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CHAIN CONDITIONS ON SEMIRINGS

T. K. MUKHERJEE, M. K. SEN and SHAMIK GHOSH

Department of Pure Mathematics, University of Calcutta,
35, Ballygunge Circular Road, Calcutta - 700 019, India.

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ABSTRACT . In this paper we characterize the class of semirings S for which the semirings of square matrices $M_n(S)$ over S are (left) k-artinian. Also an analogue of the Hilbert basis theorem for semirings is obtained.

KEY WORDS AND PHRASES . Semiring, halfring, k-ideal, h-ideal, artinian semiring, noetherian semiring.

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0. INTRODUCTION

A *semiring* S is defined as an algebraic system $(S,+,\cdot)$ such that $(S,+)$ and $(S,\cdot)$ are semigroups, connected by $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$. An *(absorbing) zero* element of a semiring S is an element 0 such that $0 + x = x + 0 = x$ and $0x = x0 = 0$ for all $x \in S$. For the rest of the paper we assume that a semiring is additively commutative and has a zero element. If moreover a semiring S is additively cancellative, then it is called a *halfring*. A *semifield* is a semiring in which non-zero elements form a group under multiplication.

We know that the ring $M_n(R)$ of $n \times n$ matrices over a ring R is (left) artinian iff R is (left) artinian. But examples show that there are (left) artinian semirings (even semifields) S for which $M_n(S)$ is not (left) artinian.

In this paper we characterize the class of semirings S such that all $M_n(S)$ are (left) k-artinian (cf. Definition 1.1). Another characterization of the class of semirings S for which all $M_n(S)$ are (left) h-artinian is obtained.

We also obtain an analogue of the Hilbert basis theorem for semirings which generalizes a result of H. E. Stone **[1]**.

1. CHAIN CONDITIONS ON MATRIX SEMIRINGS

Let S be a semiring. A subsemiring I of S is said to be a *left ideal* of S if $ra \in I$ for all $r \in S$ and $a \in I$. A *left k-ideal* [*left h-ideal*] is a left ideal of S for which $x \in S$ [$x, z \in S$], $a, b \in I$ and $x + a = b$ [$x + a + z = b + z$] imply $x \in I$ **[2]**.

DEFINITION 1.1 A semiring S is said to be *(left) artinian* [*k-artinian, h-artinian*] if S satisfies the descending chain condition on left ideals [k-ideals, h-ideals] of S.

Obviously, artinian implies k-artinian and the latter implies h-artinian.

A (*left*) *semimodule* M over a semiring S is a commutative additive semigroup which has a zero element, together with a mapping from $S \times M$ into M (sending (r,m) to rm) such that

(i) $(r + s)\, m = rm + sm$, (ii) $r\,(m + p) = rm + rp$,

(iii) $r\,(sm) = (rs)\, m$, (iv) $0m = r0 = 0$

for all $m, p \in M$ and $r, s \in S$ [3]. We define *k-subsemimodules* and *h-subsemimodules* of an S-semimodule and *k-artinian* and *h-artinian semimodule* over a semiring S in a similar fashion.

Let S be a semiring with a multiplicative identity 1 and $M_n(S)$ be the semiring of $n \times n$ matrices over S. Let E_{ij} be the matrix in $M_n(S)$ such that its (i,j)*th.* entry is 1 and all other entries are zero. Henceforth for any matrix $A = (a_{ij}) \in M_n(S)$ and $i = 1,2,\ldots,n$, we introduce

$$A_i = E_{ii}A = \sum_{j=1}^{n} a_{ij}E_{ij}.$$

Then A can be written as $A_1 + A_2 + \ldots + A_n$. The *ith.* row matrix $(a_{i1}\, a_{i2} \ldots a_{in})$ of A will be denoted by $\mathbf{a}_i$, $i = 1, 2, \ldots, n$. Let I be any left ideal of $M_n(S)$. We define for each $i = 1, 2, \ldots, n$, $I_{0i} = \{A_i : A \in I\}$ and $I_i = \{\mathbf{a}_i : A = (a_{ij}) \in I\}$. Now $I_{0i} \subseteq I$, as I is a left ideal of $M_n(S)$. Therefore $I = I_{01} \oplus I_{02} \oplus \ldots \oplus I_{0n}$, where $\oplus$ means the internal direct sum (as in the case of a ring).

Now we verify that $I_1 = I_2 = \ldots = I_n = M$ (say). Let $i \neq j$ and $\mathbf{a} \in I_j$. Let $A \in I$ be any matrix corresponding to $\mathbf{a}$. Now $E_{ij} A \in I$ and the *ith.* row matrix of $E_{ij}A$ is $\mathbf{a}$. Thus $\mathbf{a} \in I_i$. This implies $I_j \subseteq I_i$ for any $i \neq j$. This completes the verification.

Straightforward calculations show : If I is a left k-ideal [h- ideal] of $M_n(S)$, the same holds for all I_{0i}, and M is a k- subsemimodule [h-subsemimodule] of the S-semimodule S^n (considering elements of S^n as row matrices).

Conversely, let M be a subsemimodule [k-subsemimodule, h- subsemimodule] of the S-semimodule S^n and $I = M_{01} + M_{02} + \ldots + M_{0n}$, where $M_{0i} = \{\sum_{j=1}^{n} a_{ij}\, E_{ij} \in M_n(S) : \mathbf{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in}) \in M\}$.

Clearly, I is closed under addition. Let $A \in I$, $C \in M_n(S)$. We have

$$CA = (C_1 + C_2 + \ldots + C_n)(A_1 + A_2 + \ldots + A_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} C_iA_j .$$

Now C_iA_j is a matrix in $M_n(S)$ whose all rows are zero except the *ith.* row, which is $(c_{ij}\, a_{j1}\; c_{ij}a_{j2} \ldots c_{ij}a_{jn}) = c_{ij}\mathbf{a}_j \in M$, as M is a subsemimodule of S^n. Thus $C_iA_j \in M_{0i}$ and hence $CA \in I$. Therefore I is a left ideal of $M_n(S)$. Also it is trivial to show that I is a left k-ideal [h-ideal] of $M_n(S)$ if M is a k- subsemimodule [h-subsemimodule] of S^n. Thus we have proved the following lemma.

LEMMA 1.2. *Let* S *be a semiring with a multiplicative identity* 1. *Then* I *is a left ideal [k-ideal, h-ideal] of* $M_n(S)$ *iff there exists a subsemimodule [k-subsemimodule, h-subsemimodule]* M *of the S-semimodule* S^n *such that* $I = M_{01} \oplus M_{02} \oplus \ldots \oplus M_{0n}$, *where* M_{0i} *is defined as above.*

COROLLARY 1.3 *Let* S *be a semiring with multiplicative identity* 1. *Then* $M_n(S)$ *is artinian* / *k-artinian, h-artinian* / *iff* S^n *is an artinian* / *k-artinian, h-artinian* / *S-semimodule.*

EXAMPLE 1.4. Let S be the set of all sequences of positive rationals and the constant sequence (0, 0, 0, . .) with pointwise addition and multiplication. Clearly S is a commutative semifield and hence artinian Now let

$M_r = \{ (x , y) \in S^2 \cdot x = (x_1, x_2, ..), y =(y_1, y_2, ...), x_i = y_i, 1 \leq i \leq r \}$.

Then $M_1 \supset M_2 \supset M_3 \supset .$ is an infinite descending chain of k- subsemimodules of the S-semimodule S^2. Thus $M_2(S)$ is not k- artinian, which yields the same for all $M_n(S)$, $n \geq 2$.

Now we present another characterization of semirings over which semirings of matrices are h-artinian. It is easier to handle and it shows that actually the descending chain condition on h-ideals of $M_n(S)$ ($n > 1$) over a semiring S does not depend on n (cf. Corollary 1.11). We proceed through some preliminary lemmata.

LEMMA 1.5. *Let* S *be a semiring. Then every homomorphic image of a k-artinian S-semimodule is also k-artinian.*

PROOF. Let M be an S-semimodule and $\Psi : M \longrightarrow N$ be a homomorphism of M onto an S-semimodule N. It is well known that ψ^{-1} (K) is a k-subsemimodule of M for each k-subsemimodule K of N. This yields the assertion as in the corresponding proof in the case of rings ■

Let H be a halfring. We recall that H can be embedded into a ring and that the smallest ring of this kind is uniquely determined (upto isomorphism). Since the latter consists of all differences a - b for a , b∈ H, it is called the *difference ring* of H and is denoted by D(H) [**4**].

LEMMA 1.6. *If* M *is an H-semimodule for a halfring* H *such that* (M , +) *is a group, then* M *is also a* D(H)*-module under the definition* $(r_1 - r_2) m = r_1m - r_2m$ *for all* $r_i \in H$ *and* $m \in M$. *Moreover any k-subsemimodule of* M *is a submodule of the* D(H)*-module* M *and conversely.*

PROOF. One can easily check that $(r_1 - r_2) m$ is well defined and satisfies (i) to (iv) of the definition of semimodules. Let K be a k- subsemimodule of M and $m_1,m_2 \in K$. Then $(m_1 - m_2) + m_2 = m_1$ implies $m_1 - m_2 \in K$. For $m \in K$ and $r = r_1 - r_2 \in D(H)$, from $(r_1m - r_2m) + r_2m = r_1m$, we get $rm = r_1m - r_2m \in K$. Thus K is a submodule of the D(H)-module M.

Conversely, let K be a submodule of the D(H)-module M. Clearly K is also a subsemimodule of the H-semimodule M. Also if $u + m_1 = m_2$ for some $u \in M$, $m_1 , m_2 \in K$, then $u = m_2 - m_1 \in K$. Thus K is a k-subsemimodule of M, as required ■

LEMMA 1.7. *Let* H *be a halfring. If* H^n *is a k-artinian* H- *semimodule* ($n > 1$), *then* D(H) *is an artinian ring.*

PROOF. We define a mapping $\psi : H^n \longrightarrow D(H)$ by

$$\psi((a_1, a_2, ..., a_n)) = \begin{cases} a_1 - a_2 + a_3 - ... + a_n, & \text{when n is even} \\ a_1 - a_2 + a_3 - ... - a_n, & \text{when n is odd.} \end{cases}$$

Clearly ψ is a well defined (H-) semimodule homomorphism of H^n into D(H). We show that ψ is also surjective. Let $x \in D(H)$. Then there are $x' , x'' \in H$ such that $x = x' - x''$. Thus

ψ ((x ′,x ′′,0,0,...,0)) = x. Therefore D(H) is a k-artinian H-semimodule by Lemma 1.5, which implies D(H) is an artinian D(H)-module by Lemma 1.6. Hence D(H) is an artinian ring ∎

It is well known that the left k-ideals of a halfring H are precisely the intersection with H of the left ideals of D(H) [1] From this fact it is obvious that if D(H) is artinian then H is k- artinian But the converse is not true. For example, let us consider the semifield S described in the Example 1.4. S is an artinian halfring but D(S), being the countably infinite copies of all rationals, is not artinian.

LEMMA 1.8. *Let* H *be a halfring. If* D(H) *is artinian, then* $M_n(H)$ *is k-artinian* ($n \geq 1$).

PROOF. If D(H) is artinian, then so is $M_n(D(H))$. Moreover $D(M_n(H)) = M_n(D(H))$ [5]. Hence $M_n(H)$ is k-artinian ∎

The following theorem follows from Corollary 1.3, Lemma 1.7 and Lemma 1.8.

THEOREM 1.9. *Let* H *be a halfring with multiplicative identity* 1. *Then the following three statements are equivalent for* $n > 1$:

(i) $M_n(H)$ *is a k-artinian halfring.*

(ii) H^n *is a k-artinian* H-*semimodule.*

(iii) D(H) *is an artinian ring.*

Let S be a semiring. We know that $\overline{\Delta}_S = \{ (x,y) \in S \times S : x + z = y + z \text{ for some } z \in S \}$ is the least additively cancellative congruence on S and hence $S/\overline{\Delta}_S$ is a halfring. Generalizing this concept, $D(S/\overline{\Delta}_S)$ is also called the *difference ring* of the semiring S and denoted by $\tilde{S}$. We denote the $\overline{\Delta}_S$-class of any element $a \in S$ by [a]. Now straightforward calculations show that $M_n(S/\overline{\Delta}_S)$ is isomorphic to $M_n(S)/\overline{\Delta}_{M_n(S)}$ through a semiring - isomorphism which sends the matrix ([a_{ij}]) in Mn ($S/\overline{\Delta}_S$) to the element [(a_{ij})] in $M_n(S)/\overline{\Delta}_{M_n(S)}$. Also routine computations prove the following :

LEMMA 1.10. *Let* S *be a semiring. Let* H *be a left h-ideal of* S. *Then* $H' = \{ [x] \in S/\overline{\Delta}_S : x \in H \}$ *is a left k-ideal of* $S/\overline{\Delta}_S$. *Conversely, if* K *is a left k-ideal of* $S/\overline{\Delta}_S$, *then* $K_o = \{ x \in S : [x] \in K \}$ *is a left h-ideal of* S. *Moreover, one has* $(H')_o = H$ *and* $(K_o)' = K$ *and hence a bijective correspondence between the sets of all left h-ideals of* S *and all left k-ideals of* $S/\overline{\Delta}_S$. *In particular* S *is h-artinian iff* $S/\overline{\Delta}_S$ *is k-artinian.*

COROLLARY 1.11. *Let* S *be a semiring with multiplicative identity* 1. *Then the following three statements are equivalent* ($n > 1$) :

(i) $M_n(S)$ *is an h-artinian semiring.*

(ii) S^n *is an h-artinian S-semimodule.*

(iii) $\tilde{S}$ *is an artinian ring.*

PROOF. (i) ⇔ (ii) follows from Corollary 1.3.

(i) ⇔ (iii) :

$M_n(S)$ is h-artinian

⇔ $M_n(S)/\overline{\Delta}_{M_n(S)}$ is k-artinian (by Lemma 1.10)

⇔ $M_n(S/\overline{\Delta}_S)$ is k-artinian (by the above isomorphism)

⇔ $\tilde{S} = D(S/\overline{\Delta}_S)$ is artinian (by Theorem 1.9) ∎

2. HILBERT BASIS THEOREM

DEFINITION 2.1. A semiring S is called *(left) noetherian* [*k- noetherian, h-noetherian*] if it satisfies the ascending chain condition on left ideals [k-ideals, h-ideals] of S.

It is clear that every k-noetherian semiring is h-noetherian. But the following example shows that the converse is not true.

EXAMPLE 2.2. Let Z_0^+ be the set of non-negative integers. Then (Z_0^+ , max., min.) is an h-noetherian semiring, but not k-noetherian.

Let S be a semiring and $A \subseteq S$. The smallest left h-ideal of S containing A is called the left h-ideal of S generated by A. The following lemma is obvious :

LEMMA 2.3. *Let* $A = \{ a_i \in S : i = 1, 2, \ldots, n \}$ *and*

$(A)_h = \{ x \in S : x + \sum_{i=1}^{n} c_i a_i + \sum_{i=1}^{n} n_i a_i + r = \sum_{i=1}^{n} \bar{c}_i a_i + \sum_{i=1}^{n} \bar{n}_i a_i + r,$ *for some* $c_i, \bar{c}_i, r \in S$ and $n_i, \bar{n}_i \in Z_0^+,$ $i = 1, 2, \ldots, n \}$

Then $(A)_h$ *is the left h-ideal of* S *generated by* A.

One can easily prove the following statements :

THEOREM 2.4. *The following three conditions on left h-ideals of a semiring* S *are equivalent :*

(i) S *is h-noetherian.*

(ii) *Every non-empty set of left h-ideals of* S *has a maximal element.*

(iii) *Every left h-ideal of* S *is finitely generated, i.e., for any left h-ideal* I *of* S, *there is a finite set* $A \subseteq I$ *such that* $I = (A)_h$.

LEMMA 2.5. *Any homomorphic image of an h-noetherian semiring is h- noetherian.*

A halfring H is called unital [1] if D(H) is a ring with identity. Stone [1] has obtained the following analogue of the Hilbert basis theorem for halfrings :

Let H be a unital halfring. Then H[x] is k-noetherian iff D(H) is noetherian.

We first show that the condition "unital" is not essential.

THEOREM 2.6. *Let* H *be a halfring. Then* H[x] *is k-noetherian iff* D(H) *is noetherian.*

PROOF. Let D(H) be noetherian. Then H[x] is k-noetherian [1].

Conversely, let H[x] be k-noetherian. We define a mapping $\Psi : H[x] \longrightarrow D(H)$ by $\Psi(p(x)) = p_0 - p_1 + p_2 - p_3 + \ldots$, for each $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \ldots + p_n x^n$. Clearly, Ψ is a well- defined semiring-homomorphism. Also let $u \in D(H)$. Then $u = a - b$, $a, b \in H$. Now $\Psi(a + bx) = u$. Thus Ψ is surjective and hence D(H) is noetherian ■

EXAMPLE 2.7. [1] Let S be the halfring described in the Example 1.4. Then S is k-noetherian. But D(S) is not noetherian.

To prove the main result of this section we note that there is a semiring-isomorphism ψ on $(S/\bar{\Delta}_S)[x]$ onto $(S[x])/\bar{\Delta}_{S[x]}$ defined by

$\Psi\ ([p_0] + [p_1]x + [p_2]x^2 + \ldots + [p_n]x^n) = [p_0 + p_1 x + p_2 x^2 + \ldots + p_n x^n]$

THEOREM 2.8. *Let* S *be a semiring. Then* S[x] *is h-noetherian iff* $\tilde{S}$ *is noetherian.*

PROOF. S[x] is h-noetherian.

$\Leftrightarrow$ $(S[x])/\bar{\Delta}_{S[x]}$ is k-noetherian (by Lemma 1.10)

$\Leftrightarrow$ $(S/\bar{\Delta}_{S[x]})[x]$ is k-noetherian (by the above said isomorphism)

$\Leftrightarrow$ $\tilde{S} = D(S/\bar{\Delta}_S)$ is noetherian (by Theorem 2.6) ■

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