+\nu}^m$$

where

$$\begin{aligned} b_\nu &= g_\nu^m + a_\nu (mod\ r) \\ b_{\nu-1} &= g_{\nu-1}^m + a_{\nu-1} (mod\ r) \\ &- - - - - - - - - - - - - - \\ b_2 &= g_2^m + a_2 (mod\ r) \\ b_1 &= g_1^m + a_1 (mod\ r) \end{aligned} \tag{2.17}$$

Since $0 \le l' < r^{\nu-1}$, we shall search l' in the form $l' = (l'_{\nu-1} \cdots l_1')_r$, where $l_1', \cdots, l'_{\nu-1}$ are unknown quantities. Then $l'r = (l'_{\nu-1} \cdots l_1'\ 0)_r$. When $l'\ r \le i < (l+1)r$ then $i = (l'_{\nu-1} \cdots l_1'\ l_0')_r$, for $0 \le l_0 < r$.

Let us denote $\varphi(i) = 0, c_1 c_2 \cdots c_\nu$ where

$$\begin{aligned} c_1 &= l_0' + \textstyle\sum_{h=1}^{\nu-1} l_h'\ v_{h+1,1}\ (mod\ r) \\ c_2 &= l_1' + \textstyle\sum_{h=2}^{\nu-1} l_h'\ v_{h+1,2}\ (mod\ r) \\ &- - - - - - - - - - - - - - - - \\ c_{\nu-1} &= l'_{\nu-2} + l'_{\nu-1}\ v_{\nu,\nu-1}\ (mod\ r) \\ c_\nu &= l'_{\nu-1}. \end{aligned}$$

Then we have

$$\begin{array}{rl} & \quad\ \ \delta_1\ \ \delta_2 \qquad\quad \delta_{\nu-1} \\ \varphi(m) + \varphi(i)(mod\ 1) = & 0,\ g_1^m\ g_2^m \cdots g_{\nu-1}^m\ g_\nu^m\ g_{\nu+1}^m \cdots g_{\nu+n}^m \\ & +\ 0,\ c_1\ c_2 \cdots c_{\nu-1}\ c_\nu \\ \hline & 0,\ d_1\ d_2 \cdots d_{\nu-1}\ d_\nu\ g_{\nu+1}^m \cdots g_{\nu+n}^m \end{array}$$

where $\delta_1,\ \delta_2, \cdots, \delta_{\nu-1}$ are the step-by-step carries and else

$$\begin{aligned} d_\nu &= g_\nu^m + c_\nu\ (mod\ r) \\ d_{\nu-1} &= g_{\nu-1}^m + \delta_{\nu-1} + c_{\nu-1}\ (mod\ r) \\ &- - - - - - - - - - - - - - - - - - - \\ d_2 &= g_2^m + \delta_2 + c_2\ (mod\ r) \\ d_1 &= g_1^m + \delta_1 + c_1\ (mod\ r) \end{aligned} \tag{2.18}$$

For the demonstration of the equality (2.16) we make equal the numbers, constructed in (2.17) and (2.18), and we get

$$\begin{aligned} l'_{\nu-1} &\equiv l_{\nu-1}\ (mod\ r) \\ l'_{\nu-2} + l'_{\nu-1}\ v_{\nu,\nu-1} + \delta_{\nu-1} &\equiv l_{\nu-2} + l_{\nu-1}\ v_{\nu,\nu-1}\ (mod\ r) \\ - - - - - - - - - - - - - - - - - &- - - - - - - - - - - - - - - - \\ l_1' + \textstyle\sum_{h=2}^{\nu-1} l_h'\ v_{h+1,2} + \delta_2 &\equiv l_1 + \textstyle\sum_{h=2}^{\nu-1} l_h\ v_{h+1,2}\ (mod\ r) \\ l_0' + \textstyle\sum_{h=1}^{\nu-1} l_h'\ v_{h+1,1} + \delta_1 &\equiv l_0 + \textstyle\sum_{h=1}^{\nu-1} l_h\ v_{h+1,1}\ (mod\ r). \end{aligned}$$

Since $0 \le l_{\nu-1},\ l'_{\nu-1} < r$, then equation $l_{\nu-1} \equiv l_{\nu-1}\ (mod\ r)$ has the only solution $l'_{\nu-1} = l_{\nu-1}$. Consecutively we solve the left over equations and get uniquely integer number $l' = (l'_{\nu-1} \cdots l_1')_r$, such that $0 \le l' < r^{\nu-1}$.

Since l_0 takes r different values in the set $\{0, 1, ..., r-1\}$, then and l_0' takes r different values in the set $\{0, 1, ..., r-1\}$ and $l'r \le i < (l+1)r$.

Finally, we establish a bijection between the sets from the two sides of the equation (2.16).

Let p and q be such that $0 \le p, q < r^s, p \ne q$ and p' and q' are the numbers, satisfying the equality (2.16). We shall prove that $p' \ne q'$. Let us admit that $p' = q' = \alpha$. Then we have

$$\{\varphi(m+j) : m \equiv 0\ (mod\ r^\nu), pr \le j < (p+1)r\}$$
$$= \{\varphi(m) + \varphi(i)\ (mod\ 1) : m \equiv 0\ (mod\ r^\nu), \alpha r \le i < (\alpha+1)r\}.$$

and

$$\{\varphi(m+j) : m \equiv 0 \ (mod\ r^{\nu}), qr \leq j < (q+1)r\}$$
$$= \{\varphi(m) + \varphi(i) \ (mod\ 1) : m \equiv 0 \ (mod\ r^{\nu}), \alpha r \leq i < (\alpha+1)r\}.$$

Then we have

$$\{\varphi(m+j) : m \equiv 0 \ (mod\ r^{\nu}), pr \leq j < (p+1)r\}$$
$$= \{\varphi(m+j) : m \equiv 0 \ (mod\ r^{\nu}), qr \leq j < (q+1)r\}.$$

This is a contradiction, since the function φ is an injection; so the equation (2.16) is proved.

From (2.15) and (2.16) we get

$\{\varphi(m+j) : m \equiv 0 \ (mod\ r^{\nu}), 0 \leq j < r^{\nu}\} = \bigcup_{l'=0}^{r^{\nu-1}-1} \{\varphi(m) + \varphi(i) \ (mod\ 1) : m \equiv 0$ $(mod\ r^{\nu}),\ l'r \leq i < (l'+1)r\} = \{\varphi(m) + \varphi(j) \ (mod\ 1) : m \equiv 0 \ (mod\ r^{\nu}), 0 \leq j < r^{\nu}\}$.

The lemma is proved.

3. AN ESTIMATION FROM ABOVE FOR THE DIAPHONY OF P_r-SEQUENCES.

THEOREM 3.1. Let in the guiding matrix $(v_{s,j})$ every $v_{s,s} = 1$ and for $j > s$ every $v_{s,j} = 0$ and let $\sigma = (\varphi(i))_{i=0}^{\infty}$ be the P_r-sequence which is produced by the $(v_{s,j})$. Then for every positive integer N we have

$$F_N(\sigma) \leq c(r) N^{-1} (log((r-1)N+1))^{1/2},$$

where the constant $c(r)$ is given by

$$c(r) = \pi((r^2-1)/3\ log\ r)^{1/2}. \tag{3.1}$$

The proof of this theorem is based on a non-trivial estimate for the trigonometric sum of an arbitrary P_r − sequence.

3.1. AN ESTIMATION OF THE TRIGONOMETRIC SUM OF ARBITRARY P_r-SEQUENCE.

Let $X = (x_n)_{n=0}^{\infty}$ is arbitrary sequence in interval E.A trigonometric sum, $S_N(X{:}\,h)$, of the sequence X, where h is an integer is the quantity

$$S_N(X;h) = \sum_{n=0}^{N-1} exp\ (2\pi i h x_n).$$

LEMMA 3.1. Let $N = P + Q$, where P and Q are arbitrary integers. Then for every integer h and arbitrary sequence $X = (x_n)_{n=0}^{\infty}$ we have

$$|\ S_N\ (X;h)\ | \ \leq \ |\ S_P\ (X;h)\ | \ + \ |\ S_P^Q\ (X;h)\ |,$$

where

$$S_P^Q\ (X;h) = \sum_{n=P}^{P+Q-1} exp\ (2\pi i x_n).$$

The proof of lemma is obvious.

LEMMA 3.2. Let $N = ar^n$, where $a \geq 1$ and $n \geq 0$ are integers.

Then for every integer h we have

$$|\ S_N\ (X;h)\ | \ \leq \sum_{i=1}^{a} |\ S_{(i-1)r^n}^{r^n}\ (X;h)\ |,$$

The proof of lemma is based of Lemma 3.1 and is done by induction on a.

Let a be an arbitrary integer and q a positive integer. We define the function $\delta_q(a)$ by

$$\delta_q(a) = \begin{cases} 1, \text{ if } a \equiv 0 (mod\ q) \\ 0, \text{ if } a \not\equiv 0 (mod\ q) \end{cases}$$

It is well known that for every integer a and every natural q we have

$$\sum_{x=0}^{q-1} exp\ (2\pi i a x/q) = q_q^{\delta}(a)$$

LEMMA 3.3. Let $N \geq 1$ be an integer and

$$N = \sum_{j=0}^{\infty} a_j\, r^j, a_j \in \{0, 1, \cdots, r-1\}\ (j = 0, 1, \cdots) \tag{3.2}$$

be its r-adic representation.

Let in the guiding matrix $(v_{s,j})$ every $v_{s,s} = 1$ and for $j > s$ every $v_{s,j} = 0$ and $\sigma = (\varphi(n))_{n=0}^{\infty}$ be the P_r-sequence which is product of $(v_{s,j})$.

Then for every integer h we have

$$|\ S_N\ (\sigma;h)\ | \ \leq \sum_{j=0}^{\infty} a_j\, r^j\, \delta_{r^j}\ (h)$$

PROOF. Let $N \geq 1$ be an integer with r-adic representation of a type (3.2).

We shall prove that for every integer h and for every sequence X in interval E we have the estimation

$$| S_N (X; h) | \leq \sum_{j=0}^{\infty} \sum_{m=1}^{a_j} | S_{(m-1)r^j}^{r^j} (X; h) | , \tag{3.3}$$

where we have the supposition that when $a_j = 0$, the inside sum is 0.

Let h be an integer. For every $N \geq 1$ exists an integer n, such that $N < r^n$. We shall prove the lemma by the induction on n

If $n = 1$, then the estimation (3.3) is trivial.

We suppose, that (3.2) is true for every integer N, $1 \leq N < r^n$, where n is some integer.

Let now N such that $r^n \leq N < r^{n+1}$. By here we have, that in (3.2) $a_j = 0$ for $j > n$ Let $N = P + Q$ where $P = a_n r^n$ and $Q = \sum_{j=0}^{n-1} a_j r^j$.

By Lemma 3.1, Lemma 3.2 and the induction supposition we get

$$| S_N (X; h) | \leq | S_{a_n r^n} (X; h) | + | S_P^Q (X; h) | \leq \sum_{m=1}^{a_n} | S_{(m-1)r^n}^{r^n} (X; h) |$$
$$+ \sum_{j=0}^{n-1} \sum_{m=1}^{a_j} | S_{(m-1)r^j}^{r^j} (X; h) | = \sum_{j=0}^{n} \sum_{m=1}^{a_j} | S_{(m-1)r^j}^{r^j} (X; h) |$$
$$= \sum_{j=0}^{\infty} \sum_{m=1}^{a_j} | S_{(m-1)r^j}^{r^j} (X; h) | ,$$

such that (3.3) is proved If $Q = 0$, then (3 3) is got by Lemma 3.2.

Let now $j, 0 \leq j \leq n$ be an arbitrary fixed number and consider that $1 \leq m \leq a_j$. If $m = 1$, then by Lemma 2.2 for the trigonometric sum $S_0^{r^j} (\sigma; h)$ we have

$$S_0^{r^j} (\sigma; h) = r^j \, \delta_{r^j} (h) \tag{3.4}$$

Let now $2 \leq m \leq a_j$. Then for the trigonometric sum $S_{(m-1)r^j}^{r^j} (\sigma; h)$, by Lemma 2.4, we have

$$S_{(m-1)r^j}^{r^j} (\sigma; h) = \sum_{n=(m-1)r^j}^{mr^j-1} exp\, (2\pi i h \varphi(n)) = \sum_{k=0}^{r^j-1} exp\, (2\pi i h(\varphi((m-1)r^j)$$
$$+ \varphi(k))) = exp\, (2\pi i h \varphi((m-1)r^j)) \sum_{k=0}^{r^j-1} exp\, (2\pi i h \varphi(k)).$$

Thus for the module of the trigonometric sum $S_{(m-1)r^j}^{r^j} (\sigma; h)$ we get

$$| S_{(m-1)r^j}^{r^j} (\sigma; h) | = r^j \, \delta_{r^j} (h). \tag{3.5}$$

From (3.3), (3.4) and (3.5) we get

$$| S_N (\sigma; h) | \leq \sum_{j=0}^{\infty} a_j \, r^j \, \delta_{r^j} (h).$$

The lemma is proved.

3.2. PROOF OF THEOREM 3.1.

Let $(v_{s,j})$ is an arbitrary guiding matrix, such that on principal diagonal there stand ones, and over him zeros and $\sigma = (\varphi(n))_{n=0}^{\infty}$ is P_r-sequence, which is bred by the matrix $(v_{s,j})$.

We choose $N \geq 1$ arbitrary integer and let has r-adic representation in the form

$$N = \sum_{j=1}^{k} a_j \, r^{n_j} \; (a_j \in \{1, \cdots, r-1\}, \; j = 1, 2, \cdots, k), \tag{3.6}$$

where

$$0 \leq n_1 < n_2 < \cdots < n_k.$$

are integer numbers.

From Lemma 3.3 for every integer h we have

$$| S_N (\sigma; h) | \leq \sum_{j=1}^{k} a_j \, r^{n_j} \, \delta_{r^{n_j}} (h) \leq (r-1) \sum_{j=1}^{k} r^{n_j} \, \delta_{r^{n_j}} (h).$$

By the last inequality for the diaphony $F_N(\sigma)$ of σ we have

$$(NF_N(\sigma))^2 = 2 \sum_{h=1}^{\infty} h^{-2} \, | S_N \, \sigma; h)^2 \leq$$
$$2(r-1)^2 \sum_{h=1}^{\infty} h^{-2} \sum_{j=1}^{k} \sum_{\nu=1}^{k} r^{n_j + n_\nu} \, \delta_{r^{n_j}} (h) \, \delta_{r^{n_\nu}} (h)$$
$$= 2(r-1)^2 \sum_{j=1}^{k} \sum_{\nu=1}^{k} r^{n_j + n_\nu} \sum_{h=1}^{\infty} h^{-2} \, \delta_{r^{n_j}} (h) \, \delta_{r^{n_\nu}} (h). \tag{3.7}$$

If the matrix $\| \, a_{j,\nu} \, \| \; (1 \leq j, \nu \leq k)$ is symmetric then we have

$$\sum_{j=1}^{k} \sum_{\nu=1}^{k} a_{j,\nu} = 2 \sum_{j=1}^{k} \sum_{\nu=1}^{j} a_{j,\nu} - \sum_{j=1}^{k} a_{j,j}.$$

By here and (3.7) we get

$$(NF_N (\sigma))^2 \leq 4(r-1)^2 \sum_{j=1}^{k} \sum_{\nu=1}^{j} r^{n_j + n_\nu} \sum_{h=1}^{\infty} h^{-2} \, \delta_{r^{n_j}}(h) \, \delta_{r^{n_\nu}}(h)$$
$$- 2(r-1)^2 \sum_{j=1}^{k} r^{2n_j} \sum_{h=1}^{\infty} h^{-2} \, \delta_{r^{n_j}}(h). \tag{3.8}$$

For j and ν such that $1 \leq \nu \leq j \leq k$, we have

$$\delta_{r^{n_j}}(h) \, \delta_{r^{n_\nu}}(h) = \delta_{r^{n_j}}(h), \tag{3.9}$$

for every integer h.

Beside this we have

$$\sum_{h=1}^{\infty} h^{-2}\, \delta_{r^{n_j}}(h) = \pi^2/6r^{2n_j}. \tag{3.10}$$

By (3.8), (3.9) and (3.10) we have

$$(NF_N(\sigma))^2 \leq (2\pi^2\,(r-1)^2/3) \sum_{j=1}^{k} \sum_{\nu=1}^{j} r^{n_\nu - n_j} - (\pi^2/3)(r-1)^2 k \tag{3.11}$$

For the sum in last equality holds, that

$$\sum_{j=1}^{k}\sum_{\nu=1}^{j} r^{n_\nu - n_j} = \sum_{\nu=1}^{k} r^{n_\nu} \sum_{j=\nu}^{k} r^{-n_j} < \sum_{\nu=1}^{k} r^{n_\nu} \sum_{n=n_\nu}^{\infty} r^{-n} = (rk)/(r-1) \tag{3.12}$$

From (3.11) and (3.12) we have

$$(NF_N(\sigma))^2 \leq (\pi^2/3)(r^2-1)k \tag{3.13}$$

From (3.6) we get that

$$N \geq \sum_{j=1}^{k} r^{n_j} \geq \sum_{j=0}^{k-1} r^j = (r^k - 1)/(r-1).$$

Thus we discover

$$k \leq (log((r-1)N+1))/\,log\, r \tag{3.14}$$

From (3.13) and (3.14) we have

$$F_N(\sigma) \leq \pi((r^2-1)/3\, log\, r)^{1/2}\, N^{-1}\, (log((r-1)N+1))^{1/2}.$$

The Theorem 3.1 is proved.

In the case, where the guiding matrix $(v_{s,j})$ is a unit matrix I, the sequence which is bred by I is called Van der Corput-Halton's sequence. In 1935 it was first introduced by Van der Corput [18] and generalized in 1960 by Halton [19].

In this case the operation * turns out to be a simple addition.

By $\varphi_r(i)(i = 0, 1, \cdots)$ we signify the general term of the Van der Corput-Halton-sequence.

For $r = 2$ the sequence of general terms $\varphi_2(i)(i = 0, 1, \cdots)$ is called Van der Corput-sequence.

By Theorem 3.1 we can get the following corollaries.

COROLLARY 3.1. Let $\sigma = (\varphi_r(i))_{i=0}^{\infty}$ be the Van der Corput-Halton-sequence. Then for every positive integer N, we have

$$F_N(\sigma) \leq c(r)\, N^{-1}\, (log((r-1)N+1))^{1/2},$$

where the constant $c(r)$ is determined by the equality (3.1).

COROLLARY 3.2. Let $\sigma = (\varphi(i))_{i=0}^{\infty}$ be the Van der Corput-sequence. Then for every $N \geq 1$ we have

$$F_N(\sigma) < 4N^{-1}\, (log\, N)^{1/2}.$$

COROLLARY 3.3. Let $\sigma = (\varphi(i))_{i=0}^{\infty}$ be arbitrary binary P_r-sequence. Then

$$\overline{lim}_{N\to\infty}\, (NF_N(\sigma))/\, (log\, N)^{1/2} \leq \pi/\, (log\, 2)^{1/2} = 3,7773\cdots.$$

We note that the Corollary 3.1 and Corollary 3.2 are announced without proof by Proinov and Grozdanov [20] and proved by Proinov and Grozdanov [21].

4. ON QUADRATIC DISCREPANCY OF THE SYMMETRIC SEQUENCE PRODUCED BY THE ARBITRARY P_r-SEQUENCE.

In this section, we given an application of Theorem 3.1 to the problem of finding infinite sequences in E, with the best possible order of magnitude for the quadratic discrepancy.

We need the notion of symmetric sequence (see [11]). A sequence $\sigma = (x_n)_{n=0}^{\infty}$ in E is called symmetric of for every integer $n \geq 0$ we have $x_{2n} + x_{2n+1} = 1$. A symmetric sequence $\breve{\sigma} = (b_n)_{n=0}^{\infty}$ in E is said to be produced by an infinite sequence $\sigma = (a_n)_{n=0}^{\infty}$, if for every integer $n \geq 0$ we either have $a_n = b_{2n}$ or $a_n = b_{2n+1}$. Obviously, every infinite sequence in E produce at least one symmetric sequence.

By Sobol ([5], p. 117) is clear that the exact order of quadratic discrepancy of P_2-sequence is $O(N^{-1}\, log\, N)$.

We shall prove that the quadratic discrepancy of arbitrary symmetric sequence, which is produced by arbitrary P_r-sequence has exact order $O(N^{-1}\, (log\, N)^{1/2})$. In the foundation of this problem stands Theorem A, proved by Proinov and Grozdanov [20].

By this and Theorem 3 1 follows

THEOREM 4 1 Let $\tilde{\sigma}$ be an arbitrary symmetric sequence in E, which is produced by an arbitrary P_r-sequence Then for every integer $N \geq 2$ we have

$$T_N(\tilde{\sigma}) < c(r)N^{-1}\,(log(r-1)N))^{1/2} + N^{-1},$$

where $c(r)$ is defined by the equality (3 1)

From Theorem 4 1 for the case $r = 2$ we have

$$\overline{lim}_{N\to\infty}\, NT_N\,(\tilde{\sigma})/\,(log\,N)^{1/2} \leq 1/\,(log\,2)^{1/2} = 1,201\cdots,$$

for every symmetric sequence $\tilde{\sigma}$ produced by the P_2-sequence

We note that Faure [22] proved that for the symmetric sequence $\tilde{\sigma}$, produced by the Van der Corput-sequence, the constant $\overline{lim}_{N\to\infty}\,(NT_N(\tilde{\sigma})/\,(log\,N)^{1/2})$ is between $0,298$ and $0,321$

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