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ON APPROXIMATION OF FUNCTIONS AND THEIR DERIVATIVES BY QUASI-HERMITE INTERPOLATION

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ABSTRACT. In this paper, we consider the simultaneous approximation of the derivatives of the functions by the corresponding derivatives of quasi-Hermite interpolation based on the zeros of $(1-x^2)p_n(x)$ (where $p_n(x)$ is a Legendre polynomial) . The corresponding approximation degrees are given. It is shown that this matrix of nodes is almost optimal.

KEY WORDS: Hermite interpolation, optimal nodes, derivatives, Legendre polynomials, best approximation.
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1 INTRODUCTION.

Let

$$-1 \le x_n < \dots < x_1 < x_0 \le 1 \tag{1.1}$$

be an arbitrary nodes system on [-1,1] and let $f \in C^1[-1,1]$. We consider the Hermite interpolation operator:

$$H_n(f,x) := \sum_{k=0}^{n} f(x_k)h_k(x) + \sum_{k=0}^{n} f'(x_k)\sigma_k(x), \tag{1.2}$$

where

$$h_k(x) = v_k(x)l_k^2(x), \quad \sigma_k(x) = (x-x_k)l_k^2(x),$$

$$l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x-x_k)},$$

$$v_k(x) = 1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x-x_k), \quad \omega(x) = \Pi_{k=0}^{n}(x-x_k).$$

It satisfies the following conditions:

$$H_n(f,x_k) = f(x_k), \quad k = 0,1,\dots,n$$

and

$$H_n'(f,x_k) = f'(x_k), \quad k = 0,1,\dots,n$$

There have been many articles considering the problem of approximation to $f(x)$ by $H_n(f,x)$, Generally, we consider approximation of $f'(x)$ by the derivative of Hermite interpolation. We know that the convergence

$$\lim_{n\to\infty} ||H_n'(f,x) - f'(x)|| = 0,$$

does not hold for all $f \in C^1[-1,1]$(here $||.||$ is the maximum norm). Pottinger [1] investigated this problem when $\{x_k\}_{k=0}^n$ are the zeros of the Tchebycheff polynomial of the first kind and obtained the following result:

$$||H_n'(f,x) - f'(x)|| = O(n)E_{2n}(f'), \tag{1.3}$$

where $E_n(f)$ is the best approximation of $f(x)$. (The factor $O(n)$ is best possible, cf. Steinhaus [2].) In [3], Szabados and Varma introduced a norm for the higher derivatives of the operator (1.2):

$$||H_n^{(r)}|| = \sup\{||H_n^{(r)}(f,x)|| : |f^{(i)}(x_k)| \le n^i(1-x_k^2)^{r-i/2}, k = 1,...,n; i = 0,1\}$$

$(r, n = 1,2,....)$ and they proved that for any system of nodes ([3, Theorem 1])

$$||H_n^{(r)}|| \ge c_r n^r \ln n, \quad (n, r = 1,2,...) \tag{1.4}$$

where $c_r > 0$ depends only on r. Moreover, for the matrix of nodes:

$$\omega(x) = P_{n-2t+1}^{(\alpha,\alpha)}(x)\Pi_{j=1}^t(x^2 - \cos^2\frac{(j-1)\pi}{3t(n-2t+1)}), \tag{1.5}$$

they obtain ([3, Theorem 3])

$$||H_n^{(r)}|| = O(n^r \ln n), \tag{1.6}$$

where $t = [\frac{r+3}{4}]$, $\alpha = 2t - \frac{r+1}{2}$ ($r \ge 1$ integer) and $P_{n-2t+1}^{(\alpha,\alpha)}(x)$ are the ultraspherical Jacobi polynomials of degree $n-2t$. Moreover, α takes only the values $-1/2, 0, 1/2, 1$ according to $r = 0,3,2,1 (\text{mod } 4)$. (See [3, Remark, P305].) Therefore for the matrix of nodes defined by (1.5) we have

$$||H_n^{(r)}(f,x) - f^{(r)}(x)|| = O(\ln n)\omega(f^{(r)}, \frac{1}{n}). \tag{1.7}$$

(see [3]) At the end of paper [3], they speculated that "it would be interesting to construct a matrix which is optimal for *all* the derivatives up to order r." This is the problem of constructing matrix nodes so that the corresponding simultaneous approximation of $f(x)$ from the first derivative to the r-th derivative is optimal by the corresponding Hermite interpolation.

Remark: With respect to Lagrange interpolation, the complete solution of minimizing the corresponding derivatives norm to (1.4) was given by Szabados [4] (also see Vértesi [5]). The main idea is that adding nodes (near ± 1) to Jacobi nodes make the similar estimates of (1.4) optimal.

In this paper, we point out that for the quasi-Hermite interpolation $R_n(f,x)$ based on the zeros of $(1-x^2)p_n(x)$ (where $p_n(x)$ is the Legendre polynomial with normalization: $p_n(1) = 1$), we have

THEOREM 1. If $f \in C^1[-1,1]$, then

$$||R_n'(f,x) - f'(x)|| = O(\ln n)E_{2n}(f'). \tag{1.8}$$

THEOREM 2. If $f \in C^r[-1,1]$ ($r \ge 2$), then

$$||R_n'(f,x) - f'(x)|| = O(\ln n)E_{2n}(f') = O(\frac{\ln n}{n})E_{2n-1}(f''), \tag{1.9}$$

$$||\sqrt{1-x^2}(R_n''(f,x) - f''(x))|| = O(\ln n)E_{2n-1}(f''), \tag{1.10}$$

and

$$||R_n^{(i)}(f,x) - f^{(i)}(x)||_{[-\sigma,\sigma]} = O(\ln n)E_{2n-i+1}(f^{(i)}), \quad i = 2,...,r \tag{1.11}$$

where $0 < \sigma < 1$.

From this we see that the zeros of $(1-x^2)p_n(x)$ are almost optimal and the corresponding simultaneous approximation is better than that of Hermite interpolation based on the zeros of Tchebysheff polynomial of the first kind.

Remark: We conjucture that the factor $\sqrt{1-x^2}$ in (1.10) cannot be removed on the whole interval $[-1,1]$, in which case the preceding results are optimal.

2 LEMMAS.

In order to prove the Theorems, we state some properties of Legendre polynomials (see Szegö [6]).

$$|p_n(x)| \leq 1, \tag{2.1}$$

$$(1-x^2)^{1/4}|p_n(x)| \leq (2/\pi n)^{-1/2}, \quad n \geq 2 \tag{2.2}$$

$$(1-x^2)^{3/4}|p_n'(x)| \leq (2n)^{1/2}, \quad n \geq 3 \tag{2.3}$$

$$\sin^2\theta_k = 1 - x_k^2 > (k-3/2)^2 n^{-2}, \quad k = 1, ..., [n/2] \tag{2.4}$$

$$|p_n'(x_k)| > c(k-3/2)^{-3/2} n^2, \quad k = 1, ..., [n/2] \tag{2.5}$$

We note that in (2.4) and (2.5) similar estimates are hold for $k = [n/2], ..., n$. On combining (2.4) and (2.5), it follows that

$$[(1-x_k^2)^{3/4}|p_n'(x)|]^2 \geq cn, \quad k = 1, ..., n \tag{2.6}$$

where c is an absolute positive constant independent of f and n, whose value may vary from line to line throught our paper.

Let

$$-1 = x_{n+1} < x_n < ... < x_1 < x_0 = 1$$

be the zeros of $(1-x^2)p_n(x)$. Then its corresponding quasi-Hermite interpolation is the following

$$R_n(f, x) = \sum_{k=0}^{n+1} f(x_k) r_k(x) + \sum_{k=1}^{n} f'(x_k)\gamma_k(x), \tag{2.7}$$

where

$$r_0(x) = \frac{1+x}{2} p_n^2(x), \quad r_{n+1} = \frac{1-x}{2} p_n^2(x),$$

$$r_k(x) = \frac{1-x^2}{1-x_k^2} l_k^2(x), \quad k = 1, ..., n$$

$$\gamma_k(x) = (x - x_k) r_k(x), \quad k = 1, ..., n$$

$$l_k(x) = \frac{p_n(x)}{p_n'(x_k)(x - x_k)}, \quad k = 1, ..., n$$

It satisfies that

$$R_n(f, x_k) = f(x_k), \quad k = 0, 1, ..., n+1.$$

and

$$R_n'(f, x_k) = f'(x_k), \quad k = 1, ..., n$$

LEMMA 1. We have

$$\sqrt{1-x_k^2} \leq \sqrt{1-x^2} + 2\frac{|x - x_k|}{\sqrt{1-x_k^2}}, \quad k = 1, ..., n.$$

PROOF. One easily sees that

$$\sqrt{1-x_k^2} = \sqrt{1-x^2} + \sqrt{1-x_k^2} - \sqrt{1-x^2}$$

$$= \sqrt{1-x^2} + \frac{x^2 - x_k^2}{\sqrt{1-x_k^2} + \sqrt{1-x^2}} \leq \sqrt{1-x^2} + 2\frac{|x - x_k|}{\sqrt{1-x_k^2}}.$$

This proves Lemma 1. □

LEMMA 2. We have

$$(i)\quad I_1 := \sum_{k=1}^{n} \frac{|x-x_k|}{1-x_k^2} l_k^2(x) = O(\ln n) \tag{2.8}$$

$$(ii)\quad I_2 := \sum_{k=1}^{n} |x-x_k| \frac{1-x^2}{1-x_k^2} |l_k(x) l_k'(x)| = O(\ln n) \tag{2.9}$$

PROOF. From Lemma 1 we have

$$I_1 \le \sum_{k=1}^{n} \frac{\sqrt{1-x^2}|x-x_k|}{(1-x_k^2)^{3/2}} l_k^2(x) + 2\sum_{k=1}^{n} \frac{|x-x_k|^2}{(1-x_k^2)^2} l_k^2(x) := A_1(x) + A_2(x) \tag{2.10}$$

Throughout this paper we assume x_j to be the zero of $p_n(x)$ which is the nearest to x and $i = |k-j|$. By using (5.8) in Prasad and Varma[7] we have

$$\sqrt{1-x^2}\frac{|x-x_j|}{1-x_j^2} l_j^2(x) \le \frac{c}{n}. \tag{2.11}$$

Notice that, with $x = \cos\theta \quad (0 \le \theta \le \pi)$

$$\sin\theta \le \sin\theta + \sin\theta_k \le 2\sin\frac{\theta+\theta_k}{2},$$

so we have

$$\begin{aligned} A_1(x) &= \frac{1}{\sqrt{1-x_j^2}} \frac{\sqrt{1-x^2}|x-x_j|}{1-x_j^2} l_j^2(x) + \sum_{k\neq j} \frac{\sqrt{1-x^2}|x-x_k|}{(1-x_k^2)^{3/2}} l_k^2(x) \\ &\le \frac{c}{n}\frac{1}{\sin\theta_j} + \sum_{k\neq j} \frac{\sqrt{1-x^2}p_n^2(x)}{[(1-x_k^2)^{3/4}|p_n'(x_k)|]^2|x-x_k|} \\ &= O(1)[1 + p_n^2(x)\sum_{k\neq j}\frac{1}{\sin|\frac{\theta-\theta_k}{2}|}] = O(1)[1+\frac{p_n^2(x)}{n}\sum_{k\neq j}\frac{n}{i}] = O(\ln n). \end{aligned}$$

Similarly,

$$A_2(x) = \sum_{k=1}^{n} \frac{p_n^2(x)}{[(1-x_k^2)^{3/4}|p_n'(x_k)|]^2\sqrt{1-x_k^2}} = O(1)\frac{p_n^2(x)}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1-x_k^2}} = O(\ln n),$$

so we obtain (2.8).
Notice that

$$l_k'(x) = \frac{p_n'(x)(x-x_k) - p_n(x)}{(x-x_k)^2 p_n'(x_k)},$$

and we have

$$I_2 \le \sum_{k=1}^{n} |x-x_k| \frac{(1-x^2)|x-x_k||p_n'(x)|}{(1-x_k^2)(x-x_k)^2|p_n'(x_k)|}|l_k(x)| + \sum_{k=1}^{n} r_k(x) := B_1(x) + B_2(x)$$

One notes Prasad and Varma [7]

$$\frac{(1-x^2)^{1/4}}{(1-x_k^2)^{1/4}}|l_k(x)| \le c,$$

so we have

$$\begin{aligned} B_1(x) &= \sum_{k=1}^{n} \frac{(1-x^2)^{3/4}|p_n'(x)|}{(1-x_k^2)^{3/4}|p_n'(x_k)|}\frac{(1-x^2)^{1/4}}{(1-x_k^2)^{1/4}}|l_k(x)| \\ &= O(1)\frac{(1-x^2)^{3/4}|p_n'(x)|}{(1-x_j^2)^{3/4}|p_n'(x_j)|} + \sum_{k\neq j}\frac{(1-x^2)|p_n(x)p_n'(x)|\sqrt{1-x_k^2}}{[(1-x_k^2)^{3/4}|p_n'(x_k)|]^2|x-x_k|} \\ &= O(1)[1 + \frac{(1-x^2)|p_n(x)p_n'(x)|}{n}\sum_{k\neq j}\frac{\sin\theta_k}{|x-x_k|}] \\ &= O(1)[1+\ln n(1-x^2)|p_n(x)p_n'(x)|] = O(\ln n). \end{aligned}$$

Obviously,

$$B_2(x) \le \sum_{k=0}^{n+1} r_k(x) \equiv 1.$$

Therefore we obtain (2.9). □

LEMMA 3. We have

$$I_3 := \sum_{k=0}^{n+1}(1 - x_k^2)|r_k(x)| = O(\ln n)(1 - x^2), \tag{2.12}$$

and

$$I_4 := \sum_{k=1}^{n} \sqrt{1 - x_k^2}|\gamma_k(x)| = O(\frac{\ln n}{n})\sqrt{1 - x^2}. \tag{2.13}$$

Proof. Since

$$I_3 = (1 - x^2)\sum_{k=1}^{n} l_k^2(x),$$

from Nevai and Vértesi [8] we have

$$\sum_{k=1}^{n} l_k^2(x) = O(1)(1 + \frac{J_n^2(x)}{n} + \frac{\ln n}{n}J_n^2(x)),$$

where $J_n(x)$ is the orthonormal Legendre polynomials:

$$\int_{-1}^{1} J_n(x)J_m(x)\,dx = \delta_{nm},$$

and notice that Natanson [9] gives

$$||J_n(x)|| = O(1)n^{1/2}.$$

It follows that

$$\sum_{k=1}^{n} l_k^2(x) = O(\ln n),$$

this implies (2.12). Also, we have

$$I_4 = \sum_{k=1}^{n} \frac{(1 - x^2)|x - x_k|}{\sqrt{1 - x_k^2}} l_k^2(x)$$

$$= (1 - x^2)\frac{(1 - x_j^2)^{1/4}|p_n(x)|}{(1 - x_j^2)^{3/4}|p_n'(x_j)|}|l_j(x)| + \sum_{k\neq j} \frac{(1 - x^2)p_n^2(x)}{[(1 - x_k^2)^{3/4}|p_n'(x_k)|]^2}\frac{1 - x_k^2}{|x - x_k|}.$$

Recall that (Erdös [10]) for $-1 \le x \le 1$,

$$|l_k(x)| \le 1, \quad k = 1, ..., n$$

therefore, similar to the estimates of I_1 and I_2, we have

$$I_4 = O(1)\frac{1 - x^2}{n} + \frac{(1 - x^2)p_n^2(x)}{n}\sum_{k\neq j}\frac{1}{\sin|\frac{\theta-\theta_k}{2}|} = O(\frac{\ln n}{n})\sqrt{1 - x^2}.$$

This proves Lemma 3. □

Remark: If we need not want to obtain the factor $(1 - x^2)$, we can obtain a better estimate of I_3.

LEMMA 4. Let $f \in C^r[-1, 1]$, then there exist polynomials $q_n(x)$ of degree $n \ge 4r + 5$ such that $(j = 0, 1, ..., r)$

$$|f^{(j)}(x) - q_n^{(j)}(x)| = O(1)(\frac{\sqrt{1 - x^2}}{n})^{r-j}E_{n-r}(f^{(r)}). \tag{2.14}$$

PROOF. From Gopengauz's Theorem [11] we know that there exist polynomials $t_n(x)$ of degree $n \geq 4r+5$ such that

$$|f^{(j)}(x) - t_n^{(j)}(x)| \leq c(\frac{\sqrt{1-x^2}}{n})^{r-j}\omega(f^{(r)}, \frac{\sqrt{1-x^2}}{n})$$

Let $s_n(x)$ be the polynomial of degree $n > r$ such that

$$||f^{(r)}(x) - s_n^{(r)}(x)|| \leq E_{n-r}(f^{(r)}),$$

then we have

$$|f^{(j)}(x) - q_n^{(j)}(x)| := |f^{((j)}(x) - (s_n^{(j)}(x) + t_n^{(j)}(x))|$$

$$\leq c(\frac{\sqrt{1-x^2}}{n})^{r-j}\omega((f-s_n)^{(r)}, \frac{1}{n}) = O(1)(\frac{\sqrt{1-x^2}}{n})^{r-j}||f^{(r)} - s_n^{(r)}||$$

$$= O(1)(\frac{\sqrt{1-x^2}}{n})^{r-j}E_{n-r}(f^{(r)}).$$

This proves Lemma 4. □

LEMMA 5. Let $s_j(x)$ be a polynomial of degree $\leq n$, and suppose that the inequality

$$\sum_{j=1}^{m}|s_j(x)| = O(1), \quad -1 \leq x \leq 1.$$

holds. Then

$$(1-x^2)^{i/2}\sum_{j=1}^{m}|s_j^{(i)}(x)| = O(1)n^i, \tag{2.15}$$

where $m \geq 1$ and $1 \leq i \leq n$.

PROOF. Although Ramm [12, Lemma 1, p285] only proved the case of i=1, (26) can be obtained by using a completely similar method. □

3 PROOFS OF THEOREMS.

PROOF OF THEOREM 1. Notice that

$$R_n(f,x) - f(x) = \sum_{k=0}^{n+1}(f(x_k) - f(x))r_k(x) + \sum_{k=1}^{n} f'(x_k)\gamma_k(x)$$

$$= \sum_{k=0}^{n+1}\int_x^{x_k} f'(t)\,dt r_k(x) + \sum_{k=1}^{n} f'(x_k)\gamma_k(x).$$

This implies

$$||R_n'|| \leq (\sum_{k=0}^{n+1}|x - x_k||r_k'(x)| + \sum_{k=1}^{n}|\gamma_k'(x)|)||f'|| \tag{3.1}$$

One easily sees that

$$(1-x)|r_0'(x)| \leq (1-x)[\frac{p_n^2(x)}{2} + (1+x)|p_n(x)p_n'(x)|] = O(1).$$

Similarly we have

$$(1+x)|r_{n+1}'(x)| = O(1).$$

Notice that

$$r_k'(x) = -\frac{2x}{1-x_k^2}l_k^2(x) + \frac{2(1-x^2)}{1-x_k^2}l_k(x)l_k'(x)$$

and

$$\gamma_k'(x) = r_k(x) + (x - x_k)r_k'(x).$$

From Lemma 2 we have

$$\sum_{k=0}^{n+1} |x - x_k||r_k'(x)| = O(\ln n) \tag{3.2}$$

and also we have

$$\sum_{k=1}^{n} |\gamma_k'(x)| = O(\ln n). \tag{3.3}$$

It now follows that

$$||R_n'|| = O(\ln n)||f'||. \tag{3.4}$$

Combining Lemma 4, (3.2) and (3.3), we obtain Theorem 1. □

PROOF OF THEOREM 2. Theorem 1 implies (9). Here we only prove the case $i = 2$. The other cases are completely similar. By using Lemma 5 (or see Borwein and Erdelyi [13]) and from Lemma 3 we obtain the following

$$\sum^{n+1} (1 - x_k^2)|r_k''(x)| = O(n^2 \ln n) \tag{3.5}$$

and

$$\sqrt{1-x^2} \sum_{k=1}^{n} \sqrt{1 - x_k^2}|\gamma_k''(x)| = O(n \ln n) \tag{3.6}$$

Notice that

$$R_n''(f, x) - f''(x) = R_n''(f - q_{2n+1}, x) + q_{2n+1}''(x) - f''(x)$$

and

$$R_n''(f - q_{2n+1}, x) = \sum_{k=0}^{n+1} (f(x_k) - q_{2n+1}(x_k))r_k''(x) + \sum_{k=1}^{n} (f'(x_k) - q_{2n+1}'(x_k))\gamma_k''(x).$$

Combining Lemma 4 , (3.5) and (3.6), we obtain (1.10). □

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