Internat. J. Math. & Math. Sci.
VOL. 19 NO. 2 (1996) 243-252

FIXED POINT THEOREMS IN METRIC SPACES AND PROBABILISTIC METRIC SPACES

YEOL JE CHO and KEUN SAENG PARK
Department of Mathematics
Gyeongsang National University
Jinju 660-701, KOREA

SHIH-SEN CHANG
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064, PEOPLE'S REPUBLIC OF CHINA

(Received January 26, 1993 and in revised form April 19, 1995)

ABSTRACT. In this paper, we prove some common fixed point theorems for compatible mappings of type (A) in metric spaces and probabilistic metric spaces Also, we extend Caristi's fixed point theorem and Ekeland's variational principle in metric spaces to probabilistic metric spaces

KEY WORDS AND PHRASES. Non-Archimedean Menger probabilistic metric spaces, compatible and compatible mappings of type (A), common fixed points
1980 AMS SUBJECT CLASSIFICATION CODES. 47H10, 54H25

1. INTRODUCTION AND PRELIMINARIES

Recently, a number of fixed point theorems for single-valued and multi-valued mappings in probabilistic metric spaces have been proved by many authors ([1]-[3], [5]-[12], [14]-[20], [22], [25]) Since every metric space is a probabilistic metric space, we can use many results in probabilistic metric spaces to prove some fixed point theorems in metric spaces

In this paper, first, we prove some common fixed point theorems in metric spaces and probabilistic metric spaces Secondly, we give some convergence theorems for sequences of self-mappings on a metric space Finally, we extend Caristi's fixed point theorem and Ekeland's variational principle in metric spaces to probabilistic metric spaces

For notations and properties of probabilistic metric spaces, refer to [6], [9], [18] and [19]

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}^+$ the set of non-negative real numbers A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is a nondecreasing and left continuous function with $\inf F = 0$ and $\sup F = 1$ We will denote D by the set of all distribution functions

DEFINITION 1.1. A probabilistic metric space (briefly, a PM-space) is a pair (X, F), where X is a nonempty set and F is a mapping from $X \times X$ to D. For $(u, v) \in X \times X$, the distribution function $F(u, v)$ is denoted by $F_{u,v}$ The functions $F_{u,v}$ are assumed to satisfy the following conditions

(P1) $F_{u,v}(x) = 1$ for every $x > 0$ if and only if $u = v$,

(P2) $F_{u,v}(0) = 0$ for every $u, v \in X$,

(P3) $F_{u,v}(x) = F_{v,u}(x)$ for every $u, v \in X$,

(P4) If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$, then $F_{u,w}(x + y) = 1$ for every $u, v, w \in X$

DEFINITION 1.2. A t-norm is a function $\triangle : [0, 1] \to [0, 1]$ which is associative, commutative, nondecreasing in each coordinate and $\triangle (a, 1) = a$ for every $a \in [0, 1]$

DEFINITION 1.3. A Menger PM-space is a triple $(X, F, \triangle)$, where (X, F) is a PM-space and $\triangle$ is a t-norm with the following condition

(P5) $F_{u,w}(x+y) \geq \triangle(F_{u,v}(x), F_{v,w}(y))$ for every $u, v, w \in X$ and $x, y \in \mathbb{R}^+$

DEFINITION 1.4. A non-Archimedean Menger PM-space (an N A Menger PM-space) is a triple $(X, F, \triangle)$, where $\triangle$ is a t-norm and the space (X, F) satisfies the conditions (P1) ~ (P3) and (P6)

(P6) $F_{u,w}(\max\{t_1, t_1\}) \geq \triangle(F_{u,v}(t_1), F_{v,w}(t_2))$ for all $u, v, w \in X$ and $t_1, t_2 \geq 0$

The concept of neighborhoods in PM-spaces was introduced by Schweizer and Sklar [18] If $u \in X$, $\epsilon > 0$ and $\lambda \in (0,1)$, then the (ϵ, λ)-neighborhood of u, denoted by $U_u(\epsilon, \lambda)$, is defined by $U_u(\epsilon, \lambda) = \{v \in X : F_{u,v}(\epsilon) > 1 - \lambda\}$

If $(X, F, \triangle)$ is a Menger PM-space with the continuous t-norm $\triangle$, then the family $\{U_u(\epsilon, \lambda) : u \in X, \epsilon > 0, \lambda \in (0,1)\}$ of neighborhoods induces a Hausdorff topology on X, which is denoted by the (ϵ, λ)-topology τ

DEFINITION 1.5. A PM-space (X, F) is said to be of type $(C)_g$ if there exists an element $g \in \Omega$ such that

$$g(F_{x,y}(t)) \leq g(F_{x,z}(t)) + g(F_{z,y}(t)) \quad \text{for all} \quad x, y, z \in X \quad \text{and} \quad t \geq 0,$$

where $\Omega = \{g : g : [0,1] \to [0,\infty]$ is continuous, strictly decreasing, $g(1) = 0$ and $g(0) < \infty\}$

DEFINITION 1.6. An N A Menger PM-space $(X, F, \triangle)$ is said to be of type $(D)_g$ if there exists an element $g \in \Omega$ such that

$$g(\triangle(s,t)) \leq g(s) + g(t) \quad \text{for all} \quad s, t \in [0,1]$$

REMARK 1. ([9]) (1) If an N A Menger PM-space $(X, F, \triangle)$ is of type $(D)_g$, then $(X, F, \triangle)$ is of type $(C)_g$

(2) If $(X, F, \triangle)$ is an N A Menger PM-space and $\triangle \geq \triangle_m$, where $\triangle_m(s,t) = \max\{s+t-1, 0\}$, then $(X, F, \triangle)$ is of type $(D)_g$ for $g \in \Omega$ defined by $g(t) = 1 - t$

(3) If a PM-space (X, F) is of type $(C)_g$, then it is metrizable, if the metric d on X is defined by

$$(*) \qquad d(x,y) = \int_0^1 g(F_{x,y}(t))dt \quad \text{for all} \quad x, y \in X$$

(4) If an N A. Menger PM-space $(X, F, \triangle)$ is of type $(D)_g$, then it is metrizable, where the metric d on X is defined by $(*)$ On the other hand, the (ϵ, λ)-topology τ coincides with the topology induced by the metric d defined by $(*)$.

(5) If $(X, F, \triangle)$ is an N A. Menger PM-space with the t-norm $\triangle$ such that $\triangle(s,t) \geq \triangle_m(s,t) = \max\{s+t-1, 0\}$ for $s, t \in [0,1]$, then (4) is also true

2. FIXED POINT THEOREMS IN METRIC SPACES

In this section, we give several fixed point theorems for compatible mappings of type (A) in a metric space (X, d). The following definitions and properties of compatible mappings and compatible mappings of type (A) are given in [17]

DEFINITION 2.1. Let $S, T : (X, d) \to (X, d)$ be mappings S and T are said to be compatible if

$$\lim_{n\to\infty} d(ST(x_n), TS(x_n)) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t$ for some t in X

DEFINITION 2.2. Let $S, T : (X, d) \to (X, d)$ be mappings. S and T are said to be compatible type (A) if

$$\lim_{n\to\infty} d(TS(x_n), SS(x_n)) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(ST(x_n), TT(x_n)) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t$ for some t in X

The following propositions show that Definitions 2 1 and 2 2 are equivalent under some conditions

PROPOSITION 2.1. Let $S, T : (X, d) \to (X, d)$ be continuous mappings If S and T are compatible, then they are compatible of type (A)

PROPOSITION 2.2. Let $S, T : (X, d) \to (X, d)$ be compatible mappings of type (A) If one of S and T is continuous, then S and T are compatible

The following is a direct consequence of Propositions 2 1 and 2 2

PROPOSITION 2.3. Let $S, T : (X, d) \to (X, d)$ be continuous mappings Then S and T are compatible if and only if they are compatible of type (A)

REMARK 2. In [17], we can find two examples that Proposition 2 3 is not true if S and T are not continuous on X

Next, we give some properties of compatible mappings of type (A) for our main theorems

PROPOSITION 2.4. Let $S, T : (X, d) \to (X, d)$ be mappings If S and T are compatible mappings of type (A) and $S(t) = T(t)$ for some $t \in X$, then $ST(t) = TT(t) = TS(t) = SS(t)$

PROPOSITION 2.5. Let $S, T : (X, d) \to (X, d)$ be mappings Let S and T be compatible mappings of type (A) and let $S(x_n), T(x_n) \to t$ as $n \to \infty$ for some $t \in X$ Then we have the following

(1) $\lim_{n\to\infty} TS(x_n)=S(t)$ if S is continuous at t,

(2) $ST(t) = TS(t)$ and $S(t) = T(t)$ if S and T are continuous at t

Let Φ be the family of all mappings $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ such that ϕ is upper semicontinuous, non-decreasing in each coordinate variable, and for any $t > 0$,

$$\phi(t, t, 0, \alpha t, t) \leq \beta t \quad \text{and} \quad \phi(t, t, 0, 0, \alpha t) \leq \beta t\,,$$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$, and

$$\gamma(t) = \phi(t, t, a_1 t, a_2 t, a_3 t) < t\,,$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping and $a_1 + a_2 + a_3 = 4$

For convenience, we shall write Sx for $S(x)$

LEMMA 2.1 ([21]) For any $t > 0$, $\gamma(t) < 1$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the n-times composition of γ

Let A, B, S, T be mappings from a metric space (X, d) into itself such that

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X)\,, \tag{2 1}$$

there exists $\phi \in \Phi$ such that (2 2)

$$d(Ax, By) \leq \phi(d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx), d(Sx, Ty)) \quad \text{for all} \quad x, y \in X$$

Then, by (2 1), since $A(X) \subset T(X)$, for any point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$ Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{2 3}$$

LEMMA 2.2. $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, where $\{y_n\}$ is the sequence in X defined by (2 3)

PROOF. Let $d_n = d(y_n, y_{n+1})$, $n = 0, 1, 2, \ldots$ Now, we shall prove that the sequence $\{d_n\}$ is non-decreasing in $\mathbb{R}^+$, that is, $d_n \leq d_{n-1}$ for $n = 0, 1, 2, \ldots$ By (2 2), we have

$$\begin{aligned} d_{2n} &= d(y_{2n}, y_{2n+1}) \\ &= d(Ax_{2n}, Bx_{2n+1}) \\ &\le \phi(d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), \\ &\quad d(Ax_{2n}, Tx_{2n+1}), d(Bx_{2n+1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1})) \\ &= \phi(d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1}), d(y_{2n-1}, y_{2n})) \\ &\le \phi(d_{2n-1}, d_{2n}, 0, d_{2n-1} + d_{2n}, d_{2n-1}). \end{aligned} \tag{2 4}$$

Suppose that $d_{n-1} < d_n$ for some n Then, for some $\alpha < 2$, $d_{n-1} + d_n = \alpha d_n$ Since ϕ is non-decreasing in each coordinate variable and $\beta < 1$ for some $\alpha < 2$, by (2 4), we have

$$d_{2n} \le \phi(d_{2n}, d_{2n}, 0, \alpha d_{2n}, d_{2n}) \le \beta d_{2n} < d_{2n}.$$

Similarly,

$$d_{2n+1} \le \phi(d_{2n+1}, d_{2n+1}, 0, \alpha d_{2n+1}, d_{2n+1}) \le \beta d_{2n+1} < d_{2n+1}.$$

Hence, for every $n = 0, 1, 2, \ldots, d_n \le \beta d_n < d_n$, which is a contradiction Therefore, $\{d_{2n}\}$ is a non-increasing sequence in $\mathbb{R}^+$ Now, again by (2 2),

$$\begin{aligned} d_1 &= d(y_1, y_2) \\ &= d(Ax_1, Bx_2) \\ &\le \phi(d(Ax_2, Sx_2), d(Bx_1, Tx_1), d(Ax_2, Tx_1), d(Bx_1, Sx_2), d(Sx_2, Tx_1)) \\ &= \phi(d(y_2, y_1), d(y_1, y_0), d(y_2, y_0), d(y_1, y_1), d(y_1, y_0)) \\ &\le \phi(d_1, d_0, d_0 + d_1, 0, d_0) \\ &\le \phi(d_0, d_0, 2d_0, d_0, d_0) \\ &= \gamma(d_0). \end{aligned}$$

In general, $d_n \le \gamma^n(d_0)$ for $n = 0, 1, 2, \ldots,$ which implies that, if $d_0 > 0$, then, by Lemma 2 1, we have

$$\lim_{n\to\infty} d_n \le \lim_{n\to\infty} \gamma^n(d_0) = 0.$$

Therefore, it follows that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

For $d_0 = 0$, since $\{d_n\}$ is non-increasing, we have clearly $\lim_{n\to\infty} d_n = 0$ This completes the proof

LEMMA 2.3. The sequence $\{y_n\}$ defined by (2.3) is a Cauchy sequence in X.

PROOF. By Lemma 2.2, it is sufficient to prove that $\{y_{2n}\}$ is a Cauchy sequence in X. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence in X Then there is an $\epsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) \ge 2k$ such that

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon. \tag{2 5}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (2.5), that is,

$$d(y_{2n(k)}, y_{2m(k)-2}) \le \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \tag{2 6}$$

Then for each even integer $2k$, we have

$$\epsilon < d(y_{2n(k)}, y_{2m(k)}) \le d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}).$$

It follows from Lemma 2 2 and (2.6) that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{2 7}$$

By the triangle inequality, we obtain

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d(y_{2m(k)-1}, y_{2m(k)})$$

$$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2n(k)}, y_{2n(k)+1})$$

From Lemma 2 2 and (2 7), as $k \to \infty$, it follows that

$$d(y_{2n(k)+1}, y_{2m(k)-1}) \to \epsilon \quad \text{and} \quad d(y_{2n(k)+1}, y_{2m(k)-1}) \to \epsilon\,. \tag{2 8}$$

Therefore, by (2 2) and (2 3), we have

$$\begin{aligned} d(y_{2n(k)}, y_{2m(k)}) &\le d(y_{2n(k)}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2m(k)}) \\ &= d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2m(k)}, Bx_{2n(k)+1}) \\ &\le d(y_{2n(k)}, y_{2n(k)+1}) + \phi(d(Ax_{2m(k)}, Sx_{2m(k)}), d(Bx_{2n(k)+1}, Tx_{2n(k)+1}), \\ &\quad d(Ax_{2m(k)}, Tx_{2n(k)+1}), d(Bx_{2n(k)+1}, Sx_{2m(k)}), d(Sx_{2m(k)}, Tx_{2n(k)+1})) \\ &= d(y_{2n(k)}, y_{2n(k)+1}) + \phi(d(y_{2m(k)}, y_{2m(k)-1}), d(y_{2n(k)+1}, y_{2n(k)}), \\ &\quad d(y_{2m(k)}, y_{2n(k)}), d(y_{2n(k)+1}, y_{2m(k)-1}), d(y_{2m(k)-1}, y_{2n(k)}))\,. \end{aligned} \tag{2 9}$$

Since ϕ is upper semicontinuous, as $k \to \infty$ in (3 9), by Lemma 2 2, (2 7) and (2 8), we have

$$\epsilon \le \phi(0, 0, \epsilon, \epsilon, \epsilon) \le \gamma(\epsilon) < \epsilon\,,$$

which is a contradiction Therefore, the sequence $\{y_{2n}\}$ is a Cauchy sequence in X and so is $\{y_n\}$ This completes the proof

Now, we are ready to prove a main theorem in this section

THEOREM 2.4. Let A, B, S, and T be mappings from a complete metric space (X, d) into itself satisfying the conditions (2 1), (2 2), (2 10) and (2 11)

one of A, B, S, and T is continuous, (2 10)

the pairs A, S and B, T are compatible of type (A) (2 11)

PROOF. By Lemma 2 3, the sequence $\{y_n\}$ defined by (2 3) is a Cauchy sequence in X and so, since (X, d) is complete, it converges to a point z in X On the other hand, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to the point z

Now, suppose that T is continuous. Since B and T are compatible of type (A), by Proposition 2.5, $BTx_{2n+1}, TTx_{2n+1} \to Tz$ as $n \to \infty$ Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (2 2), we have

$$\begin{aligned} d(Ax_{2n}, BTx_{2n+1}) \le \phi(&d(Ax_{2n}, Sx_{2n}), d(BTx_{2n+1}, TTx_{2n+1}), \\ &d(Ax_{2n}, TTx_{2n+1}), d(BTx_{2n+1}, Sx_{2n}), d(Sx_{2n}, TTx_{2n+1}))\,. \end{aligned} \tag{2 12}$$

Taking $n \to \infty$ in (3 12), since $\phi \in \Phi$, we have

$$d(z, Tz) \le \phi(0, 0, d(z, Tz), d(z, Tz), d(z, Tz)) < \gamma(d(z, Tz)) < d(z, Tz)\,,$$

which is a contradiction Thus, we have $Tz = z$ Similarly, if we replace x by x_{2n} and y by z in (2 2), respectively, and take $n \to \infty$, then we have $Bz = z$ Since $B(X) \subset S(X)$, there exists a point u in X such that $Bz = Su = z$ By using (2 2) again, we have

$$\begin{aligned} d(Au, z) = d(Au, Bz) &\le \phi(d(Au, Su), d(Bz, Tz), d(Au, Tz), d(Bz, Su), d(Su, Tz)) \\ &= \phi(d(Au, z), 0, d(Au, z), 0, 0) < \gamma(d(Au, z)) < d(Au, z)\,, \end{aligned}$$

which is a contradiction and so $Au = z$ Since A and S are compatible mappings of type (A) and $Au = Su = z$, by Proposition 2 4, $d(ASu, SSu) = 0$ and hence $Az = ASu = SSu = Sz$ Finally, by (2 2) again, we have

$$\begin{aligned} d(Az, z) = d(Az, Bz) &\le \phi(d(Az, Sz), d(Bz, Tz), d(Az, Tz), d(Bz, Sz), d(Sz, Tz)) \\ &= \phi(d(Az, z), 0, d(Az, z), 0, 0) < \gamma(d(Az, z)) < d(Az, z)\,, \end{aligned}$$

which implies that $Az = z$. Therefore, $Az = Bz = Tz = z$, that is, z is a common fixed point of the given mappings A, B, S and T The uniqueness of the common fixed point z follows easily from (2 2)

Similarly, we can prove Theorem 2 4 when A or B or T is continuous This completes the proof

Next, we give convergence theorems for sequences of self-mappings on a metric space

THEOREM 2.5. Let $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ be sequences of mappings from a metric space (X, d) into itself such that $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to self-mappings A, B, S and T on X, respectively Suppose that, for $n = 1, 2, \dots, z_n$ is a unique common fixed point of A_n, B_n, S_n and T_n and the self-mappings A, B, S and T satisfy the following conditions

$$d(Ax, By) \leq \phi(d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx), d(Sx, Ty)) \tag{2 13}$$

for all $x, y \in X$, where $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ is a mapping such that ϕ is upper semicontinuous, non-decreasing in each variable and for any $t > 0, \phi(t, t, t, t, t) \leq \beta t$ for $0 < \beta < 1$

If z is a unique common fixed point of A, B, S and T and $\sup\{d(z_n, z)\} < +\infty$, then the sequence $\{z_n\}$ converges to z

PROOF. Let $\epsilon_i > 0$ for $i = 1, 2$ Since $\{A_n\}$ and $\{S_n\}$ converge uniformly to self-mappings A and S on X, respectively, there exist positive integers N_1, N_2 such that for all $x \in X$

$$d(A_n x, Ax) < \epsilon_1 \quad \text{for} \quad n \geq N_1 \quad \text{and} \quad d(S_n x, Sx) < \epsilon_2 \quad \text{for} \quad n \geq N_2\,,$$

respectively Choose $N = \max\{N_1, N_2\}$ and $\epsilon = \max\{\epsilon_1, \epsilon_2\}$ For $n \geq N$, we have

$$\begin{aligned} d(z_n, z) &= d(A_n z_n, Bz) \leq d(A_n z_n, Az_n) + d(Az_n, Bz) \\ &\leq d(A_n z_n, Az_n) + \phi(d(Az_n, Sz_n), d(Bz, Tz), d(Az_n, Tz), d(Bz, Sz_n), d(S_n, Tz)) \\ &\leq d(A_n z_n, Az_n) + \phi(d(Az_n, A_n z_n) + d(A_n z_n, Sz_n), 0, d(Az_n, A_n z_n) \\ &\quad + d(A_n z_n, Tz), d(Bz, S_n z_n) + d(S_n z_n, Sz_n), d(Sz_n, S_n z_n) + d(S_n z_n, Tz)) \\ &= d(A_n z_n, Az_n) + \phi(d(Az_n, A_n z_n) + d(S_n z_n, Sz_n), 0, d(A_n z_n, Az_n) + d(z_n, z), \\ &< \epsilon + \phi(2\epsilon, 0, \epsilon + d(z_n, z), \epsilon + d(z_n, z), \epsilon + d(z_n, z))\,. \end{aligned} \tag{2 14}$$

From (2 14), if $d(z_n, z) > \epsilon$, then we have

$$\begin{aligned} d(z_n, z) &< \epsilon + \phi(\epsilon + d(z_n, z), \epsilon + d(z_n, z), \epsilon + d(z_n, z), \epsilon + d(z_n, z), \epsilon + d(z_n, z)) \\ &\leq \epsilon + \beta(\epsilon + d(z_n, z)) = \epsilon + \beta\epsilon + \beta\, d(z_n, z)\,. \end{aligned}$$

This implies that

$$(1 - \beta)d(z_n, z) < (1 + \beta)\epsilon \quad \text{or} \quad d(z_n, z) < \left(\frac{1 + \beta}{1 - \beta}\right)\epsilon\,. \tag{2 15}$$

Thus, letting $\beta \to 0^+$ in (2 15), then $\epsilon < d(z_n, z) \leq \epsilon$, which is a contradiction Therefore, for $n \geq N$, $d(z_n, z) < \epsilon$, which means that $\{z_n\}$ converges to z This completes the proof.

Similarly, we have the following

THEOREM 2.6. Let $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ be sequences of mappings from a metric space (X, d) into itself satisfying the following condition

$$d(A_n x, B_n y) \leq \phi(d(A_n x, S_n x), d(B_n y, T_n y), d(A_n x, T_n y), d(B_n y, S_n x), d(S_n x, T_n y)) \tag{2 16}$$

for all x, $y \in X$, where the mapping ϕ is as in the condition (2 14)

If $\{A_n\}$, $\{B_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to self-mappings A, B, S and T on X, respectively, then A, B, S and T satisfy the condition (2 14)

Further, the sequence $\{z_n\}$ of unique common fixed points z_n of A_n, B_n, S_n and T_n converges to a unique common fixed point z of A, B, S and T if $\sup\{d(z_n, z)\} < +\infty$

REMARK 3. Our main theorems extend and improve a number of fixed point theorems for commuting, weakly commuting and compatible mappings in metric spaces

3. FIXED POINT THEOREMS IN PM-SPACES

In this section, we extend the Caristi's fixed point theorem and the Ekeland's variational principle in PM-spaces Also, we prove some common fixed point theorems in PM-spaces by using the results in Section 2 In [4] and [13], K Caristi and I Ekeland proved the following theorems, respectively

THEOREM 3.1. Let (X, d) be a complete metric space and T be a mapping from X into itself If there exists a lower semicontinuous function ζ $X \to \mathbb{R}^+$ such that $d(x, Tx) \leq \zeta(x) - \zeta(Tx)$ for all $x \in X$, then T has a fixed point in X

THEOREM 3.2. Let (X, d) be a complete metric space and f be a proper, bounded below and lower semicontinuous function from X into $(-\infty, +\infty]$ Then for each $\epsilon > 0$ and $u \in X$ such that $f(u) \leq \inf\{f(x) \cdot x \in X\} + \epsilon$, there exists a point $v \in X$ such that

$$f(v) \leq f(u), \tag{3 1}$$

$$d(u, v) \leq 1, \tag{3 2}$$

$$f(w) > f(v) - \epsilon d(v, w) \quad \text{for all} \quad w \in X, \ w \neq v \tag{3 3}$$

First, we prove the following

THEOREM 3.3. Let (X, F) be a PM-space of type $(C)_g$ and (X, d) be a complete metric space, where the metric d on X is defined by $(*)$ If $\zeta : X \to \mathbb{R}$ is a lower semicontinuous and bounded below function and a mapping $T : X \to X$ satisfies the following condition

$$g(F_{x,Tx}(t)) \leq \zeta(x) - \zeta(Tx) \quad \text{for all} \quad x \in X \quad \text{and} \quad t \geq 0, \tag{3 4}$$

then T has a fixed point in X

PROOF. From (3 4), we have

$$d(x, Tx) = \int_0^1 g(F_{x,Tx}(t))dt \leq \int_0^1 (\zeta(x) - \zeta(Tx))dt = \zeta(x) - \zeta(Tx)$$

and thus, by Theorem 3 1, T has a fixed point in X

COROLLARY 3.4. Let (X, F) be a PM-space of type $(C)_g$, (X, d) be a complete metric space, where the metric d on X is defined by $(*)$, and a function $\eta(x, t) : X \times \mathbb{R}^+ \to \mathbb{R}^+$ be integrable in t If a function $\psi(x) = \int_0^1 \eta(x, t)dt$ is lower semicontinuous and bounded below and a mapping $T : X \to X$ satisfies the following condition

$$g(F_{x,Tx}(t)) \leq \eta(x, t) - \eta(Tx, t) \quad \text{for all} \quad x \in X \quad \text{and} \quad t \geq 0, \tag{3 5}$$

then T has a fixed point in X

PROOF. From (4 5), we have

$$\begin{aligned} d(x, Tx) &= \int_0^1 g(F_{x,Tx}(t))dt \leq \int_0^1 (\eta(x, t) - \eta(Tx, t))dt \\ &= \int_0^1 \eta(x, t)dt - \int_0^1 \eta(Tx, t)dt \\ &= \psi(x) - \psi(Tx) \end{aligned}$$

Therefore, by Theorem 3 3, T has a fixed point in X

THEOREM 3.5. Let (X, F) be a PM-space of type $(C)_g$ and (X, d) be a complete metric space, where the metric d on X is defined by $(*)$ If a function $\zeta : X \to \mathbb{R}$ is proper, lower semicontinuous and bounded below, and T is a multi-valued mapping from X into 2^X such that for each $x \in X$, there exists a point $fx \in Tx$ satisfying that $f : X \to X$ is a function satisfying the following condition

$$g(F_{x,Tx}(t)) \leq \zeta(x) - \zeta(fx) \quad \text{for all} \quad x \in X \quad \text{and} \quad t \geq 0, \tag{3 6}$$

then f and T have a common fixed point in X

PROOF. Since ζ is proper, there exists a point $u \in X$ such that $\zeta(x) < +\infty$ and so let $A = \{x \in X : g(F_{x,u}(t)) \leq \zeta(x)\}$ Then A is a nonempty closed set in X Since $g(F_{x,fx}(t)) \leq \zeta(x) - \zeta(fx)$ for each $x \in X$, $fx \in A$ and so we have

$$\zeta(x) + g(F_{x,fx}(t)) \leq \zeta(x) \leq \zeta(u) - g(F_{x,u}(t)) .$$

Thus we have

$$\begin{aligned} g(F_{u,fx}(t) &\leq g(F_{u,x}(t)) + g(F_{x,fx}(t)) \\ &\leq \zeta(u) - \zeta(x) + \zeta(x) - \zeta(fx) \\ &= \zeta(u) - \zeta(fx) \end{aligned}$$

Therefore, by Theorem 3 3, the function $f : A \to A$ has a fixed point in A, say x_0, and so $x_0 = fx_0 \in Tx_0$, that is, the point x_0 is a common fixed point of f and T This completes the proof

By Theorem 3 5, we have Ekeland's variational principle in PM-spaces

THEOREM 3.6. Let (X, F) be a PM-space of type $(C)_g$ and (X, d) be a complete metric space, where the metric d on X is defined by $(*)$ If a function $\zeta : X \to \mathbb{R}$ is proper, lower semicontinuous and bounded below and, for each $\epsilon > 0$, there exists a point $u \in X$ such that $\zeta(u) \leq \inf\{\zeta(x) : x \in X\} + \epsilon$, then there exists a point $v \in X$ such that

$$\zeta(v) \leq \zeta(u) , \tag{3 7}$$

$$g(F_{u,v}(t)) \leq 1 , \tag{3 8}$$

$$\zeta(v) - \zeta(x) \leq \epsilon g(F_{u,x}(t)) \quad \text{for all} \quad x \in X \quad \text{and} \quad t \geq 0 . \tag{3 9}$$

PROOF. Let $\epsilon > 0$ and let a point $u \in X$ such that $\zeta(u) \leq \inf\{\zeta(u) : x \in X\} + \epsilon$ Letting $A = \{x \in X : \zeta(x) \leq \zeta(u) - \epsilon g(F_{u,x}(t))\}$, then A is a nonempty closed set in X and so, since (X, d) is complete, A is complete For each $x \in A$, let

$$Sx = \{y \in X : \zeta(y) \leq \zeta(x) - \epsilon g(F_{x,y}(t)), x \neq y\}$$

and define

$$Tx = \begin{cases} x & \text{if} \quad Sx \quad \text{is empty,} \\ Sx & \text{if} \quad Sx \quad \text{is nonempty} \end{cases}$$

Then T is a multi-valued mapping from A into 2^A Since $Tx = x \in A$ if $Sx = \emptyset$ and $Tx = Sx$ if $Sx \neq \emptyset$, we have, for each $y \in Tx = Sx$,

$$\zeta(y) \leq \zeta(x) - \epsilon g(F_{x,y}(t))$$

and

$$\begin{aligned} \epsilon g(F_{u,y}(t)) &\leq \epsilon g(F_{u,x}(t)) + \epsilon g(F_{x,y}(t)) \\ &\leq \zeta(u) - \zeta(x) + \zeta(x) - \zeta(y) \\ &= \zeta(u) - \zeta(y) , \end{aligned}$$

which implies $y \in A$ and so we have $Tx = Sx \subset A$ Assume that T has no fixed point in A Then for each $x \in A$ and $y \in Tx = Sx$, we obtain

$$\epsilon g(F_{x,y}(t)) \leq \zeta(x) - \zeta(y) , \quad \text{and} \quad g(F_{x,y}(t)) \leq \frac{1}{\epsilon}\zeta(x) - \frac{1}{\epsilon}\zeta(y) .$$

Thus, by Theorem 4 5, T has a fixed point v in A, which is a contradiction Therefore, $Sv = \emptyset$, that is, for each $x \in X$, $x \neq v$, $\zeta(x) > \zeta(v) - \epsilon g(F_{v,x}(t))$ Since $v \in A$, $\zeta(v) \leq \zeta(u) - \epsilon g(F_{u,v}(t))$ and so $\zeta(v) \leq \zeta(u)$ On the other hand, we have

$$\begin{aligned} \epsilon g(F_{u,v}(t)) &\leq \zeta(u) - \zeta(v) \\ &\leq \zeta(u) - \inf\{\zeta(x) : x \in X\} < \epsilon \end{aligned}$$

and so $g(F_{u,v}(t)) \leq 1$ This completes the proof

Next, by using Theorem 2.4, we prove common fixed point theorems in PM-spaces. Now, we introduce some definitions and properties of compatible mappings of type (A) in PM-spaces ([11]).

DEFINITION 3.1. Let $(X, F, \triangle)$ be an N.A. Menger PM-space of type $(D)_g$ and A, S be mappings from X into itself. A and S are said to be compatible if

$$\lim_{n\to\infty} g(F_{ASx_n,SAx_n}(t)) = 0 \quad \text{for all} \quad t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

DEFINITION 3.2. Let $(X, F, \triangle)$ be an N.A. Menger PM-space of type $(D)_g$ and A, S be mappings from X into itself. A and S are said to be compatible of type (A) if

$$\lim_{n\to\infty} g(F_{ASx_n,SSx_n}(t) = 0 \quad \text{and} \quad \lim_{n\to\infty} g(F_{SAx_n,AAx_n}(t)) = 0$$

for all $t > 0$, whenever $\{x_n\}$is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

REMARK 4. (1) In fact, since $(X, F, \triangle)$ is an N.A. Menger PM-space of type $(D)_g$ and it is metrizable by the metric d defined by $(*)$, Definitions 2.1 and 3.1, 2.2 and 3.2 are equivalent to each other, respectively.

(2) By using Definitions 3.1 and 3.2, we can obtain same properties, that is, Propositions 2.1 ~ 2.5, between compatible mappings and compatible mappings of type (A) in PM-spaces.

THEOREM 3.7. Let $(X, F, \triangle)$ be a τ-complete N.A. Menger PM-space with the t-norm $\triangle$ such that $\triangle(s,t) \geq \triangle_m(s,t) = \max\{s+t-1, 0\}$, s, $t \in [0,1]$. Let A, B, S and T be mappings from X into itself such that

(ii) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(ii) one of A, B, S and T is τ-continuous,

(iii) the pairs A, S and B, T are compatible mappings of type (A),

(iv) there exists $\phi \in \Phi$ such that

$$\int_0^1 F_{SAx_n,AAx_n}(t)dt \geq 1 - \phi\left(1 - \int_0^1 F_{SAx,Sx}(t)dt, 1 - \int_0^1 F_{By,Ty}(t)dt, 1 - \int_0^1 F_{Ax,Ty}(t)dt,\right.$$
$$\left. 1 - \int_0^1 F_{Ax,Sx}(t)dt, 1 - \int_0^1 F_{Ax,Sx}(t)dt\right) \text{ for all } x,\ y \in X \text{ and } t \geq 0$$

Then A, B, S and T have a unique common fixed point in X.

PROOF. Since $(X, F, \triangle)$ is an N.A. Menger PM-space with the t-norm $\triangle$ such that $\triangle(s,t) \geq \triangle_m(s,t) = \max\{s+t-1, 0\}$, $s, t \in [0,1]$, by Remark 1 (5), it is metrizable by the metric d defined by $(*)$. Thus, if we define $g(t) = 1 - t$, from (3.12), we have

$$d(Ax, By) \leq \phi(d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx), d(Sx, Ty))$$

for all $x, y \in X$. Therefore, by Theorem 2.4, A, B, S and T have a unique common fixed point in X. This completes the proof.

As an immediate consequence of Theorem 3.7, we have the following.

COROLLARY 3.8. Let $(X, F, \triangle)$ be as in Theorem 3.7. Let A, B, S and T be mappings from X into itself satisfying the conditions (i)-(iv) and (v)

there exists $c \in (0,1)$ such that

$$\int_0^1 F_{Ax,By}(t)dt \geq 1 - c\left(1 - \int_0^1 F_{Ax,Sx}(t)dt, 1 - \int_0^1 F_{By,Ty}(t)dt, 1 - \int_0^1 F_{Ax,Ty}(t)dt,\right.$$
$$\left. 1 - \int_0^1 F_{Ax,Sx}(t)dt, 1 - \int_0^1 F_{Sx,Ty}(t)dt\right) \text{ for all } x,\ y \in X \text{ and } t \geq 0$$

Then A, B, S and T have a unique common fixed point in X

REFERENCES

[1] BHARUCHA-REID, A T , Fixed point theorems in probabilistic analysis, *Bull. Amer Math. Soc.* **82** (1976), 641-657

[2] BOCSAN, Gh , On some fixed point theorems in probabilistic metric spaces, *Math Balkanica* **4** (1974), 67-70

[3] CAIN, G L Jr and KASRIEL, R H , Fixed and periodic points of local contraction mappings of probabilistic metric spaces, *Math. Systems Theory* **9** (1976), 289-297

[4] CARISTI, J , Fixed point theorems for mappings satisfying inwardness conditions, *Trans. Amer. Math. Soc.* **215** (1976), 241-251

[5] CHANG, S S , On some fixed point theorems in probabilistic metric spaces and its applications, *Z. Wahr. verw Gebiete* **63** (1983), 463-474

[6] CHANG, S S , On the theory of probabilistic metric spaces with applications, *Acta Math. Sinica*, New Series, **1** (4) (1985), 366-377

[7] CHANG, S S , Probabilistic metric spaces and fixed point theorems for mappings, *J. Math. Research Expos.* **3** (1985), 23-28

[8] CHANG, S S , Fixed point theorems for single-valued and multi-valued mappings in non-Archimedean Menger probabilistic metric spaces, *Math. Japonica* **35** (5) (1990), 875-885

[9] CHANG, S S and XIANG, S W , Topological structure and metrization problem of probabilistic metric spaces and application, *J. Qufu Normal Univ.* **16** (3) (1990), 1-8

[10] CHANG, S S , CHEN, Y Q and GUO, J L , Ekeland's variational principle and Caristi's fixed point theorem in probabilistic metric spaces, *Acta Math. Appl. Sinica* **7** (3) (1991), 217-229

[11] CHO, Y J , MURTHY, P P and STOJAKOVIC, M , Compatible mappings of type (A) and c mmon fixed points in Menger spaces, *Comm. of Korean Math. Soc.* **7** (2) (1992), 325-339

[12] CIRIC, Lj B , On fixed points of generalized contractions of probabilistic metric spaces, *Publ. Inst. Math. Beograd* **18** (32) (1975), 71-78

[13] EKELAND, I , Nonconvex minimization problems, *Bull. Amer. Math. Soc.* **1** (1979), 443-474

[14] HADZIC. O , A fixed point theorem in Menger spaces, *Publ. Inst. Math. Beograd* **20** (40) (1979), 107-112

[15] HADZIC, O , Some theorems on the fixed points in probabilistic metric and random normed spaces, *Boll. Un. Mat. Ital.* **13** (5) 18 (1981), 1-11

[16] HICKS, T L , Fixed point theory in probabilistic metric spaces, *Review of Research, Fac. Sci. Math. Series,* Univ of Novi Sad , **13** (1983), 63-72

[17] JUNGCK, G , MURTHY, P P and CHO, Y J , Compatible mappings of type (A) and common fixed points, *Math. Japon.* **38** (2) (1993), 381-390

[18] SCHWEIZER, B. and SKLAR, A , Statistical metric spaces, *Pacific J. Math.* **10** (1960), 313-334.

[19] SCHWEIZER, B and SKLAR, A , Probabilistic Metric Spaces, *North-Holland Series in Probability and Applied Math* **5**, 1983

[20] SEHGAL, V and RHARUCHA-REID, A T , Fixed points of contraction mappings of probabilistic metric spaces, *Math. Systems Theory* **6** (1972), 92-102

[21] SINGH, S P and MEADE, B A , On common fixed point theorems, *Bull. Austral. Math. Soc.* **16** (1977), 49-53

[22] STOJAKOVIC, M , Fixed point theorem in probabilistic metric spaces, *Kobe J. Math.* **2** (1985), 1-9

[23] STOJAKOVIC, M , Common fixed point theorems in complete metric and probabilistic metric spaces, *Bull. Austral. Math. Soc.* **36** (1987), 73-88

[24] STOJAKOVIC, M , A common fixed point theorem in probabilistic metric spaces and its applications, *Glasnik Mat.* **23** (43) (1988), 203-211

[25] TAN, N X , Generalized probabilistic metric spaces and fixed point theorems, *Math. Nachr.* **129** (1986), 205-218