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ON THE DISTRIBUTIONAL ORTHOGONALITY OF THE GENERAL POLLACZEK POLYNOMIALS

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ABSTRACT. A distributional representation of the moment functional of the general Pollaczek polynomials is established. This representation holds for a wider range of parameters than the representation by a positive measure.

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1. INTRODUCTION.

A regular moment functional (Chihara [7], Chap I) is a complex linear map $\mathcal{L}$ of the space of complex polynomials into the field of complex numbers for which there is a system $\{ P_n(x) \mid n \geq 0\}$ of monic polynomials determined by a recurrence relation

$$xP_n(x) = P_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}(x), \ n \geq 0, \tag{1.1}$$

with $P_{-1}(x) = 0, \ P_0(x) = 1$ and

$$C_{n+1} \neq 0, \ n \geq 0, \tag{1.2}$$

such that

$$\mathcal{L}(1) = \mathcal{L}(P_0(x)) = 1; \qquad \mathcal{L}(P_n(x)) = 0, \ n \geq 1. \tag{1.3}$$

and, with

$$\lambda_0 = 1; \ \lambda_n = C_1 \cdots C_n, \ n \geq 1,$$

that

$$\mathcal{L}(P_n(x).P_m(x) = \lambda_n\delta_{mn}, \quad m,n \geq 0 \tag{1.5}$$

Observe that $\lambda_n \neq 0, \ n \geq 0$. The system $\{P_n(x)\}$ is uniquely determined by $\mathcal{L}$, and because of (1.5), it is called the monic orthogonal system of $\mathcal{L}$. Conversely (Chihara [7], Chap. I), if

$\{P_n(x)\}$ is determined by a recurrence relation (1.1) satisfying (1.2), and if $\mathcal{L}$ is defined through (1.3) and linear extension, $\mathcal{L}$ is regular and $\{P_n(x)\}$is its monic orthogonal polynomial system. The functional $\mathcal{L}$ is called the moment functional of $\{P_n(x)\}$.

If the recurrence relation (1.1) is bounded, i.e., if there is a constant $M > 0$ such that

$$|B_n| \leq \frac{M}{3}, \qquad |C_{n+1}| \leq \frac{M^2}{9},\ n \geq 0, \tag{1.6}$$

and if (1–2) holds, the continued fraction (Wall [16], Chap. V)

$$\frac{1\ |}{|x - B_0} - \frac{C_1\ |}{|x - B_1} - \frac{C_2\ |}{|x - B_2} - \cdots \tag{1.7}$$

of the system $\{P_n(x)\}$ converges uniformly on $|z| \geq M'$, for all $M' > M$, to a limit $X(z)$, which is an analytic function on $|z| > M$. Furthermore, if $\mathcal{L}$ is the moment functional of $\{P_n(x)\}$, then (Charris and Soriano [6], Ismail te al. [8])

$$\mathcal{L}(P(x)) = \frac{1}{2\pi i} \int_C X(z)P(z)\,dz, \tag{1.8}$$

for any positively oriented contour of $|z| > M$ enclosing $z = 0$. This is a very useful representation of $\mathcal{L}$.

If a regular moment functional is positive (Chihara [7], Chap. I) i.e., if B_n, C_n in (1–1) are real numbers and

$$C_{n+1} > 0,\ n \geq 0, \tag{1.9}$$

then (Chihara [7], Chap. II) a positive measure μ on the real line can be found such that

$$\mathcal{L}(P(x)) = \int_{-\infty}^{+\infty} P(x)\,d\mu(x). \tag{1.10}$$

If in addition (1.6) holds, μ is unique, $Supp\mu \subseteq [-M, M]$ and (1.7) converges to $X(z)$ on compact subsets of $\mathbb{C} - [-M, M]$. In practice, B_n, C_n in (1.1) usually depend on some parameters, $\alpha, \beta, \lambda, \ldots$. In the classical theory of orthogonal polynomials only the positive case is dealt with, which generally imposes strong restrictions on the ranges of these parameters in order to ensure that (1.8) holds. For many examples of how μ can be explicitely determined in such circumstances, see Askey and Ismail [1].

If (1.9) does not hold, representation (1.10) of $\mathcal{L}$ is out of the question. However, if (1.2) and (1.6) are still valid, which usually happens under less restrictive assumptions on the parameters B_n and C_n, representation (1.8) still holds, and can be used to derive other types of representations of $\mathcal{L}$. Among those, distributional representations are likely to be the most useful. For example, if a polynomial $q(x) = a(x - \alpha_1)^{p_1}(x - \alpha_2)^{p_2} \cdots (x - \alpha_m)^{p_m}$ with real roots can be found such that the functional $\mathcal{U} = q(x)\mathcal{L}$ defined by

$$\mathcal{U}(P(x)) = \mathcal{L}(q(x)P(x)) \tag{1.11}$$

is positive, then writing

$$\mathcal{L}(P(x)) = \mathcal{L}\left(q(x)\frac{P(x)}{q(x)}\right) \tag{1.12}$$

we obtain, by means of the partial fraction decomposition

$$\frac{P(x)}{q(x)} = \sum_{j=1}^{m} \sum_{k=1}^{p_j} \frac{\alpha_{jk}}{(x - \alpha_j)^k} + R_m(x) \tag{1.13}$$

($R_m(x)$ a polynomial) and (1.8), that if ν is the positive measure representing $\mathcal{U}$ then

$$\mathcal{L}(P(x)) = \sum_{j=1}^{m}\sum_{k=1}^{p_j}\left\{\frac{1}{2\pi i}\int_C \frac{q(z)}{(z-\alpha_j)^k} X(z)\,dz\right\}a_{jk} + \int_{-\infty}^{\infty} R_m(x)\,d\nu(x) \tag{1.14}$$

is, since

$$a_{jk} = \frac{1}{(p_j-k)!}\frac{d^{p_j-k}}{dx^{p_j-k}}\left[\frac{P(x)(x-\alpha_j)^{p_j}}{q(x)}\right](\alpha_j), \tag{1.15}$$

a representation of $\mathcal{L}$ by distributions supported by the real line.

The above argument provides an alternative approach to that of Krall [9] and Morton and Krall [10] to establish distributional representations of regular functionals, and can be applied to systems, such as the Pollaczek polynomials, that fall outside the scope of Krall [9] and Morton and Krall [10].

The aim of this paper is in fact to establish such a representation for the moment functional of the general Pollaczek polynomials (Section 2). To this purpose we recall some properties of the funtional $\mathcal{U} = q(x)\mathcal{L}$ defined by (1.11). We follow Belmehdi [3], where the history of this subject is briefly reviewed. We restrict ourselves to the case of $q(x) = a(x-\alpha_0)(x-\alpha_1)$, and assume that $\mathcal{L}$ is regular with monic orthogonal system $\{P_n(x)\}$ determined by (1.1). For $n \geq 0$, let

$$\Delta_n = \begin{vmatrix} P_n(\alpha_0) & P_{n+1}(\alpha_0) \\ P_n(\alpha_1) & P_{n+1}(\alpha_1)\end{vmatrix},\ \alpha_0 \neq \alpha_1;\ \Delta_n = \begin{vmatrix} P_n(\alpha_0) & P_{n+1}(\alpha_0) \\ P'_n(\alpha_0) & P'_{n+1}(\alpha_0)\end{vmatrix},\ \alpha_0 = \alpha_1. \tag{1.16}$$

Observe that $\Delta_0 = \alpha_1 - \alpha_0$ if $\alpha_0 \neq \alpha_1$, $\Delta_0 = 1$ if $\alpha_0 = \alpha_1$. Also let

$$D_n = \begin{vmatrix} P_{n-1}(\alpha_0) & P_{n+1}(\alpha_0) \\ P_{n-1}(\alpha_1) & P_{n+1}(\alpha_1)\end{vmatrix},\ \alpha_0 \neq \alpha_1;\ D_n = \begin{vmatrix} P_{n-1}(\alpha_0) & P_{n+1}(\alpha_0) \\ P'_{n-1}(\alpha_0) & P'_{n+1}(\alpha_0)\end{vmatrix},\ \alpha_0 = \alpha_1. \tag{1.17}$$

Then (Belmehdi [3])

LEMMA 1. The functional $\mathcal{U}$ is regular if and only if $\Delta_n \neq 0$ for all $n \geq 0$. If such is the case and $\{Q_n(x)\}$ is the monic system of orthogonal polynomials for $\mathcal{U}$, and if a is so chosen that $\mathcal{L}(q(x)) = 1$ (i.e., $\mathcal{U}(1) = 1$), then $\{Q_n(x)\}$ satisfies the recurrence relation

$$xQ_n(x) = Q_{n+1}(x) + \tilde{B}_nQ_n(x) + \tilde{C}_nQ_{n-1}(x),\ n \geq 0, \tag{1.18}$$

with $Q_{-1}(x) = 0,\ Q_0(x) = 1$,

$$\tilde{B}_n = B_n + C_n\frac{D_n}{\Delta_n} - C_{n+1}\frac{D_{n+1}}{\Delta_n},\ n \geq 0, \tag{1.19}$$

and

$$\tilde{C}_n = \frac{\Delta_{n-1}\Delta_{n+1}}{\Delta_n^2}C_n,\ n \geq 1 \tag{1.20}$$

Furthermore

$$\mathcal{U}(Q_n^2(x)) = a\frac{\Delta_{n+1}}{\Delta_n}\mathcal{L}(P_n^2(x)),\ n \geq 0, \tag{1.21}$$

so that $a = \Delta_0/\Delta_1$.

From (1.21) it follows that

$$\Delta_{n+1} = \left[\frac{1}{a}\right]^{n+1}\frac{\tilde{\lambda}_0\cdots\tilde{\lambda}_n}{\lambda_0\cdots\lambda_n}\Delta_0,\ n \geq 0, \tag{1.22}$$

where $\lambda_n = \mathcal{L}(P_n^2(x))$, $\tilde{\lambda}_n = \mathcal{U}(Q_n^2(x))$.

2. THE MONIC POLLACZEK POLYNOMIALS.

This system, denoted with $\{P_n^\lambda(x;\, a,\, b)\}$, is determined by (1.3) with $P_{-1}^\lambda(x;a,b) = 0$, $P_0^\lambda(x;a,\,b) = 1$, and

$$B_n = -\frac{b}{\lambda + a + n}, \qquad C_{n+1} = \frac{(n+1)(n+2\lambda)}{4(\lambda+a+n)(\lambda+a+n+1)}, \; n \geq 0. \tag{2.1}$$

For appropriate values of the parameters λ, a, b (see Charris and Ismail [5]), the continued fraction limit function $X_\lambda(z)$ of these polynomials may have an infinite number of poles. This rather exotic property makes it difficult to study the spectral measure or distributional representations of their moment functional. The Pollaczek polynomials were introduced by F. Pollaczek [11-13]. See also Szegö [15]. Singular cases of the Pollaczek polynomials are studied in Charris and Ismail [5]. Special cases of the pollaczek polynomials are useful in the description of certain physical phenomena (Bank and Ismail [2]).

We will assume a, b to be fixed throughout most arguments. So, we will only emphasize the parameter λ, and write $P_n^\lambda(x)$ instead $P_n^\lambda(x;\, a, b)$. We will assume hereafter that

$$2\lambda \text{ and } \lambda \pm a \quad \text{are not integers} \leq 0. \tag{2.2}$$

This guaranties that (1.2) holds. Observe that (1.9) demands in addition that λ, a, b should be real numbers and that

$$\lambda > 0 \text{ and } \lambda + a > 0, \text{ or, } -\frac{1}{2} < \lambda < 0 \text{ and } 0 < \lambda + a + 1 < 1 \tag{2.3}$$

The continued fraction limit function $X_\lambda(z)$ of $\{P_n^\lambda(x)\}$ is (see Charris and Ismail [5])

$$X_\lambda(z) = -\frac{2(\lambda+a)\beta}{B_\lambda}\, {}_2F_1\left(\left.\begin{matrix} A_\lambda + 1, 1 \\ -B_\lambda + 1 \end{matrix}\,\right|\, \beta^2\right), \qquad |z| > M_\lambda, \tag{2.4}$$

where ${}_2F_1$ is the hypergeometric function (Rainville [14], Chap.IV),

$$M_\lambda = 3\sup_n \left\{ \left|\frac{b}{n+\lambda+a}\right|, \left|\sqrt{\frac{(n+1)(n+2\lambda)}{4(n+\lambda+a)(n+\lambda+a+1)}}\right| \right\} \tag{2.5}$$

$$A_\lambda = -\lambda + \frac{az+b}{(z^2-1)^{\frac{1}{2}}}, \qquad B_\lambda = -\lambda - \frac{az+b}{(z^2-1)^{\frac{1}{2}}}, \tag{2.6}$$

and

$$\alpha = z + (z^2-1)^{\frac{1}{2}}, \qquad \beta = z - (z^2-1)^{\frac{1}{2}}, \tag{2.7}$$

with $(z^2-1)^{\frac{1}{2}}$ denoting the branch of the square root of z^2-1 in $\mathbb{C}$ that behaves as z when $z \to \infty$.

Clearly $M_\lambda < +\infty$, and it can be shown (see Charris and Ismail [5]) that $(z^2-1)^{\frac{1}{2}}$ and thus α, β, A_λ, B_λ are analytic functions on $\mathbb{C} - [-1,1]$, with branch discontinuities on $[-1,1]$. Also $\alpha + \beta = 2z$, $\alpha\beta = 1$, $\alpha - \beta = 2(z^2-1)^{\frac{1}{2}}$, $A_\lambda + B_\lambda = -2\lambda$ and $|\beta(z)| \leq 1 \leq |\alpha(z)|$, with $|\beta(z)| = 1 = |\alpha(z)|$ if and only if $-1 \leq z \leq 1$. From (2.4) and the above properties of α, β, A_λ, B_λ it follows that $X_\lambda(z)$ is in fact analytic on $\mathbb{C} - [-1,1]$, except, perhaps, for simple poles on the set

$$Z_\lambda = \{z \in \mathbb{C} - [-1,1]\, |\, B_\lambda(z) = n,\ n = 0,1,2,\dots\} \tag{2.8}$$

Since $B_\lambda(z) = n$ implies that

$$\left[a^2 - (n+\lambda)^2\right] x^2 + 2abx + b^2 + (n+\lambda)^2 = 0, \tag{2.9}$$

there are at most two values $x = x_{2n}, x_{2n+1}$ such that $B_\lambda(x) = n$.

For the branch of the square root with $\sqrt{1} = 1$ we can write

$$\begin{aligned} x_{2n} &= \frac{-ab - (n+\lambda)\sqrt{(n+\lambda)^2 + b^2 - a^2}}{a^2 - (n+\lambda)^2}, \\ x_{2n+1} &= \frac{-ab + (n+\lambda)\sqrt{(n+\lambda)^2 + b^2 - a^2}}{a^2 - (n+\lambda)^2}, \qquad n \geq 0. \end{aligned} \tag{2.10}$$

and observe that $x_{2n} \to 1, \quad x_{2n+1} \to -1$ as $n \to \infty$.

The set Z_λ can be empty, finite or infinite countable with no limit points in $\mathbb{C} - [-1, 1]$, according to the relative values of λ, a, b. If λ, a, b are real numbers and (2.3) holds, $\mathcal{L}_\lambda$ is positive and we have the representation (see Charris and Ismail [5])

$$\mathcal{L}_\lambda(P(x)) = \int_{-\infty}^{+\infty} P(x)\omega_\lambda(x)\,dx + \sum_{\zeta \in Z_\lambda} Res(X_\lambda, \zeta)P(\zeta) \tag{2.11}$$

where

$$\omega_\lambda(x) = 2^{2\lambda-1}\frac{\lambda + a}{\pi\Gamma(2\lambda)}(1 - x^2)^{\lambda - \frac{1}{2}}\left|\Gamma(-B_\lambda)\right|^2\left|(1 - \alpha^2)^{-B_\lambda - \lambda}\right|^2 \tag{2.12}$$

with Γ denoting the Gamma function (Rainville [14], Chap.II). Thus, if δ is the Dirac measure at $\zeta = 0$,

$$d\mu_\lambda = \omega_\lambda(x)dx + \sum_{\zeta \in Z_\lambda} Res(X_\lambda, \zeta)\delta(x - \zeta)dx \tag{2.13}$$

is the positive measure representing $\mathcal{L}_\lambda$. Explicit formulae for $Res(X_\lambda, \zeta)$ can be found in Charris and Ismail [5]. We mention that provided (2.2) holds,

$$\lambda_n = \mathcal{L}_\lambda((P_n^\lambda(x))^2) = \frac{(2\lambda)_n n!}{(\lambda + a)_n(\lambda + a + 1)_n}, \ n \geq 0, \tag{2.14}$$

as follows from (1.4) and (2.1).Here, $(\alpha)_n$, defined by $(\alpha)_0 = 1$, $(\alpha)_1 = \alpha$ and $(\alpha)_n = \alpha(\alpha + 1)\cdots\alpha(\alpha + n - 1)$ if $n \geq 2$, is the Pochhammer symbol. We also mention that Euler's transformation (Rainville [14], p.60) applied to (2.4) yields

$$X_\lambda(z) = -\frac{2(\lambda + a)\beta}{B_\lambda}(1 - \beta^2)^{2\lambda - 1}\ {}_2F_1\left(\begin{matrix} 2\lambda,\ B_\lambda \\ -B_\lambda + 1 \end{matrix}\middle| \beta^2\right). \tag{2.15}$$

Now let

$$q_\lambda(x) = \frac{2(\lambda + a + 1)}{(\lambda + a)(2\lambda + 1)\lambda}(1 - x^2)(-A_\lambda)(-B_\lambda),\ \lambda \notin Z, \tag{2.16}$$

where

$$Z = \{\lambda \mid 2\lambda \ \text{ or } \ \lambda \pm a \text{ is an integer} \leq 0\} \tag{2.17}$$

Then $q_\lambda(x)$ is a polynomial. We have

LEMMA 2. Assume λ, a, b are real numbers and that (2.2) and (2.3) hold. Then $\mathcal{L}_{\lambda+1} = q_\lambda(x)\mathcal{L}_\lambda$.

PROOF. Let $A_\lambda = A$, $B_\lambda = B$. Then $A_{\lambda+1} = A - 1$, $B_{\lambda+1} = B - 1$. Now, the poles, if any, of X_λ in $B(z) = 0$, are wiped out in $q_\lambda X_\lambda$; and since $B_{\lambda+1} = B - 1$, the poles of both $X_{\lambda+1}$ and of $q_\lambda X_\lambda$ lie on $Z_{\lambda+1}$. Now assume $B_{\lambda+1}(z_m) = m$, so that $B(z_m) = m + 1$. Since

$$q_\lambda(x) = -\frac{(\lambda + a + 1)\alpha^2}{(\lambda + a)(2\lambda)(2\lambda + 1)}(1 - \beta^2)^2 AB,$$

so that, in view of (2.15),

$$q_\lambda(z)X_\lambda(z) = \frac{2(\lambda+a+1)A}{(2\lambda)(2\lambda+1)}\alpha(1-\beta^2)^{2\lambda+1}{}_2F_1\left(\begin{matrix}2\lambda, -B\\ -B+1\end{matrix}\middle|\beta^2\right),$$

a simple calculation shows that

$$\begin{aligned}Res(q_\lambda X_\lambda, z_m) &= -2(\lambda+a+1)\frac{(2\lambda)_{m+1}(2\lambda+m+1)(m+1)}{(m+1)!(2\lambda)(2\lambda+1)}(1-\beta_m^2)^{2\lambda+1}\frac{\beta_m^{2m+1}}{B'(z_m)}\\ &= -2(\lambda+a+1)\frac{(2\lambda+2)_m}{m!}(1-\beta_m^2)^{2\lambda+1}\frac{\beta_m^{2m+1}}{B'(z_m)},\end{aligned}$$

where $\beta_m = \beta(z_m)$ and $B'(z_m)$ is the derivate of $B(z)$ at $z = z_m$. This is precisely the residue of $X_{\lambda+1}(z)$ at $z = z_m$. Thus, since obviously $q_\lambda(x)\omega_\lambda(x) = \omega_{\lambda+1}(x)$, the assertion follows from (2.11).□

Now let

$$q_{\lambda,m}(x) = \frac{4^m(\lambda+a+m)}{(\lambda+a)(2\lambda)_{2m}}(1-x^2)^m(-A_\lambda)_m(-B_\lambda)_m, \qquad m \geq 0, \tag{2.18}$$

so that $q_{\lambda,0}(x) = 1$ and $q_{\lambda,1}(x) = q_\lambda(x)$. Also $q_{\lambda,m}(x) = q_\lambda(x)q_{\lambda+1}(x)\cdots q_{\lambda+m-1}(x)$, and is therefore a polynomial. Induction on Lemma 2 shows that

COROLLARY 1. If λ, a, b are real numbers and (2.2), (2.3) hold, then $\mathcal{L}_{\lambda+m} = q_{\lambda,m}(x)\mathcal{L}_\lambda$.

Now write

$$q_\lambda(x) = a(\lambda)(x-\alpha_0(\lambda))(x-\alpha_1(\lambda)), \tag{2.19}$$

so that $a(\lambda) = \dfrac{(\lambda-a)(\lambda+a+1)}{\lambda(2\lambda+1)}$ and $\alpha_0(\lambda) = x_0$, $\alpha_1(\lambda) = x_1$ are as in (2.10). Let $\Delta_n(\lambda)$, $D_n(\lambda)$ be Δ_n, D_n in (1.16) and (1.17), respectively, with $P_n^\lambda(x)$ in the place of $P_n(x)$. A simple calculation based on (1.22), (2.14) and Lemma 2.1 shows that

$$\frac{1}{\Delta_0(\lambda)}\Delta_n(\lambda) = \frac{(2\lambda)_n(2\lambda+1)_n(\lambda+a)^n}{4^n(\lambda+a)_n(2\lambda+a+1)_n(a-\lambda)^n}, \; \lambda > 0, \; \lambda+a > 0, \tag{2.20}$$

and all $n \geq 0$.

Let Z be as in (2.17). Now we extend (2.20) to all λ not in Z.□

LEMMA 3. Provided a, b are real numbers, (2.20) holds for $\lambda \notin Z$.

PROOF. From (1.1) and (2.1) it follows that $P_n^\lambda(x)$ is a polynomial whose coefficients are rational functions of λ. Let $\tau = \frac{1}{2}(\alpha_0(\lambda)+\alpha_1(\lambda))$ and $\sigma = \frac{1}{2}(\alpha_1(\lambda)-\alpha_0(\lambda))$, so that $\alpha_0(\lambda) = \tau+\sigma$, $\alpha_1(\lambda) = \tau-\sigma$. Clearly τ is a rational function of λ. If $\sigma = 0$, i.e., if $\alpha_0 = \alpha_1$, $\Delta_n(\lambda)$ obviously is a rational function of σ. If $\sigma \neq 0$, let $\tilde{\Delta}_n(\sigma) = \Delta_n(\tau+\sigma)$. Then $\tilde{\Delta}_n(-\sigma) = -\tilde{\Delta}_n(\sigma)$, so that $\tilde{\Delta}_n(\sigma) = \sigma\tilde{\tilde{\Delta}}_n(\sigma^2)$, where $\tilde{\tilde{\Delta}}_n(x)$ is a polynomial whose coefficients are rational functions of λ. Thus $\Delta_n(\lambda)/\Delta_0(\lambda)$ is, in any case, a rational function of λ. Since also the right hand side of (2.20) is a rational function of λ, the uniqueness principle of analytic continuation shows that (2.20) holds for all λ not in Z.□

COROLLARY 2. For $\lambda \notin Z$, $\Delta_n(\lambda) \neq 0$ for all $n \geq 0$. Hence, $\mathcal{U}_\lambda = q_\lambda(x)\mathcal{L}_\lambda$ is a regular moment functional for $\lambda \notin Z$.

Let $\tilde{B}_n(\lambda)$, $\tilde{C}_n(\lambda)$ be the coefficients in the recurrence relation of the monic orthogonal system $\{Q_n^\lambda(x)\}$ of $\mathcal{U}_\lambda$, $\lambda \notin Z$. From Lemma 2,

$$\tilde{B}_n(\lambda) = -\frac{b}{\lambda+a+n+1}, \; \tilde{C}_{n+1}(\lambda) = \frac{(n+1)(n+2\lambda+2)}{4(\lambda+a+n+1)(\lambda+a+n+2)} \tag{2.21}$$

for all $n \geq 0$, provided λ, a, b are real numbers and $\lambda > 0$, $\lambda + a > 0$. Since also $D_n(\lambda)/\Delta_0(\lambda)$ is a rational function of λ, (2.21) holds for all $\lambda \notin Z$. Thus $Q_n^\lambda(x) = P_n^{\lambda+1}(x)$, and

LEMMA 4. For $\lambda \notin Z$, $\mathcal{U}_\lambda = \mathcal{L}_{\lambda+1}$.

Induction gives

THEOREM 1. For $\lambda \notin Z$, $\mathcal{U}_{\lambda+m} = q_{\lambda,m}(x)\mathcal{L}_\lambda$ for all $m \geq 0$. Hence, if λ, a, b are real numbers and

$$\begin{aligned} &\lambda + m > 0 \text{ and } \lambda + a + m > 0, \text{ or, } -1/2 < \lambda + m < 0 \text{ and} \\ &0 < \lambda + a + m < 1, \end{aligned} \tag{2.22}$$

then $q_{\lambda,m}(x)\mathcal{L}_\lambda$ is a positive moment functional.

Now write

$$q_{\lambda,m}(x) = a(\lambda)(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_p)^{m_p} \tag{2.23}$$

with $m_1 + m_2 + \cdots + m_p = 2m$. From (1.14), (1.15) (2.23) and Theorem 1 we obtain

THEOREM 2. Assume λ, a, b are real numbers and that 2λ and $\lambda \pm a$ are not integers ≤ 0. Also assume that $m \geq 0$ is such that (2.22) holds and that

$$(\lambda + j)^2 + b^2 \geq a^2, \; j = 0,\ 1,\ 2,\ \ldots,\ m-1. \tag{2.24}$$

Then, $\mathcal{L}_\lambda$ has the distributional representation

$$\mathcal{L}_\lambda = T_1 + T_2 \tag{2.25}$$

where, for any test function φ on the real line,

$$T_1(\varphi) = \sum_{j=1}^{p}\sum_{k=1}^{m_j} A_{jk}\frac{d^{m_j-k}}{dx^{m_j-k}}\left[\frac{\varphi(x)(x-\alpha_j)^{m_j}}{q_{\lambda,m}(x)}\right](\alpha_j) \tag{2.26}$$

with

$$A_{jk} = \frac{1}{2\pi i(m_j-k)!}\int_C \frac{q_{\lambda,m}(z)X_\lambda(z)}{(z-\alpha_j)^k}\,dz \tag{2.27}$$

and C any positivily oriented closed contour in $|z| > M = \max\{M_\lambda,\ M_{\lambda+m}\}$, and

$$T_2(\varphi) = \int_{-\infty}^{+\infty} \varphi_m(x)\,d\mu_{\lambda+m}(x), \tag{2.28}$$

where

$$\varphi_m(x) = \frac{\varphi(x)}{q_{\lambda,m}} - \sum_{j=1}^{p}\sum_{k=0}^{m_j-1}\frac{1}{k!}\frac{d^k}{dx^k}\left[\frac{\varphi(x)(x-\alpha_j)^{m_j}}{q_{\lambda,m}(x)}\right](\alpha_j)(x-\alpha_j)^{k-m_j} \tag{2.29}$$

and $\mu_{\lambda+m}$ is the positive measure representing $\mathcal{L}_{\lambda+m}$. Both distributions T_1 and T_2 have compact support on the real line and can act on polynomials.

REMARK 1. That T_2 is a distribution follows from

$$|\varphi_m(x)| \leq C\sum_{k=0}^{2m}\sup_{t\in\mathbb{R}}\left|\varphi^{(k)}(t)\right|, \qquad x \in [-M_{\lambda+m},\ M_{\lambda+m}], \tag{2.30}$$

where $C > 0$ is a constant (independent of φ). This is a consequence of the Taylor Remainder theorem. For a description of $\mu_{\lambda+m}$ (which may have an infinite number of real isolated mass points), see Charris and Ismail [5]. We also observe that

$$Supp\ T_1 = \{\alpha_1, \ldots, \alpha_p\}, \qquad Supp\ T_2 = Supp\ \mu_{\lambda+m}. \tag{2.31}$$

REMARK 2. In (2.23), $m_j = 1$, provided λ, a, b are real numbers, $a \neq \pm b$ and $(\lambda+j)^2 > a^2 - b^2$ for $j = 0, 1, \ldots, m-1$. To see this, let x_{2j}, x_{2j+1} be the roots of $(1-x^2)(-A_\lambda + j)(-B_\lambda + j)$ as in (2.10). We observe that since (2.2) holds, $x_{2j} \neq 1$ and $x_{2j+1} \neq -1$, $j = 0, 1, \ldots, m-1$. For $x = x_{2j}$, x_{2j+1} one of the possibilities $A_\lambda(x) = j$, $A_\lambda(x) = -2\lambda - j$ holds, and similarly for B_λ. We intend to prove that if $j \neq k$ then $x_{2k}, x_{2j}, x_{2k+1}, x_{2j+1}$ are all distinct. By the symmetric roles played by x_{2j} and x_{2j+1} and by A_λ and B_λ, it is enough to prove that $x_{2j} \neq x_{2k}$, and for this, that, under the assumptions $x_{2j} = x_{2k}$, both choices $A_\lambda(x_{2j}) = j$ and $A_\lambda(x_{2j}) = -2\lambda - j$ lead to a contradiction. So assume $A_\lambda(x_{2j}) = j$; since $j \neq k$ then $A_\lambda(x_{2j}) = -2\lambda - k$, so that $-2\lambda = j + k$; this is contradictory. Now assume $A_\lambda(x_{2j}) = -2\lambda - j$; then $A_\lambda(x_{2k}) = k$ and $B_\lambda(x_{2k}) = B_\lambda(x_{2j}) = -2\lambda - k$, so that $-2\lambda = A_\lambda(x_{2j}) + B_\lambda(x_{2j}) = -4\lambda - j - k$ and $2\lambda = -j - k$; also this is contradictory.

Hence

COROLLARY 3. Under the assumptions of the theorem, and if $a \neq \pm b$ and $(\lambda+i)^2 > a^2 - b^2$, $i = 0, 1, 2, \ldots, m-1$, then $p = 2m$ and $m_k = 1$ for all $k = 1, 2, \ldots, 2m$. Furthermore, for any test function φ,

$$T_1(\varphi) = \sum_{k=1}^{2m} A_k \left[\frac{\varphi(x)(x-\alpha_k)}{q_{\lambda,m}(x)}\right](\alpha_k), \; A_k = \frac{1}{2\pi i}\int_C \frac{q_{\lambda,m}(z)X_\lambda(z)}{(z-\alpha_k)}\,dz, \tag{2.32}$$

and T_1 is a measure.

REMARK 3. For (2.24) to hold it is sufficient that $|b| \geq |a|$. If (2.24) fails, at least one of the α_i is not real, and representation (2.25) is not possible.

REMARK 4. If $m = 0$ in (2.22), then T_1 vanishes and T_2 reduces to μ_λ. A basic idea of our approach has been to let the possibly infinite masses of $\mathcal{L}_\lambda$ be absorbed by $\mu_{\lambda+m}$. Then we can quote Charris and Ismail [5] for a closer exam of them.

REMARK 5. Now we observe that in spite of the apparent freedom of choice of m in Theorems 1 and 2, the distributional representation of $\mathcal{L}$ is unique, as far as only distributions with compact support are taken into account. This follows from general results in the theory of representations of distributions on the real line by analytic functions on $\mathbb{C} - \mathbb{R}$ (Bremermann [4], Chap. 5). In fact, if T is a distribution with compact support K on $\mathbb{R}$, the *Cauchy-Stieljes transform of T*,

$$\hat{T}(z) = T_\zeta\left(\frac{1}{z-\zeta}\right) \tag{2.33}$$

in an analytic function off K, and if $K \subseteq (-M, M)$ and $|z| > 2M$, from the uniform convergence of $\sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}}$ on $(-M, M)$ it follows that

$$\hat{T}(z) = \sum_{n=0}^{\infty} \frac{T(\zeta^n)}{z^{n+1}}. \tag{2.34}$$

Hence, if T represents $\mathcal{L}$, $T(\zeta^n) = \mathcal{L}(\zeta^n) = c_n$ is the n^{th}-moment of $\mathcal{L}$, and

$$\hat{T}(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}} = X(z), \qquad |z| > 2M, \tag{2.35}$$

where $X(z)$ is the limit function of the continued fraction of the monic orthogonal system of $\mathcal{L}$ (as in (1.7). For a proof of 2.35, see [16], Chap. XI). Hance, $\hat{T}(z)$ is an analytic continuation of $X(z)$ from $|z| > 2M$ to $\mathbb{C} - K$. This implies, in view of the Stieljes inversion formula (Bremermann [4], Chap. 5), that

$$\langle T, \varphi \rangle = \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{\infty}^{\infty} \{X(x + i\epsilon) - X(x - i\epsilon)\} \varphi(x)\, dx \tag{2.36}$$

for any test function φ, which ensures the uniqueness of T.

REMARK 6. Under the assumptions of Theorem 2, a result of R. P. Boas ensures that a non-positive measure on the line can be found wich represents $\mathcal{L}$ (Chihara [7], Chap. II). Since the distributions representing $\mathcal{L}$ in (2.25) are not measures when the positivity conditions fail, Boas' measures cannot be supported by a compact set under such circumstances.

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