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COMPARISON AND OSCILLATION RESULTS FOR DELAY DIFFERENCE EQUATIONS WITH OSCILLATING COEFFICIENTS

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ABSTRACT. In this paper we consider the oscillation of the delay difference equation with oscillating coefficients

$$x_{n+1} - x_n + \sum_{i=1}^{n} p_i(n) x_{n-k_i(n)} = 0, \qquad n \geqslant 0.$$

Some comparison and oscillation results are obtained.

KEY WORDS AND PHRASES. Oscillation, delay difference equation, oscillating coefficient
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1. INTRODUCTION.

Let $R=(-\infty,\infty)$ and $Z=\{0,1,2,\cdots\}$. Consider the delay difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i(n)} = 0, n \geqslant 0, \tag{1.1}$$

and the delay difference inequality

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_i(n)} \leqslant 0, n \geqslant 0, \tag{1.2}$$

where

$$\begin{aligned} &k_i(n) \in Z \qquad for\ n \in Z, \\ &p_i(n) \in R \qquad for\ n \in Z, \end{aligned} \tag{1.3}$$

and there exist positive $k_1, k_m \in Z$, such that

$$k_1 \geqslant k_1(n) \geqslant k_2(n) \geqslant \cdots \geqslant k_m(n) \geqslant k_m > 0, \tag{1.4}$$

and the following condition(A) is satisfied for k_1,

$$(A) \quad \begin{cases} (i) & p_1(n) \in R^+, p_1(n) + p_2(n) \in R^+, \cdots, \sum_{i=1}^{m} p_i(n) \in R^+ = [0,\infty); \\ (ii) & For\ any\ N \in Z, there\ exists\ N_1 \in Z\ such\ that\ p_i(n) \in R^+ \\ & for\ any\ n \in [N_1, N_1+k_1], where\ i = 1,2,\cdots,m, \end{cases}$$

where $[N_1, N_1+k_1] = \{N_1, N_1+1, \cdots, N_1+k_1\}$.

Let $n_0 - k = \inf_{n \in Z}\{n - k_1(n)\}$ and $n_0 \geqslant 0$. By a solution of (1.1)(or (1.2)) we mean a sequence $\{x_n\}$ which is defined for $n \geqslant n_0 - k$ and satisfies (1.1)(or (2.2)) for $n \geqslant n_0$. With Eq. (1.1) and with a given "initial point" $n_0 \geqslant 0$ and "initial condition" $a_{n_0} - k, a_{n_0}$

$-k+1,\cdots,a_{n_0}$, Eq. (1.1) has a unique solution $\{x_n\}$ which satisfies

$$x_j = a_j \quad for \quad j = n_0 - k, n_0 - k + 1, \cdots, n_0.$$

A solution $\{x_n\}$ of Eq. (1.1) is said to be oscillatory if the terms x_n of the sequence are not eventually positive or eventually negative. Otherwise, the solution is called nonoscillatory. Eq. (1.1) is called oscillatory if every solution of the equation oscillates.

A solution $\{x_n\}$ of Eq. (1.1) through an initial point n_0 is said to be positive if the terms x_n of the solution $\{x_n\}$ are posivive for all $n \geqslant n_0 - k$.

Recently there has been a lot of interest in the oscillations of delay difference equations . See, for example, [1]—[5] and the references cited therein. Our aim in this paper is to study the oscillation of Eq. (1.1). Some necessary and sufficient conditions and some easily verifiable sufficient conditions are established for oscillation of Eq. (1.1).

2. MAIN RESULTS.

Consider a sequence $\{A_n^{(r)}\}_r^\infty = 0$, which is defined as

$$A_n^{(0)} = 0, \qquad for\ n \geqslant n_0 - k_1,$$

and for $r \geqslant 1$,

$$A_n^{(r)} = \begin{cases} 0, & for\ n = n_0 - k_1, n_0 - k_1 + 1, \cdots, n_0 - 1, \\ \sum_{i=1}^{m} p_i(n) \prod_{j=n-k_i(n)}^{n-1} (1 - A_j^{(r-1)})^{-1}, & for\ n \geqslant n_0. \end{cases} \tag{2.1}$$

First, we introduce the following Lemmas.

LEMMA 1. Assume that condition (A) holds for k_1 and $\{x_n\}$ is an eventually positive solution of (1.1). Then, $\{x_n\}$ must be eventually nonincerasing. And, we have

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_m} \leqslant 0. \tag{2.2}$$

PROOF. By condition (A), there exists $N_1 \geqslant n_0$ such that

$$x_{n+1} - x_n \leqslant x_{n+1} - x_n + \sum_{n=1}^{m} p_i(n) x_{n-k_i(n)} = 0, \qquad for\ n \in [N_1, N_1 + k_1],$$

that is, $\{x_n\}$ is nonincreasing on $[N_1, N_1 + k_1]$.

We claim that $\{x_n\}$ is nonincreasing for $n \in [N_1 + k_1, N_1 + k_1 + k_m]$.

In fact, for any $n \in [N_1 + k_1, N_1 + k_1 + k_m]$, we have $n - k_i(n) \in [N_1, N_1 + k_1]$. By (1.4) and nonincreasing property of $\{x_n\}$ on $[N_1, N_1 + k_1]$ we have that

$$x_{n-k_1} \geqslant x_{n-k_1(n)} \geqslant \cdots \geqslant x_{n-k_m(n)} \geqslant x_{n-k_m} > 0, \tag{2.3}$$

for any $n \in [N_1 + k_1, N_1 + k_1 + k_m]$.

So, we get that

$$\begin{aligned} x_{n+1} - x_n &= - \sum_{i=1}^{m} p_i(n) x_{n-k_i(n)} \\ &= - p_1(n) x_{n-k_1(n)} - \sum_{i=1}^{m} p_i(n) x_{n-k_2(n)} \end{aligned}$$

$$\leqslant - p_1(n)x_{n-k_1(n)} - \sum_{i=2}^{m} p_i(n)x_{n-k_i(n)}$$
$$\leqslant - (p_1(n)x_n + p_2(n))x_{n-k_2(n)} - \sum_{i=3}^{m} p_i(n)x_{n-k_i(n)}$$
$$\leqslant \cdots\cdots$$
$$\leqslant - (p_1(n) + p_2(n) + \cdots + p_m(n))x_{n-k_m(n)}$$
$$\leqslant - (p_1(n) + p_2(n) + \cdots + p_m(n))x_{n-k_m} \leqslant 0.$$

Therefore, x_n is nonincreasing on $[N_1+k_1, N_1+k_1+k_m]$. Similarly, we can show that $\{x_n\}$ is nonincreasing for all $n \geqslant N_1+k_1$ and the proof is complete.

LEMMA 2. Assume that condition (A) holds for k_1 and (1.1) has a positive solution. Then there exists a sequence $\{\alpha_n\}_{n=n_0-k}$ such that the following statements are true:

(i) $\alpha_n = \sum_{i=1}^{m} p_i(n) \prod_{j=n-k_i(n)}^{n-1} (1-\alpha_j)^{-1}$, for $n \geqslant n_0$;

(ii) $\alpha_n < 1$ for $n = n_0-k_1, n_0-k_1+1, \cdots, n_0-1$ and eventually $0 \leqslant \alpha_n < 1$ for $n \geqslant N_1$.

PROOF. Assume that $\{x_n\}$ is a solution of (1.1) and $x_n > 0$ for all $n \geqslant n_0-k$. Set

$$\alpha_n = 1 - \frac{x_{n+1}}{x_n}, \qquad for\ n \geqslant n_0 - k. \tag{2.4}$$

Then

$$\frac{x_{n-k_i(n)}}{x_n} = \frac{x_{n-k_i(n)}}{x_{n-k_i(n)+1}} \cdot \frac{x_{n-k_i(n)+1}}{x_{n-k_i(n)+2}} \cdots \cdots \frac{x_{n-1}}{x_n}$$
$$= \prod_{j=n-k_i(n)}^{n-1} (1-\alpha_j)^{-1}, n \geqslant n_0. \tag{2.5}$$

From (1.1), we have that

$$\frac{x_{n+1}}{x_n} - 1 + \sum_{i=1}^{m} p_i(n) \frac{x_{n-k_i(n)}}{x_n} = 0, \qquad n \geqslant n_0. \tag{2.6}$$

Hence, by substituting (2.4) and (2.5) into (2.6), we get

$$\alpha_n = \sum_{i=1}^{m} p_i(n) \prod_{j=n-k_i(n)}^{n-1} (1-\alpha_j)^{-1}, \qquad n \geqslant n_0, \tag{2.7}$$

that is, (i) is satisfied. Clearly, $\alpha_n < 1$ for $n \geqslant n_0-k_1$. By Lemma 1, we have eventually $0 \leqslant \alpha_n < 1$. The proof of Lemma 2 is completed.

LEMMA 3. Assume that condition (A) holds for k_1 and (1.1) has a positive solution through n_0. Then the sequence (2.1) is well defined for $n \geqslant n_0$ and satisfies

(i) $0 \leqslant A_n^{(r)} \leqslant A_n^{(r+1)}$, for $n \geqslant n_0$ and $r \geqslant 0$;

(ii) $\lim_{r\to\infty} A_n^{(r)} \stackrel{\text{def.}}{=\!=} A_n < 1$, for $n \geqslant n_0$.

PROOF. Assume that $\{x_n\}$ is a positive solution of (1.1) through n_0. By Lemma 1, without loss of generality, we assume $\{x_n\}$ nonincreasing as $n \geqslant n_0-k_1$.

Set $\alpha_n = 1 - \frac{x_{n+1}}{x_n}$ for $n \geqslant n_0-k_1$. Then from (2.4), and Lemma 1, and by a simple induction, it can be seen that

$$0 \leqslant A_n^{(r)} \leqslant A_n^{(r+1)} \leqslant \alpha_n < 1, \qquad r \geqslant 0. \tag{2.8}$$

In fact, $A^{(0)}_n = 0$, and

$$A_n^{(1)} = \begin{cases} 0, & for\ n = n_0 - k_1, n_0 - k_1 + 1, \cdots, n_0 - 1. \\ \sum_{i=1}^{n} p_i(n), & for\ n \geqslant n_0. \end{cases}$$

So, we have $A_n^{(1)} \geqslant A_n^{(0)} \geqslant 0$. Assume that $A_n^{(r)} \geqslant A_n^{(r-1)} \geqslant 0$. Then, for $n \geqslant n_0$, we have

$$A_n^{(r+1)} = \sum_{i=1}^{m} p_i(n) \prod_{j=n-k_i(n)}^{n-1} (1 - A_j^{(r)})^{-1},$$

$$A_n^{(r)} = \sum_{i=1}^{m} p_i(n) \prod_{j=n-k_i(n)}^{n-1} (1 - A_j^{(r-1)})^{-1},$$

Hence, we get

$$A_n^{(r+1)} - A_n^{(r)} = \sum_{i=1}^{m} p_i(n) \prod_{j=n-k_i(n)}^{n-1} (1 - A_j^{(r-1)})^{-1} \left[\prod_{j=n-k_i(n)}^{n-1} \frac{(1 - A_j^{(r-1)})}{(1 - A_j^{(r)})} - 1 \right]$$
$$\geqslant \left(\sum_{i=1}^{m} p_i(n) \right) \prod_{j=n-k_m}^{n-1} (1 - A_j^{(r-1)})^{-1} \left[\prod_{j=n-k_m}^{n-1} \frac{(1 - A_j^{(r-1)})}{(1 - A_j^{(r)})} - 1 \right] \geqslant 0,$$

so, we know that $0 \leqslant A_n^{(r)} \leqslant A_n^{(r+1)}$ for all $r \geqslant 0$.

By (2.7), we use the induction to get

$$A_n^{(r)} \leqslant \alpha_n < 1, \qquad for\ any\ r \geqslant 0, n \geqslant n_0 - k_1,$$

which implies that (2.8) holds. Hence it is easy to get that

$$\lim_{r \to \infty} A_n^{(r)} = A_n < 1, n \geqslant n_0,$$

and the proof is complete.

The next result is a generalization of Theorem 1 in [5].

THEOREM 1. Assume that (1.3), (1.4) and condition (A) hold for k_1. Then the following statements are equivalent:

(a) Eq. (1.1) has a positive solution through the initial point $n_0 \geqslant 0$;

(b) The inequality (1.2) has an eventually positive solution;

(c) The sequence $\{A_n^{(r)}\}_{r=0}^{\infty}$ which is well defined by (2.1) converges to a limit A_n with $0 \leqslant A_n < 1$ for each $n \geqslant n_0 > 0$.

PROOF. (a)$\Rightarrow$(b). This is obvious.

(b)$\Rightarrow$(c). Assume that $x_n > 0$ for $n \geqslant n_0 - k$ which is a solution of (1.2). Set

$$\tilde{\alpha}_n = 1 - \frac{x_{n+1}}{x_n}, n \geqslant n_0 - k.$$

Then,

$$\frac{x_{n-k_i(n)}}{x_n} = \prod_{j=n-k_i(n)}^{n-1} (1 - \tilde{\alpha}_j)^{-1}, \qquad n \geqslant n_0. \tag{2.9}$$

Thus from (1.2), it follows that

$$\sum_{i=1}^{m} p_i(n) \prod_{j=n-k_i(n)}^{n-1} (1 - \tilde{\alpha}_j)^{-1} \leqslant \tilde{\alpha}_n, \qquad n \geqslant n_0.$$

By (2.1) and a simple induction which is the same as that of Lemma 3, we hvae that

$$0 \leqslant A_n^{(r)} \leqslant A_n^{(r+1)} \leqslant \tilde{\alpha}_n < 1 \quad for\ r \geqslant 0\ and\ n \geqslant n_0,$$

which implies that the sequence $\{A_n^{(r)}\}$ converges to finite limit A_n with $0 \leqslant A_n < 1$ for each fixed $n \geqslant n_0$.

(c)$\Rightarrow$(a). It is similar to that of Theorem 1 in [5].

The proof of Theorem 1 is complete.

COROLLARY 1. Assume that (1. 3), (1. 4) and condition (A) hold for k_1. Then the following statements are equivalent:

(a) Eq. (1. 1) is oscillating;

(b) Inequality (1. 2) has no eventually positive solution.

Assume that $P_n \in R^+$, $n \in Z$, and $k \in Z$. The following theorem of oscillation was obtained in [3].

THEOREM A. Consider the delay difference equation

$$A_{n+1} - A_n + P_n A_{n-k} = 0, \qquad n = 0,1,2,\cdots. \tag{$*$}$$

If

$$\liminf_{n\to\infty} [\frac{1}{k}\sum_{i=n-k}^{n-1} P_i] > \frac{k^k}{(k+1)^{k+1}}, \tag{2.10}$$

then all solutions of (*) are oscillatory.

In [5], the following conclusion was obtained.

THEOREM B. Consider the following inequality

$$A_{n+1} - A_n + P_n A_{n-k} \leqslant 0. \tag{$**$}$$

The following conclusions are equivalent:

(i) (*) is oscillatory;

(ii) (* *) has no eventually positive solution.

We can obtain the following theorem.

THEOREM 2. Assume that (1. 3), (1. 4) and condition (A) hold for k_1, and the equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_m} = 0 \tag{2.11}$$

is oscillatory, then (1. 1) must be oscillatory.

PROOF. Let $\{x_n\}$ be a nonoscillatory solution of (1. 1). As the opposite of a solution of (1. 1) is still a solution of (1. 1), we can assume that $x_n > 0$ for $n \geqslant n_0$. By Lemma 1, we have

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i(n) x_{n-k_m} \leqslant 0, \tag{2.12}$$

that is, inequality (2. 12) has a positive solution. On the other hand, by Theorem B we know that (2. 12) has no eventually positive solution. This is a contradiction. So, (1. 1) must be oscillatory. The proof of Theorem 2 is complete.

COROLLARY 2. Assume that (1. 1), (1. 4), condition (A) hold for k_1, and

$$\liminf_{n\to\infty} \frac{1}{k_m}\sum_{i=n-k_m}^{n-1}\sum_{j=1}^{m} p_j(i) > \frac{k_m^{k_m}}{(k_m+1)^{k_m+1}}. \tag{2.13}$$

Then (1. 1) is oscillatory.

PROOF. By Theorem 2 and Theorem A, we obtain the conclusion.

COROLLARY 3. Assume that (1.3), (1.4), condition(A) hold for k_1, and

$$\sum_{i=1}^{m} p_i(n) \geqslant P_n \quad \textit{for } n \textit{ sufficiently large.} \tag{2.14}$$

Then, if (*) is oscillatory, (1.1) must be oscillatory.

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