Internat. J. Math. & Math. Sci.
VOL. 19 NO. 3 (1996) 521-528

ON MINIMAX THEORY IN TWO HILBERT SPACES

E. M. EL-KHOLY

Faculty of Science
Department of Mathematics
Tanta University
Tanta - Egypt

HANAN ALI ABDOU

Faculty of Education
Department of Mathematics
Ain Shams University
Cairo - Egypt

(Received February 1, 1994 and in revised form April 19, 1995)

ABSTRACT. In this paper, we investigated the minimax of the bifunction

$$J: H^1(\Omega)xV_2 \to R^m xR^n,$$

such that

$$J(v_1, v_2) = ((\frac{1}{2}a(v_1, v_1) - L(v_1)), v_2)$$

where

$a(.,.)$ is a finite symmetric bilinear bicontinuous, coercive form on $H^1(\Omega)$ and L belongs to the dual of $H^1(\Omega)$.

In order to obtain the minimax point we use lagrangian functional.

KEY WORDS AND PHRASES. Hilbert spaces, dual spaces, minimization of functionals, minimax point, saddle point, concave functions, convex function, bicontinuous form, coercive form.

1. INTRODUCTION.

The minimization of functonals defined only on one Hilbert space has been studied by Jean Cea[1], and the saddle functions has been introduced and investigated by Rockafellar [2]. Our aim, is to generalize the above work in order to obtain a general minimax of functionals defined on two Hilbert spaces. We consider the Sobolev space $H^1(\Omega)$ with inner-product,

$$((u, v)) = \int_\Omega \{uv + \sum_{j=1}^{n} (D_j u)(D_j v)\}dx,$$

as a Hilbert space.

The Green's formula will be used to obtain a minimal solution in this work.

2. PRIMILARIES

Let Ω be a bounded open subset in R^n and Γ denote its boundary. The Sobolev space $H^1(\Omega)$ can be defined as follows, [1]:

$$H^1(\Omega) = \{v: v \in L^2(\Omega), \partial v/\partial x_j \in L^2(\Omega), J = 1, 2, \ldots, n\}$$

where $D_j v = \partial v / \partial x_j$ are taken in the sense of distributions i.e.

$$< D_j v, \varphi > = - < v, D_j \varphi > \qquad \text{for all } \varphi \in D(\Omega)$$

are $\mathcal{D}(\Omega)$ denotes the space of all C^∞-functions with compact support in Ω, Also $<, >$ denotes the duality between $\mathcal{D}(\Omega)$ and the space of distribution $\mathcal{D}'(\Omega)$ on Ω. The space $H^1(\Omega)$ is provided with the inner-product.

$$((u,v)) = (u,v)_{L^2(\Omega)} + \sum_{j=1}^{n}(D_j u, D_j v)_{L^2(\Omega)}$$
$$= \int_\Omega \{uv + \sum_{j=1}^{n}(D_j u)(D_j v)\}dx.$$

The vector $\varphi \in V, \varphi \neq 0$ is called a direction in V.

If Γ is " regular" (for instance, Γ is a C^1 (or C^∞)-manifold of dimension n-1) then the linear mapping $v \to v$ of $C^1(\bar{\Omega}) \to C^1(\Gamma)$ (resp. of $C^\infty(\bar{\Omega}) \to C^\infty(\Gamma)$ extends to a continuous linear map of $H^1(\Omega)$ into $L^2(\Gamma)$ denoted by γ and for $v \in H^1(\Omega)$, $\gamma(v)$ is called the trace of v on Γ.

A linear tranformation $L: v \to L(v)$ is continuous if there exists a constant N such that

$$L(v) \le N\|v\|_V \qquad \text{for } v \in V.$$

A bilinear function $a(u,v): V \times V \to R^m$, is bicontinuous if there exists a constant M>0 such that

$$a(u,v) \le M\|u\|_V\|v\|_V \qquad \text{for all } u, v \in V.$$

A biliniear function a(u, v) is V-coercive if there exists a constant $\alpha > 0$ such that

$$a(u,v) \ge \alpha\|v\|_V^2 \qquad \text{for all } v \in V.$$

Let E be a vector space, a cone with vertex at o in E is a subset Λ of E such that, if λ belongs to Λ and if α belongs to R with $\alpha \ge 0$ then $\alpha\lambda$ also belongs to Λ.

In the following we have collected the Lemmas and Theorems we needed to obtain our main results. It should be noted that these are based on [1] except corollary (4) on [2].

1. LEMMA. The following equality

$$\sum_{j=1}^{n}(D_j u, D_j v)_{L^2(\Omega)} = -\int_\Omega (\Delta u)v dx + \int_\Gamma \frac{\partial u}{\partial \underline{n}} v d\sigma \tag{1}$$

where

$(D_j u, D_j v)_{L^2(\Omega)}$ is the inner product defined in $L^2(\Omega)$ is true.

PROOF. If $u, v \in C^1(\Omega)$, and by using Green's formula [3], we get

$$\int_\Omega (D_j u)v dx = -\int_\Omega u(D_j v)dx + \int_\Gamma u\, v\, n_j d\sigma \tag{2}$$

where $d\sigma$ is the area element on Γ.

We define the operator of exterior normal derivation formally as

$$\frac{\partial}{\partial \underline{n}} = \sum_{j=1}^{n} n_j(x)D_j,$$

such that $\underline{n}(x)$ is the unique outer normal vector at each point x on Γ, and $(\underline{n}_1(x),\ldots,\underline{n}_n(x))$ are the direction cosines of $\underline{n}(x)$.

Next, if $u, v \in C^2(\bar{\Omega})$, then applying the above formula to $D_j u, D_j v$ becomes

$$\int_\Omega (D_j^2 u)v dx = -\int_\Omega (D_j u)(D_j v)dx + \int_\Gamma (D_j u)\, v\, n_j d\sigma$$

and

$$\sum_{j=1}^{n}\int_\Omega (D_j u)(D_j v)dx = -\sum_{j=1}^{n}\int_\Omega (D_j^2 u)v dx + \sum_{j=1}^{n}\int_\Gamma (D_j u)v\, n_j d\sigma.$$

Then

$$\sum_{j=1}^{n}(D_j u, D_j v)_{L^2(\Omega)} = -\sum_{j=1}^{n}\int_\Omega (D_j^2 u)v dx + \int_\Gamma \frac{\partial u}{\partial \underline{n}} v\, d\sigma$$

$$\sum_{j=1}^{n}(D_j u, D_j v)_{L^2(\Omega)} = -\int_{\Omega}(\Delta u)v\,dx + \int_{\Gamma}\frac{\partial u}{\partial n}v\,d\sigma.$$

2. LEMMA. Let V be a Hilbert space, V' be its strong dual, and $J: V \to R$ be a functional such that

$$J(v) = \frac{1}{2}a(v,v) - L(v)$$

where $a(\,,\,)$ is a symmetric bilinear, bicontinuous, coercive form on V and $L \in V'$. Further, let K be a closed convex subset of V then for all $v \in K$ there exist $u \in K$ such that

$$J(u) \le J(v)$$

3. THEOREM. There exists a unique solution $u \in K$ which minimizes J on K, and this problem is equivalent to the following variational problem: Find $u \in K$ such that $a(u, v-u) \ge L(v-u)$

4. COROLLARY. Let C and D be non-empty closed convex sets in R^m and R^n, respectively, and let K be any finite continuous concave-convex function on C x D.

Let $\underline{K}$ and $\overline{K}$ be the lower and upper simple extensions of K to $R^m \times R^n$, respectively. Then $\underline{K}$ is lower closed, $\overline{K}$ is upper closed, and there exists a unique closed convex bifunction from R^m to R^n such that

$$\underline{K}(u,x^*) = \langle Fu, x^* \rangle, \quad \overline{K}(u,x^*) = \langle u, F^*x^* \rangle.$$

The bifunctions F and F^* are expressed in terms of K by

$$(Fu)(x) = \begin{cases} \sup\{\langle x,x^* \rangle - K(u,x^*);\, x^* \in D\} & \text{if } u \in C, \\ +\infty & \text{if } u \notin C, \end{cases}$$

$$(F^*x^*)(u^*) = \begin{cases} \inf\{\langle u,u^* \rangle - K(u,x^*);\, u \in C\} & \text{if } x^* \in D, \\ -\infty & \text{if } x^* \notin D. \end{cases}$$

In particular, dom F=C and dom F^* =D.

5. THEOREM. (Ky Fan and Sion)

Let V and E be two Hausdorff topological vector spaces, U be a convex compact subset of V and Λ be a convex compact subset of E. Suppose

$$\mathcal{L}: U \times \Lambda \to R$$

be a functional such that

i) for every $v \in U$ the functional

$$\mathcal{L}(v,.): \mu \to \mathcal{L}(v, \mu)$$

is upper-semi-continuous and concave,

ii) for every $\mu \in \Lambda$ the functional

$$\mathcal{L}(.,\mu): v \to \mathcal{L}(v, \mu)$$

is lower-semi-continuous and convex. Then there exists a saddle point $(u,\lambda) \in U \times \Lambda$ for $\mathcal{L}$.

3. MAIN RESULTS.

6. THEOREM. Let Ω be a bounded open set in R^n with smooth boundary Γ. Let V_1 and V_2 be two Hilbert spaces, $V_1 = H^1(\Omega)$, and V_1', V_2' be two (strongly) duals. If the bifunction

$$J: V_1 \times V_2 \to R^m \times R^n$$

is defined by

$$J(v_1, v_2) = ((\tfrac{1}{2}a(v_1, v_1) - L(v_1)), v_2)$$

where $a(\ ,\ )$ is a finite symmetric bilinear, bicontinuous, coercive form on $V_1, L \in V_1'$, and K_1 is the following subset:

$$K_1 = \{v_1; v_1 \in H^1(\Omega), \|\gamma\, v_1\|_{L^2(\Gamma)} \le 1; \gamma(v_1) \text{ is a trace}\}.$$

Then there exist a minimax of J.

PROOF. We first prove that K_1 is a closed convex set. To show that it is closed, we suppose $(v_1^n) \in K_1$ is a sequence such that $(v_1^n) \to v_1$ in V_1, and since

$$\gamma: H^1(\Omega) \to L^2(\Gamma)$$

is a continuous linear map. Then

$$\gamma(v_1^n) \to \gamma(v_1) \text{ in } L^2(\Gamma).$$

If $\varphi \in L^2(\Gamma)$ is such that $\varphi > 0$ on Γ, then

$$\int_\Gamma |\gamma(v_1)\varphi| d\sigma = \lim_{n\to\infty} \int_\Gamma |\gamma(v_1^n)\varphi| d\sigma$$

$$\le \lim_{n\to\infty} \left(\int_\Gamma |\gamma(v_1^n)|^2 d\sigma\right)^{1/2} \left(\int_\Gamma |\varphi|^2 d\sigma\right)^{1/2},$$

we choose $\|\varphi\|_{L^2(\mu)}$ such that

$$\lim_{n\to\infty} \|\gamma(v_1^n)\|_{L^2(\Gamma)} \|\varphi\|_{L^2(\Gamma)} \le 1 \qquad \text{for } (v_1^n) \in K.$$

From which we deduce that $\|\gamma(v_1)\|_{L^2(\Gamma)} \le 1$.

Also, to show that K_1 is convex, let v_1 and v_2 be two elements in K_1, i.e.

$$\|\gamma(v_1)\|_{L^2(\Gamma)} \le 1 \quad \text{and} \quad \|\gamma(v_2)\|_{L^2(\Gamma)} \le 1.$$

Let λ be a number such that $0 \le \lambda \le 1$. Now $\lambda v_1 + (1-\lambda)v_2 \in K_1; \gamma$ is a continuous linear map, therefore

$$\|\gamma(\lambda v_1 + (1-\lambda)v_2)\|_{L^2(\Gamma)} \le \|\lambda\gamma(v_1) + (1-\lambda)\gamma(v_2)\|_{L^2(\Gamma)} \le \|\lambda\gamma(v_1)\|_{L^2(\Gamma)} + \|(1-\lambda)\gamma(v_2)\|_{L^2(\Gamma)}$$

$$\le \lambda\|\gamma(v_1)\|_{L^2(\Gamma)} + (1-\lambda)\|\gamma(v_2)\|_{L^2(\Gamma)} \le \lambda + (1-\lambda) = 1.$$

Now consider $f \in L^2(\Gamma)$, then the problem of minimizing J where

$$J(v_1, v_2) = ((\tfrac{1}{2}(v_1, v_1)_{v_1}, -(f, v_1)_{v_1}), v_2)$$

on the closed convex set K_1 is equivalent to finding $u \in K_1$ satisfying the inequality

$$(a(u, v_1 - u), v_2) \equiv ((u, v_1 - u)_{V_1}, v_2)$$

$$\ge \left((f, v_1 - u)_{L^2(\Gamma)}, v_2\right) \qquad \forall\, v_1 \in K_1$$

where $(f, v_1 - u)_{L^2(\Gamma)} = L(v_1 - u)$.

Therefore, assuming the solution u (which exists and unique by Theorem 3) is sufficiently regular, we can interpret u as follows:

By Green's formula (1), we have

$$\int_{\Omega}(-\Delta u+u-f)(v_1-u)dx+\int_{\Gamma}\frac{\partial u}{\partial \underline{n}}(v_1-u)d\sigma \geq 0, \quad \forall\, v_1 \in K_1.$$

Ω, Γ are subsets of R^m.

If $\varphi \in \mathcal{D}(\Omega)$ then the boundary integral vanishes for $v_1 = u \pm \varphi$ which belongs to K_1 and

$$\int_{\Omega}(-\Delta u+u-f)\varphi dx \geq 0$$

which implies that $-\Delta u+u-f \geq 0$ in Ω.

Next since $v_1 = 2u$ and $v_1 = \frac{1}{2}u$ are both belong to K_1, we have

$$\int_{\Gamma}\frac{\partial u}{\partial \underline{n}} u d\sigma = 0$$

which implies that $\frac{\partial u}{\partial \underline{n}} u = 0$ on Γ.

Thus the minimal of J is equivalent to the solution of the following problem

$$\begin{cases} -\Delta u+u-f \geq 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \underline{n}} u = 0 & \text{on } \Gamma, \\ \frac{\partial u}{\partial \underline{n}} u \geq 0 & \text{on } \Gamma, \\ u \geq 0 & \text{on } \Gamma, \end{cases} \qquad \text{(I)}$$

One can also deduce from (I) that on the subset of Γ where $u > 0$, u satisfies the homogeneous Neumann condition $\frac{\partial u}{\partial \underline{n}} = 0$.

From the Corollary 4, and since K_2 is a nonempty closed convex set in V_2, we then have

$$J(v_1,u_2)=\begin{cases} \max_{v_2}\{J(v_1,v_2);\ v_2 \in K_2\} & \text{if } v_1 \in K_1, \\ +\infty & \text{if } v_1 \notin K_1 \end{cases}$$

i.e.,

$$J(v_1,u_2)=\begin{cases} \max_{v_2}\{((\frac{1}{2}a(v_1,v_1)-L(v_1)),v_2);\ v_2 \in K_2\} & \text{if } v_1 \in K_1, \\ +\infty & \text{if } v_1 \notin K_1 \end{cases}$$

7. THEOREM. Suppose the functional

$$J: H^1(\Omega) \to R$$

is given by

$$J(v) = \frac{1}{2}a(v,v) - L(v)$$

and the closed convex set K of $V = H^1(\Omega)$ is given by

$$K = \{v;\ v \in H^1(\Omega),\quad 1-\text{grad}^2\ v(x) \geq 0 \text{ a.e. in } \Omega\}$$

where

$$1-\text{grad}^2 u(x) \in L^1(\Omega) \qquad \text{and} \quad (L^1(\Omega))' = L^\infty(\Omega).$$

Then the lagrangian

$$\mathcal{L}(v,\mu) = J(v) + \langle \mu, v \rangle_{L^\infty(\Omega) \times L^1(\Omega)}$$

associated to the primal problem (finding $u \in K$ such that $J(u) \leq J(v), \forall \; v \in K$), has a minimax point (saddle point).

PROOF. Let $\ell > 0$ be any real number. We consider the subsets K_ℓ and Λ_ℓ of $H^1(\Omega)$ and Λ respectively are defined by

$$K_\ell = \{v;\ v \in H^1(\Omega), 0 \leq 1 - \text{grad}^2 \quad v(x) \leq \ell \ \text{ a.e. in } \Omega\}$$

$$\Lambda_\ell = \{\mu;\ \ \mu \in L^\infty(\Omega), \quad 0 \leq \mu \leq \ell \quad \text{in} \quad \Omega\}$$

such that Λ_ℓ being the cone in infinite dimensional Banach space.

Frist, we show that K_ℓ and Λ_ℓ are convex sets in Ω.

Let $v_1, \ v_2 \in K_\ell$ i.e.,

$$\int_\Omega \left|1 - \text{grad}^2 v_1(x)\right| dx \leq \ell^{1/2} \quad \text{a.e. in } \Omega,$$

and

$$\int_\Omega \left|1 - \text{grad}^2 v_2(x)\right| dx \leq \ell^{1/2} \quad \text{a.e. in } \Omega.$$

Let $0 \leq \lambda \leq 1$ we have

$$\begin{aligned}\int_\Omega \left|1 - \text{grad}^2(\lambda v_1(x) + (1-\lambda) v_2(x))\right| dx &= \int_\Omega \left|1 - \lambda \ \text{grad}^2 v_1(x) - (1-\lambda)\text{grad}^2 v_2(x)\right| dx \\ &= \int_\Omega \left|\lambda(1 - \text{grad}^2 v_1(x)) + (1-\lambda)(1 - \text{grad}^2 v_2(x))\right| dx \\ &\leq \int_\Omega \left|\lambda(1 - \text{grad}^2 v_1(x))\right| dx + \int_\Omega \left|(1-\lambda)(1 - \text{grad}^2 v_2(x))\right| dx \\ &\leq \lambda\sqrt{\ell} + (1-\lambda)\sqrt{\ell} = \sqrt{\ell} \qquad \text{a.e. in } \Omega.\end{aligned}$$

and hence is a convex set.

Now let $\mu_1, \mu_2 \in \Lambda_\ell$ i.e., $\text{Sup}\left|\mu_1\right| \leq \ell$ in Ω and $\text{Sup}\left|\mu_2\right| \leq \ell$ in Ω , for $0 \leq \lambda \leq 1$ we get

$$\begin{aligned}\text{Sup}\left|\lambda\mu_1 + (1-\lambda)\mu_2\right| &\leq \text{Sup}(\left|\lambda\mu_1\right| + \left|(1-\lambda)\mu_2\right|) \qquad \text{in } \Omega \\ &\leq \text{Sup}\left|\lambda\mu_1\right| + \text{Sup}\left|(1-\lambda)\mu_2\right| \leq \lambda\,\ell + (1-\lambda)\ell = \ell \quad \text{in } \Omega\end{aligned}$$

Thus Λ_ℓ is a convex set in Ω. And Λ_ℓ is compact in the dual weak topology of $L^\infty(\Omega)$. Since K_ℓ is a closed bounded set in the Hilbert space $H^1(\Omega)$, K_ℓ is weakly compact. We consider $H^1(\Omega)$ with its weak topology.

Now $H^1(\Omega) = V$ with the weak topology is a Hausdorff topological vector space.

All the hypothesis of the theorem of Ky Fan and Sion are satisfied by K_ℓ, Λ_ℓ and $\mathcal{L}$ in view of (I) and (II). Hence

$$\mathcal{L}: K_\ell \times \Lambda_\ell \to R$$

has a saddle point (u_ℓ, λ_ℓ), i.e., there exist $(u_\ell, \lambda_\ell) \in K_\ell \times \Lambda_\ell$ such that

$$\text{(II)} \begin{cases} J(u_\ell) + \langle \mu, u_\ell \rangle_{L^\infty(\Omega) \times L^1(\Omega)} \leq J(u_\ell) + \langle \lambda_\ell, u_\ell \rangle_{L^\infty(\Omega) \times L^1(\Omega)} \\ \leq J(v) + \langle \lambda_\ell, v \rangle_{L^\infty(\Omega) \times L^1(\Omega)}, \qquad \forall (v, \mu) \in K_\ell \times \Lambda_\ell. \end{cases}$$

We shall show that if we choose $\ell > 0$ sufficiently large then such a saddle point can be obtained independent of ℓ.

For this we shall first prove $\|u_\ell\|_{H^1(\Omega)}$ and λ_ℓ are bounded by constants independent of ℓ.

If we take $\mu = 0 \in \Lambda_\ell$ in (II) we get

$$J(u_\ell) \le J(v) + <\lambda_\ell, v>_{L^\infty(\Omega)xL^1(\Omega)}, \quad \forall \quad v \in K_\ell \tag{3}$$

taking $v = 0 \in K_\ell \cap K$ such that

$$1 - \mathrm{grad}^2 v(x) \ge 0 \qquad \text{a.e. in } \Omega$$

we get

$$J(u_\ell) \le J(0) \quad \text{for} \quad 1 - \mathrm{grad}^2 v(x) \ge 0 \qquad \text{a.e. in } \Omega.$$

On the other hand, since $a(u_\ell, u_\ell) \ge 0$ and since $u_\ell \in K_\ell$

$$L(u_\ell) \le \|L\|_{v'} \|1 - \mathrm{grad}^2 v(x)\|_{L^1(\Omega)} \le \ell \|L\|_{v'} \qquad \text{a.e. in } \Omega$$

and hence

$$J(u_\ell) = \frac{1}{2} a(u_\ell, u_\ell) - L(u_\ell) \ge -\ell \|L\|_{v'} \qquad \text{a.e. in } \Omega$$

Thus we have

$$-\ell \|L\|_{v'} \le J(u_\ell) \le J(0) \qquad \text{a.e. in } \Omega \tag{4}$$

Now by coercivity of $a(.,.)$ and (4) we find

$$\alpha \||u_\ell|\|^2 \le a(u_\ell, u_\ell) = 2(J(u_\ell) + L(u_\ell)) \le 2(J(0) + \|L\|_{v'} \||u_\ell|\|) \text{ a.e. in } \Omega$$

with a constant $\alpha > 0$ (independent of ℓ) and $\||.|\|$ defined the norm $\|.\|_{H^1(\Omega)}$.

By using the inequality

$$\|L\|_{v'} \||u_\ell|\| \le \varepsilon \||u_\ell|\|^2 + \frac{1}{\varepsilon} \|L\|^2_{v'} \qquad \text{for any } \varepsilon > 0$$

with $\varepsilon = \alpha/4 > 0$, we optain

$$\alpha \||u_\ell|\| \le 2J(0) + \frac{\alpha}{2} \||u_\ell|\|^2 + \frac{8}{\alpha} \|L\|^2_{v'}$$

Therefore,

$$\||u_\ell|\|^2 \le \frac{4}{\alpha}(J(0) + \frac{4}{\alpha} \|L\|^2_{v'}).$$

This proves that there exists a constant $C_1 > 0$ such that

$$\||u_\ell|\|^2 \le C_1 \qquad \forall \ \ell, \qquad \text{a.e. i } \Omega. \tag{5}$$

We observe that since J satisfies all assumptions of Theorem (3), there will exists a unique global minimum in $V = H^1(\Omega)$ i.e., there exists a unique $\tilde{u} \in H^1(\Omega)$ such that

$$J(\tilde{U}) \le J(v), \qquad \forall \ v \in V \tag{6}$$

But, if we take $v = 1 \in K_\ell$ in the inequality (3) we get

$$J(u_\ell) + \sup \lambda_\ell \le J(1).$$

The last two inequalities imply that

$$C_2 = -J(\tilde{u}) + J(1) \ge \sup \lambda_\ell \ge 0 \tag{7}$$

which proves that λ_ℓ is also bounded, ℓ may be taken as follows

$$\ell > \max(C_1, 2C_2) > 0 \tag{8}$$

Next we show that (II) holds for any $\mu \in \Lambda$ For this we write the frist inequality of (II) in the form

$$<\mu, u_\ell>_{L^\infty(\Omega)\times L^1(\Omega)} \le <\lambda_\ell, u_\ell>_{L^\infty(\Omega)\times L^1(\Omega)},$$

which implies that

i) taking $\mu = 0$ we have $<\lambda_\ell, u_\ell>_{L^\infty(\Omega)\times L^1(\Omega)} \ge 0$, and

ii) taking $\mu = 2\lambda_\ell \le 2C_2 < \ell$ we have $<\lambda_\ell, u_\ell>_{L^\infty(\Omega)\times L^1(\Omega)} \le 0$.

Thus

$$<\lambda_\ell, u_\ell>_{L^\infty(\Omega)\times L^1(\Omega)} = 0 \quad \text{and} \quad <\mu, u_\ell>_{L^\infty(\Omega)\times L^1(\Omega)} \le 0, \quad \forall\ \mu \in \Lambda_\ell$$

In particular, $\mu = \ell \in \Lambda_\ell$ and so $<\ell, u_\ell>_{L^\infty(\Omega)\times L^1(\Omega)} \le 0$ which means that

$$<\mu, u_\ell>_{L^\infty(\Omega)\times L^1(\Omega)} \le 0 \quad \text{for any } \mu \ge 0,$$

and therefore

$$\mathcal{L}(u_\ell, \mu) \le \mathcal{L}(u_\ell, \lambda_\ell) \le \mathcal{L}(v, \lambda_\ell) \quad \forall\ \mu \ge 0 \text{ and } v \in K_\ell \tag{9}$$

where $\ell > \max(C_1, 2C_2)$.

We have now only to show that the inequality (9) holds for any $v \in H^1(\Omega) = V$.

For this, we note that $\||\, u_\ell \,|\|^2 \le C_1 < \ell$ a.e. in Ω, and hence we can find an $r > 0$ such that the ball

$$B(u_\ell, r) = \{v; v \in H^1(\Omega): 1 - \text{grad}^2(v(x) - u_\ell(x)) < r \ \text{ a.e. in } \Omega\}$$

is contained in the ball

$$B(u_\ell, r) = \{v; v \in H^1(\Omega): 1 - \text{grad}^2 v(x) < \ell \ \text{ a.e. in } \Omega\}$$

In fact, it is enough to take $0 < r < (\ell - C_1)/2$. Now the functional

$$\mathcal{L}(\cdot, \lambda_\ell): v \to \mathcal{L}(v, \lambda_\ell)$$

$$\mathcal{L}(v, \lambda_\ell) = J(v) + <\lambda_\ell, v>_{L^\infty(\Omega)\times L^1(\Omega)}$$

has a local minimum in $B(u_\ell, r)$. But since this functional is convex such a minimum is also a global minimum. This means that

$$\inf_{v \in B(u_\ell, r)} \mathcal{L}(v, \lambda_\ell) = \inf_{v \in V} \mathcal{L}(v, \lambda_\ell).$$

On the other hand, since $B(u_\ell, r) \subset K_\ell$, we see from (9) that

$$\mathcal{L}(u_\ell, \mu) \le \mathcal{L}(u_\ell, \lambda_\ell) \le \inf_{v \in K_\ell} \mathcal{L}(v, \lambda_\ell) \le \inf_{v \in B(u_\ell, r)} \mathcal{L}(v, \lambda_\ell) = \inf_{v \in V} \mathcal{L}(v, \lambda_\ell).$$

In other words, we have

$$\mathcal{L}(u_\ell, \mu) \le \mathcal{L}(u_\ell, \lambda_\ell) \le \mathcal{L}(v, \lambda_\ell), \quad \forall\, v \in V, \text{ and } \forall\, \mu \ge 0,$$

which means that $\mathcal{L}$ has a saddle point.

REFERENCES

1. JEAN CEA, Lectures on Optimizing-Theory and Algorithms, Narosa Publishing House, New Delhi, 1978

2. Rockafellar, R.T., Convex Analysis, Princeton, New Jersy, 1970.

3. Vladimirov, V. S. Equations of Mathematical Physics, Mir Publishers, Moscow, 1984.