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A FIXED POINT THEOREM FOR GENERALIZED METRIC SPACES

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ABSTRACT. In this paper we prove two fixed point theorems for the generalized metric spaces introduced by Dhage.

In a recent paper, Dhage [1] defined a generalized metric space as follows: Let $D : X \times X \times X \to \mathbb{R}$ with the following properties:

(i) $D(x,y,z) \geq 0$ for each $x,y,z \in X$, with equality if and only if $x = y = z$,
(ii) $D(x,y,z) = D(y,x,z) = D(x,z,y) = \cdots$ (symmetry)
(iii) $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$, for each $x,y,z \in X$.

2-metric spaces are defined by a function $d : X \times X \times X \to \mathbb{R}$ with properties (ii) and (iii) above, and (i) replaced by

(i′) For each distinct pair $x,y \in X$, there exists a $z \in X$ such that $d(x,y,z) \neq 0$, and $d(x,y,z) = 0$ if any two of the triplet x,y,z are equal.

A number of fixed point theorems have been proved for 2-metric spaces. However, Hsiao [2] showed that all such theorems are trivial in the sense that the iterations of f are all colinear. The situation for D-metric spaces is quite different. Some specific examples of D-metric spaces appear in [1].

The purpose of this paper to prove two general fixed point theorems for D-metric spaces.

THEOREM 1. Let X be a complete and bounded D-metric space, f a selfmap of X satisfying

$$D(Tx,Ty,Tz) \leq q \max\{D(x,y,z), D(x,Tx,z), D(y,Ty,z), D(x,Ty,z), D(y,Tx,z)\} \tag{1}$$

for all $x,y,z \in X$, $0 \leq q < 1$. Then T has a unique fixed point p in X, and T is continuous at p.

PROOF. Let $x_0 \in X$ and define $x_{n+1} = Tx_n$. If $x_{n+1} = x_n$ for some n, then T has a fixed point. Assume that $x_{n+1} \neq x_n$ for each n. In (1), setting $x = x_{n-1},\ y = x_n,\ z = x_{n+p}$, we have

$$\begin{aligned} D(x_n, x_{n+1}, x_{n+p}) \leq q \max\{&D(x_{n-1}, x_n, x_{n+p-1}), D(x_{n-1}, x_n, x_{n+p-1}), \\ &D(x_n, x_{n+1}, x_{n+p-1}), D(x_{n-1}, x_{n+1}, x_{n+p-1}), \\ &D(x_n, x_n, x_{n+p-1})\}. \end{aligned} \tag{2}$$

But

$$\begin{aligned} D(x_{n-1}, x_n, x_{n+p-1}) \leq q \max\{&D(x_{n-2}, x_{n-1}, x_{n+p-2}), D(x_{n-2}, x_{n-1}, x_{n+p-2}), \\ &D(x_{n-1}, x_n, x_{n+p-2}), D(x_{n-2}, x_n, x_{n+p-2}), \\ &D(x_{n-1}, x_{n-1}, x_{n+p-2})\}, \end{aligned} \tag{3}$$

$$D(x_n, x_{n+1}, x_{x+p-1} \leq q \max\{D(x_{n-1}, x_n, x_{n+p-2}), D(x_{n-1}, x_n, x_{n+p-2}),$$
$$D(x_n, x_{n+1}, x_{n+p-2}), D(x_{n-1}, x_{n+1}, x_{n+p-2}),$$
$$D(x_n, x_n, x_{n+p-2})\}, \tag{4}$$
$$D(x_{n-1}, x_{n+1}, x_{n+p-1}) \leq q \max\{D(x_{n-2}, x_n, x_{n+p-2}), D(x_{n-2}, x_{n-1}, x_{n+p-2}),$$
$$D(x_n, x_{n+1}, x_{n+p-2}), D(x_{n-2}, x_{n+1}, x_{n+p-2}),$$
$$D(x_n, x_{n-1}, x_{n+p-1})\}, \tag{5}$$

and

$$D(x_n, x_n, x_{n+p-1}) \leq q \max\{D(x_{n-1}, x_{n-1}, x_{n+p-2}), D(x_{n-1}, x_n, x_{n+p-2})\}. \tag{6}$$

Substituting (3) - (6) into (2) gives

$$D(x_n, x_{n+1}, x_{n+p}) \leq q^2 \max_{a,b,c} D(x_a, x_b, x_c),$$

where $n-2 \leq a \leq n,\ n-1 \leq b \leq n+1$, and $c = n+p-2$. Continuing this process it follows that

$$D(x_n, x_{n+1}, x_{n+p-1}) \leq q^n \max_{a,b,c} D(x_a, x_b, x_c), \tag{7}$$

where now $0 \leq a \leq n,\ 1 \leq b \leq n+1$, and $c = p$. Let $M := \sup_{x,y,z \in X} D(x,y,z)$. Then, it follows from (7) that

$$D(x_n, x_{n+1}, x_{n+p}) \leq q^n M. \tag{8}$$

Using (iii) and (8),

$$\begin{aligned} D(x_n, x_{n+p}, x_{n+p+t}) &\leq D(x_n, x_{n+p}, x_{n+1}) + D(x_n, x_{n+1}, x_{n+p+t}) + D(x_{n+1}, x_{n+p}, x_{n+p+t}) \\ &\leq 2Mq^n + D(x_{n+1}, x_{n+p}, x_{n+p+t}) \\ &\leq 2Mq^n + D(x_{n+1}, x_{n+p}, x_{n+2}) + D(x_{n+1}, x_{n+2}, x_{n+p+t}) \\ &\quad + D(x_{n+2}, x_{n+p}, x_{n+p+t}) \\ &\leq 2M(q^n + q^{n+1}) + D(x_{n+2}, x_{n+p}, x_{n+p+1}) \leq \cdots \\ &\leq 2M(q^n + q^{n+1} + \cdots + q^{n+p-1}) + D(x_{n+p-1}, x_{n+p}, x_{n+p+t}) \\ &\leq 2M \sum_{k=n}^{n+p} q^k \leq \frac{2Mq^n}{1-q} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore $\{x_n\}$ is D-Cauchy. Since X is complete, $\{x_n\}$ converges. Call the limit p.

From (1),

$$D(x_n, x_{n+1}, Tp) \leq q \max\{D(x_{n-1}, x_n, p), D(x_n, x_{n+1}, p), D(x_{n-1}, x_{n+1}, p), D(x_n, x_n, p)\}.$$

Taking the limit as $n \to \infty$, and using the fact that D is continuous, yields $D(p, p, Tp) \leq 0$, which implies that $p = Tp$.

To prove uniqueness, assume that $w \neq p$ is also a fixed point of T. From (1),

$$\begin{aligned} D(p, w, p) &= D(Tp, Tw, Tp) \\ &\leq q \max\{D(p, w, p), D(p, Tp, p), D(w, Tw, p), D(p, Tw, p), D(w, Tp, p)\} \\ &= q \max\{D(p, w, p), D(w, w, p)\} = qD(w, w, p). \end{aligned} \tag{9}$$

But

$$\begin{aligned} D(w, w, p) &= D(w, p, w) = D(Tw, Tp, Tw) \\ &\leq q \max\{D(w, p, w), D(w, Tw, w), D(p, Tp, w), D(w, Tp, w), D(p, Tw, w)\} \\ &= q \max\{D(w, p, w), D(p, p, w)\} = qD(p, p, w) \end{aligned} \tag{10}$$

Combining (9) and (10) yields $D(p,w,p) \leq q^2 D(p,w,p)$, a contradiction. Therefore $p = w$.

To show that T is continuous at p, let $\{y_n\} \subseteq X$ with $\lim y_n = p$. Then, substituting in (1), with $x = z = p,\ y = y_n$, we obtain

$$D(Tp,Ty_n,Tp) \leq q\max\{D(p,y_n,p), D(p,Tp,p), D(y_n,Ty_n,p),\\ D(p,Ty_n,p), D(y_n,Tp,p)\} \tag{11}$$

Taking the lim sup of (11), we obtain

$$\limsup D(p,Ty_n,p) \leq q\max\{0,0,\limsup D(p,Ty_n,p),\ 0\},$$

which implies that $\lim Ty_n = p = Tp$, and T is continuous at p.

COROLLARY 1. Let X be a complete and bounded D-metric space, m a positive integer, T a selfmap of X satisfying

$$D(T^m x,T^m y,T^m z) \leq q\max\{D(x,y,z), D(x,T^m x,z), D(y,T^m y,z),\\ D(x,T^m y,z), D(y,T^m x,z)\} \tag{1?}$$

for all $x,y,z \in X,\ 0 \leq q < 1$. Then T has a unique fixed point p in X, and T^m is continuous at p.

PROOF. From Theorem 1, T^m has a unique fixed point p, and T^m is continuous at p. But $Tp = T(T^m p) = T^m(Tp)$, and Tp is also a fixed point of T^m. Since the fixed point is unique, $p = Tp$.

THEOREM 2. Let X be a compact D-metric space, f a continuous selfmap of X satisfying

$$D(Tx,Ty,Tz) < \max\{D(x,y,z), D(x,Tx,z), D(y,Ty,z),\\ D(x,Ty,z), D(y,Tx,z)\} \tag{12}$$

for all $x,y,z \in X$. Then T has a unique fixed point p in X.

PROOF. Since X is compact, both sides of (12) are bounded.

Case I. Suppose that the right-hand-side of (12) is positive for all x,y,z in X. Define

$$f(x,y,z) := \frac{D(Tx,Ty,Tz)}{\max\{D(x,y,z), D(x,Tx,z), D(y,Ty,z), D(x,Ty,z), D(y,Tx,z)\}}.$$

Since T and D are continuous, so is f. The compactness of X implies that f assumes its maximum at some point (u,v,w) in X. Call the value c. From (12), it follows that $0 < c < 1$. Thus T now satisfies (1) with $q = c$. By Theorem 1, T has a unique fixed point p.

Case II. Suppose there exists a point (x,y,z) such that the right-hand-side of (12) is zero. Then, in particular, $x = Tx = z$, and x is a fixed point of T. Suppose that w is also a fixed point of T. Then, using the same argument as in Theorem 1, it follows that $x = w$, and the fixed point is unique.

COROLLARY 2. Let X be a compact D-metric space, m a positive integer, T a continuous selfmap of X satisfying

$$D(T^m x,T^m y,T^m z) < \max\{D(x,y,z), D(x,T^m x,z), D(y,T^m y,z),\\ D(x,T^m y,z), D(y,T^m x,z)\} \tag{12}$$

for all $x, y, z \in X$. Then T has a unique fixed point p in X.

The proof of Corollary 2 parallels that of Corollary 1.

Theorem 2.1 and 2.2 of Dhage [1] are special cases of Theorems 1 and 2 of this paper.

There are two limitations involving fixed point theorems on D-metric spaces. The first is that the proof of the existence of a fixed point appears to require that X be bounded. The second is that there is apparently no reasonable contractive definition for a pair of maps on a D-metric space.

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