Internat. J. Math. & Math. Sci.
VOL. 19 NO. 4 (1996) 789-796

UNIVALENT FUNCTIONS MAXIMIZING $Re[f(\zeta_1)+f(\zeta_2)]$

INTISAR QUMSIYEH HIBSCHWEILER

Daemen College
4380 Main Street
Amherst New York 14226 USA

(Received April 11, 1994 and in revised form September 28, 1995)

ABSTRACT. We study the problem $max_{h \in S} \Re[h(z_1)+h(z_2)]$ with z_1, z_2 in Δ. We show that no rotation of the Koebe function is a solution for this problem except possibly its real rotation, and only when $z_1 = \bar{z}_2$ or z_1, z_2 are both real, and are in a neighborhood of the x-axis. We prove that if the omitted set of the extremal function f is part of a straight line that passes through $f(z_1)$ or $f(z_2)$ then f is the Koebe function or its real rotation. We also show the existence of solutions that are not unique and are different from the Koebe function or its real rotation. The situation where the extremal value is equal to zero can occur and it is proved, in this case, that the Koebe function is a solution if and only if z_1 and z_2 are both real numbers and $z_1 z_2 < 0$.

KEY WORDS AND PHRASES. Univalent Functions, Support Points, Quadratic Differential.

1991 AMS SUBJECT CLASSIFICATION CODES: Primary 30C55. Secondary 30C70.

1. INTRODUCTION

Let $H(\Delta)$ denote the set of all functions analytic in the open unit disk $\Delta = \{|z| < 1\}$, endowed with the topology of uniform convergence on compact subsets. Let S denote the subset of $H(\Delta)$ that consists of functions that are univalent in Δ and satisfy $f(0) = 0$ and $f'(0) = 1$. It is known [6] that S is a compact subset of $H(\Delta)$. $H(\Delta)^*$ will denote the space of all continuous linear functional on $H(\Delta)$.

A function f in S is said to be a support point of S if there is a continuous linear functional L in $H(\Delta)^*$, not constant on S, such that $\Re L(f) \geq \Re L(h)$ for all h in S. If this is the case we will simply write $f \sim L$.

An expression $Q(w)dw^2$, where $Q(w)$ is meromorphic in a region G, is called a quadratic differential in G. An analytic arc $w(t)$ for which $Q(w)dw^2 > 0$ (i.e. $Q(w(t))(w'(t))^2 > 0$) is called a trajectory arc. A trajectory is a maximal analytic arc $w(t)$ such that $Q(w(t))(w'(t))^2 > 0$. The zeros and poles for $Q(w)$ in G are called critical points. A critical point is called an infinite critical point if it has order -2 or less; otherwise it is called a finite critical point.

It is known ([4], [9], [7]) that each support point of S maps the disk onto the complement of a single analytic arc Γ with increasing modulus and an asymptotic direction at ∞. It was shown [11] that the omitted set Γ of a support point is a trajectory arc for the quadratic differential $L(f^2/(f-w))(dw/w)^2$, i.e. Γ is an analytic arc w(t) satisfying

$$L(f^2/(f-w(t))(w'(t)/w(t))^2 > 0. \tag{1}$$

This is called the Schiffer differential equation. Furthermore Γ has the property that the angle between the radius and tangent vector never exceeds $\pi/4$.

The problem of finding support points associated with a certain given functional had been studied. For example in [5], [2] point evaluation functionals were studied, it was shown that the Koebe function is the solution to the problem $max_{h\in S}\Re h(\zeta)$ if and only if ζ in $((1-e)/(1+e),1)$ and that no rotation of the Koebe function is a solution. If ζ is in $(-1,(1-e)/(1+e))$ then there are two solutions related by conjugation. For any other ζ the solution is unique.

In this paper we study the problem:

$$max_{h\in S}\Re J_{\zeta_1,\zeta_2}(h), \tag{2}$$

where

$$J_{\zeta_1,\zeta_2}(h)=h(\zeta_1)+h(\zeta_2) \qquad \text{with } 0<|\zeta_1|<1, 0<|\zeta_2|<1.$$

We show that no rotation of the Koebe function is a solution for the problem (2) except possibly its real rotation, and only when $\zeta_1=\bar{\zeta}_2$ or both ζ_1 and ζ_2 are real. We also show that if $f\sim J_{\zeta_1,\zeta_2}$ and the omitted set of f, Γ, is part of a straight line segment that passes through $f(\zeta_1)$ or $f(\zeta_2)$ then f is the Koebe function or its real rotation. We also study the case where $max_{h\in S}\Re J_{\zeta_1,\zeta_2}(h)=0$ and in this case the Koebe function is the solution if and only if ζ_1 and ζ_2 are both real and $\zeta_1\zeta_2<0$.

2. THE OMITTED SET Γ

Using the Schiffer differential equation (1) we can conclude that if $f\sim J_{\zeta_1,\zeta_2}$, then Γ satisfies

$$[\frac{a^2}{a-w}+\frac{b^2}{b-w}](\frac{dw}{w})^2>0 \qquad \text{for w in } \Gamma \tag{3}$$

where $a=f(\zeta_1)$, $b=f(\zeta_2)$.

Let k_x denote the function $z/(1-xz)^2$, or for $x=1$ simply k. Let

$$L_\zeta(h)=h(\zeta)+h(\bar{\zeta}) \qquad \text{with } 0<|\zeta|<1.$$

We will need the following Lemma:

Lemma 1: Let $F(z)$ be a function that is analytic in a neighborhood of the origin. Suppose that there exist a sequence of real numbers such that $t_n\to 0$ and $F(t_n)$ is real, then F is real on the real axis.

To prove the lemma show that all the coefficients of the Taylor series expansion of F at the origin are real.

Theorem 1:

(a) If k_x is a solution for (2), then $x=\pm 1$ and $\zeta_1=\bar{\zeta}_2$ or ζ_1 , ζ_2 are both real.

(b) If $f\sim L_\zeta$, then $\bar{f}\sim L_\zeta$ where $\bar{f}(z)=\overline{f(\bar{z})}$. In fact, if $f\sim L_\zeta$ uniquely then $f=k$ or $f=k_{-1}$.

(c) Let ζ_1 and ζ_2 both be real. If neither k nor k_{-1} are solutions for the problem (2), then the problem (2) has at least two distinct solutions related by conjugation.

Proof: Parameterize the omitted set Γ of k_x by $w=\bar{x}t$ with $t\le -1/4$. Substitution in (1) gives the inequality

$$\frac{a^2}{a-\bar{x}t}+\frac{b^2}{b-\bar{x}t}>0 \qquad \text{for } t\le \tfrac{-1}{4}. \tag{4}$$

Define

$$F(t)=\frac{a^2}{a-\bar{x}t}+\frac{b^2}{b-\bar{x}t}. \tag{5}$$

We claim that $F(t)$ is real for all t except possibly $t = a/\bar{x}$ or $t = b/\bar{x}$. The claim follows from the fact that the function $F(z)$ is meromorphic in a neighborhood of the real axis and, from (4) maps the line segment $t \leq -1/4$ onto the positive real axis. We can apply Lemma 1, if necessary twice, to show that F(t) is real in a neighborhood of $a/\bar{x}$ and $b/\bar{x}$. For small values of t, F can be rewritten as

$$F(t) = (a+b) + \bar{x}t + \bar{x}^2 t^2 (\tfrac{1}{a} + \tfrac{1}{b}) + \dots . \tag{6}$$

From this follows the fact that $\bar{x}$ is real, or $x = \pm 1$. This is because $F'(0) = \bar{x}$ is real. This also shows that $a+b$ and $1/a + 1/b$ are both real. Consequently, either $\zeta_1 = \bar{\zeta}_2$ or ζ_1 and ζ_2 are both real. This proves part (a).

To prove part (b) note that the definition of L_ζ implies that $\Re L_\zeta(f) = \Re L_\zeta(\bar{f})$. If f is a unique solution for the problem $max_{h \in S} L_\zeta(h)$, then $f = \bar{f}$, so that $f = k$ or $f = k_{-1}$.

Part (c) follows since

$$\Re[f(\zeta_1) + f(\zeta_2)] = \Re[\overline{f(\bar{\zeta}_1)} + \overline{f(\bar{\zeta}_2)}]$$

This finishes the proof of Theorem 1.

The following theorem shows that the problem (2) has solutions other than k or its real rotation.

Theorem 2: Given r in $(-1, (1-e)/(1+e))$ we can find a neighborhood U_r of r such that, whenever ζ_1, ζ_2 are in U_r, k and k_{-1} are not solutions for the problem (2).

Proof: Let f_r in S be such that $\Re f_r(r) \geq \Re h(r)$ for all h in S. It is known ([5] and [2]) that f_r is not unique, $f_r \neq k$ and $f_r \neq k_{-1}$. A continuity argument shows that there exists a neighborhood U_r of r such that $\Re f_r(\zeta) > \Re k(\zeta)$ for all ζ in U_r. Consequently $\Re[f_r(\zeta_1) + f_r(\zeta_2)] > \Re[k(\zeta_1) + k(\zeta_2)]$, whenever ζ_1 and ζ_2 are in U_r. A similar argument applies for k_{-1} .

We note that if $(1-e)/(1+e) < \zeta_1, \zeta_2 < 1$, then k is the unique solution for the problem (2). This follows because if $\Re[f(\zeta_1) + f(\zeta_2)] > \Re[k(\zeta_1) + k(\zeta_2)]$, for some f in S, then either $\Re f(\zeta_1) > \Re k(\zeta_1)$ or $\Re f(\zeta_2) > \Re k(\zeta_2)$. But k maximizes $\{\Re h(r) : h \in S\}$ uniquely for any r with $[(1-e)/(1+e)] < r < 1$ (see [5]).

Corollary: Let $r \in (-1, (1-e)/(1+e))$. Then there exists U^*, a neighborhood of r, such that whenever ζ_1, ζ_2 are real in U^* or $\zeta_1 = \bar{\zeta}_2$ in U^*, the problem (2) has at least two distinct solutions f and g related by conjugation.

The corollary follows as a consequence of the previous Theorems.

Theorem 3: If $k \sim L_\zeta$ then

$$\Re[\frac{\zeta^2}{(1-\zeta)^4}] \geq 0, \tag{7}$$

and

$$\Re[\frac{\zeta^2}{(1-\zeta)^3}] \geq 0, \tag{8}$$

and

$$\Re[\frac{\zeta^2}{(1-\zeta)^2}] \geq 0, \tag{9}$$

Proof: To prove (7) parameterize the omitted arc by $w(t) = -t$, $t \geq 1/4$, and substitute in (1), we obtain

$$\frac{a^2}{a+t} + \frac{b^2}{b+t} \geq 0 \qquad \text{for } t \geq 1/4. \tag{10}$$

Multiply (10) by t and take the limit as t tends to infinity to obtain $a^2 + b^2 \geq 0$. Since $a = \bar{b}$, from Theorem 1 it follows that $\Re a^2 \geq 0$ and this is exactly (7).

To prove inequality (8), we use the variation

$$\Re L_\zeta(k) \leq \Re L_\zeta(zk'P), \tag{11}$$

where P has a positive real part and $P(0) = 1$ (see[12] p.82). From (11) we obtain

$$2\Re k(\zeta) \leq \Re[\frac{\zeta(1+\zeta)}{(1-\zeta)^3}(P(\zeta) + \overline{P(\bar{\zeta})})]. \tag{12}$$

Substitute $P \equiv 1$, (12) becomes

$$\Re k(\zeta) \leq \Re[\frac{\zeta(\zeta+1)}{(1-\zeta)^3}],$$

or equivalently

$$\Re[\frac{\zeta}{(1-\zeta)^2} - \frac{\zeta(1+\zeta)}{(1-\zeta)^3}] \leq 0.$$

From this it follows that

$$\Re\frac{\zeta^2}{(1-\zeta)^3} \geq 0.$$

To obtain (9), note that for any c with $|c| = 1$ the function

$$g(z) = \frac{z - \frac{1}{2}(c+1)z^2}{(1-z)^2}$$

is in S (see[3]). Consequently we have the inequality (with $c = i$)

$$\Re[\frac{\zeta}{(1-\zeta)^2} + \frac{\bar{\zeta}}{(1-\bar{\zeta})^2}] \geq \Re[\frac{\zeta - \frac{1}{2}(i+1)\zeta^2}{(1-\zeta)^2} + \frac{\bar{\zeta} - \frac{1}{2}(i+1)\bar{\zeta}^2}{(1-\bar{\zeta})^2}].$$

From this inequality (9) follows. This ends the proof of Theorem 3. A similar statement holds when $k_{-1} \sim L_\zeta$.

Corollary : Theorem 3 and the results in [5] show that a necessary condition for $k \sim L_\zeta$ is that ζ is in a neighborhood of the line segments $((1-e)/(1+e), 0)$ and $(0, 1)$.

This is because $z/(1-z)$ maps the circle determined by the points $1, 0, -i$ onto the line $u = v$, (where $z/(1-z) = u + iv$). Inequality (9) then implies that

$$\frac{-\pi}{4} < arg\frac{\zeta}{(1-\zeta)} < \frac{\pi}{4} \ or\ \frac{3\pi}{4} < arg\frac{\zeta}{(1-\zeta)} < \frac{5\pi}{4}.$$

That is, ζ must be in a neighborhood of the x-axis. A similar argument gives a region for ζ in order for $k_{-1} \sim L_\zeta$.

It is not known whether k is the only rational function that maximizes the problem (2). We prove the following:

Theorem 4: Assume f is in S with $f \sim J_{\zeta_1,\zeta_2}$, and suppose that $f(z)$ is a rational function in z. Assume further that the analytic continuation of Γ passes through one of the simple poles $a = f(\zeta_1), b = f(\zeta_2)$ of the quadratic differential in (3). Then Γ is a horizontal line segment.

Proof: It is known [13] , in this case, that Γ is a straight line segment. Without loss of generality parameterize Γ by $w = a + \beta t$ and use (3) to obtain the inequality

$$[\frac{a^2}{-\beta t} + \frac{b^2}{b-a-\beta t}][\frac{\beta}{a+\beta t}]^2 \geq 0 \qquad for\ a + \beta t \in \Gamma.$$

Define

$$F(z) = [\frac{a^2}{-\beta z} + \frac{b^2}{b-a-\beta z}][\frac{\beta}{a+\beta z}]^2.$$

Notice that $F(z)$ is meromorphic in a neighborhood of the x-axis. Apply Lemma 1 and use similar argument as in Theorem 1 part (a) to conclude that $F(t)$ is real for all t except possibly when $t = (b-a)/\beta$, $t = -a/\beta$ or $t = 0$. It follows that $tF(t)$ is real for all $t \neq (b-a)/\beta$, $-a/\beta$, i.e.

$$\Big[\frac{a^2}{-\beta} + \frac{b^2 t}{b-a-\beta t}\Big]\Big[\frac{\beta}{a+\beta t}\Big]^2 \tag{13}$$

is real for t close to 0. Take the limit in (13) as t tends to zero to conclude that $[-a^2/\beta][\beta/a]^2$ is real or $-\beta$ is real. Hence Γ is a horizontal line segment.

Note that by taking the limit in (13) as t tends to infinity, we obtain $(a^2+b^2)/\beta$ is real, i.e. (a^2+b^2) is real, whenever the assumption in Theorem 4 holds.

One would like to show that the solution to problem (2) is unique for any ζ_1, ζ_2 with ζ_1, ζ_2 not real. The previous Theorems and argument support this conjecture. However, it remains an open problem.

3. THE EXTREMAL VALUE

In this section we will study the situation:

$$f \sim J_{\zeta_1,\zeta_2} \textit{ and } J_{\zeta_1,\zeta_2}(f) = 0.$$

This situation occurs, for example if $\zeta_1 = r_1 > 0$ and $\zeta_2 = r_2 < 0$ with $k(r_1) + k(r_2) = 0$ and $r_2 > (1-e)/(1+e)$. This case is of special interest for the following reason: If $J_{\zeta_1,\zeta_2}(f) = 0$, the quadratic differential in (3) becomes

$$\frac{-2a^2}{w(w-a)(w+a)}dw^2 \tag{14}$$

where $a = f(\zeta_1) = -f(\zeta_2)$. Let $a = re^{i\theta}$ and $w = ve^{i\theta}$ and substitute in (14) to obtain

$$\frac{-2r^2 e^{i\theta} dv^2}{v(v-r)(v+r)}. \tag{15}$$

The trajectories in (14) can be obtained from the trajectories in (15) by a rotation. It is known ([7],[8]) that if θ is an irrational multiple of 2π then every trajectory of (15) is dense in the whole complex plane, i.e. it comes arbitrary close to any complex number. It follows that the same is true for (14). Therefore if this situation occurs and $\arg f(\zeta_1)$ is an irrational multiple of 2π, then we can conclude that there exists a support point f in S with the property that its omitted set Γ has an analytic continuation that is dense in the whole complex plane, (this seems unlikely, but remains as a conjecture).

We prove the following

Theorem 5: Suppose $f \sim J_{\zeta_1,\zeta_2}$ and $J_{\zeta_1,\zeta_2}(f) = 0$. $f = k$ if and only if ζ_1 and ζ_2 are real and $\zeta_1\zeta_2 < 0$.

Proof: Assume first that k is a solution for (2) and $J_{\zeta_1,\zeta_2}(k) = 0$. Parameterize Γ by $w(t) = -t$, $t \geq 1/4$, and substitute in (3) to obtain that $[(a^2/(a+t)) - (a^2/(a-t))]$ is real and positive for $t \geq 1/4$. Define $F(z) = a^2/(a+z) - a^2/(a-z)$, and note that F is meromorphic in a neighborhood of the real axis. Therefore, we can conclude that $F(t)$ is real for all t except when a is real. However, if a is real then ζ_1 and ζ_2 are real and $\zeta_1\zeta_2 < 0$. Considering $F(t)$ for small t, we have

$$F(t) = a\Big[\frac{-2t}{a} - \frac{2t^3}{a^3} - \frac{2t^5}{a^5} + \ldots\Big] \qquad \text{for } t \text{ near } 0,$$

so that a^2 is real. Multiply F(t) by t and take the limit as $t \to \infty$ to conclude that a^2 is real and positive. Therefore either a is real and positive or a is real and negative. In any case we conclude that ζ_1 and ζ_2 are real and $\zeta_1\zeta_2 < 0$.

To prove the only if part, we need the following observation due to Leung.

Lemma A: Let f in S maps Δ onto the complement of an analytic arc extending to infinity. Suppose $f(1) = \infty$, and $f(e^{i\theta}) = f(e^{-i\theta})$ for all θ in $[0, \pi]$. Then $f = k$.

Proof of Lemma A: Let $g(z) = (f(z)(1-z)^2)/z$. Then $g(z)$ is analytic on $z \leq 1$ and $g(z) \neq 0$ in Δ. Also $g(1/z)$ is analytic in $z \geq 1$. The given conditions yield $g(1/z) = g(z)$ on $|z| = 1$. Thus $g(1/z) = g(z)$ for all z. This implies that $g(z)$ is bounded in the complex plane and therefore is a constant.

We also need the following well known fact about quadratic differentials.

Lemma B: Suppose $B(z)dz^2$ and $A(w)dw^2$ are quadratic differentials in the $z - plane$ and $w - plane$ respectively. Suppose under a slit mapping $w = f(z)$ where f is in S, we have $A(w)dw^2 = B(z)dz^2$. Assume further that $f(-1) = w_0$, where w_0 is the finite tip of the slit. If $f^{-1}(w) = \{e^{i\theta_1}, e^{i\theta_2}\}$ for w on the slit, then

$$\int_{\pi}^{\theta_1} \sqrt{|B(e^{i\theta})|}\, d\theta = \int_{\pi}^{\theta_2} \sqrt{|B(e^{-i\theta})|}\, d\theta.$$

We would like to prove that if ζ_1 and ζ_2 are real, $\zeta_1\zeta_2 < 0$, and $f \sim J_{\zeta_1,\zeta_2}$ with $J_{\zeta_1,\zeta_2}(f) = 0$, then $f = k$.

Substitute $w = f(e^{i\theta})$ in (14) to conclude that

$$\Phi(z) = \frac{-2a^2 f(z)}{(f(z)-a)(f(z)+a)}\left(\frac{zf(z)}{f(z)}\right)^2 \tag{16}$$

is real and nonpositive for $|z| = 1$. Recall that $f(\zeta_1) = a$ and $f(\zeta_2) = -a$ and note that by the Schwarz reflection principle,

$$\Phi(z) = \frac{Az(z-e^{i\alpha})^2}{(z-\zeta_1)(1-\zeta_1 z)(z-\zeta_2)(1-\zeta_2 z)} \tag{17}$$

where $e^{i\alpha}$ is the point on $|z| = 1$ that corresponds to the finite tip of the omitted set of f, Γ (for a similar argument see [5]). Because $\Phi(e^{i\theta}) \leq 0$, we have

$$\Phi(e^{i\theta}) = \frac{Ae^{i\theta}(e^{i\theta}-e^{i\alpha})(e^{i\theta}-e^{i\alpha})}{(e^{i\theta}-\zeta_1)(1-\zeta_1 e^{i\theta})(e^{i\theta}-\zeta_2)(1-\zeta_2 e^{i\theta})}$$

$$= \frac{Ae^{i\alpha}|1-e^{i(\theta-\alpha)}|^2}{|1-\zeta_1 e^{i\theta}|^2|1-\zeta_2 e^{i\theta}|^2}.$$

Therefore $Ae^{i\alpha} > 0$. Equate the two expressions for $\Phi(z)$ in (16) and (17) and divide the resulting equation by z, then take the limit as $z \to 0$ we obtain $Ae^{2i\alpha} = 2\zeta_1\zeta_2$. If $\zeta_1\zeta_2 < 0$, then $Ae^{2i\alpha}$ is negative. Therefore $e^{i\alpha}$ is negative. This implies that $f(-1)$ is the finite tip of Γ and also that $\Phi(e^{i\theta}) = \Phi(e^{-i\theta})$. We can apply Lemma B with with $B(e^{i\theta}) = \Phi(e^{i\theta})$ to show that if $f^{-1}(w) = \{e^{i\theta_1}, e^{i\theta_2}\}$ for w in Γ, then $\theta_1 = -\theta_2$ and hence $f(1) = \infty$. All the conditions in Lemma A are fulfilled so we may conclude that $f = k$.

Remark: The problem (2) remains undetermined for many values of ζ_1 and ζ_2. We conjecture that if $[(1-e)/(1+e)] < \zeta_1 < 1$ and $[(1-e)/(1+e)] < \zeta_2 < 1$, then the omitted set Γ has an analytic continuation that is the real axis. Otherwise, for the all other values of ζ_1 and ζ_2, Γ has an analytic continuation that spirals toward the origin.

Acknowledgments. This paper forms part of the author's doctoral dissertation written at the State University of New York at Albany. The author wishes to thank Professor Donald Wilken and Professor Y.J. Leung for their help while this work was on progress.

REFERENCES

[1] BRICKMAN, L., Extreme points of the set of univalent functions, *Bull. Amer. Math. Soc.* (1970), 372-374.

[2] BRICKMAN, L. and DUREN, P., Truncation of support points for univalent functions, *Complex variable Theory app.* **3** (1984), 71-83.

[3] BRICKMAN, L., MacGREGOR, T. and WILKEN, D.R., *Convex hulls of some classical families of univalent functions*, Trans. Amer. Math. Soc. **156** (1971), 91-107.

[4] BRICKMAN, L. and WILKEN, D.R., Support points of the set of univalent functions, *Proc. Amer. Math. Soc.* **42** (1974), 523-528.

[5] BROWN, J., Geometric properties of a class of support points of univalent functions, *Trans. Amer. Math. Soc.* **256** (1979), 371-382.

[6] DUREN, P., *Univalent Functions*, Springer-Verlag, Heidelberg and New York, 1983.

[7] JENKINS, J., *Univalent functions and Conformal Mapping*, Springer-Verlag, Berlin, 1958.

[8] PFLUGER, A., *Lectures on Conformal Mapping*, Lecture Notes, Indiana University, Dept. of Mathematics, 1969.

[9] PFLUGER, A., Lineare Extremal Probleme bei shlichten Funktionen, *Ann. Acad. Sci. Fenn.* **489** (1971), no. Ser AI, 32 pp.

[10] POMMERENKE, Ch., *Univalent Functions*, with a Chapter on Quadratic Differential by G. Jensen, Vandenhoeck and Ruprecht, Gőttingen, 1975.

[11] SCHIFFER, M., A method of variation within the family of simple functions, *Proc. London Math. Soc. (2)* **44** (1938), 432-449.

[12] SCHOBER, G., *Univalent Functions-Selected Topics*, Lecture Notes in Math, vol. 478, Springer-Verlag, Berlin and New York, 1975.

[13] SREBRO, U., Is the Slit of a Rational slit mapping in S straight?, *Proc. Amer. Math. Soc.* **96** (1986 no. 1), 65-66.