Internat. J. Math. & Math. Sci.
VOL. 19 NO. 4 (1996) 821-824

RESEARCH NOTES

A CURIOUS INTEGRAL

DAVID R. MASSON

Department of Mathematics
University of Toronto
Toronto Ontario M5S 1A1
Canada

(Received November 22, 1994)

ABSTRACT. A double integral which came from a cohomology calculation is evaluated explicitly using the properties of ${}_3F_2$ and ${}_2F_1$ hypergeometric functions.

KEY WORDS AND PHRASES: double integral. hypergeometric functions.

1992 AMS SUBJECT CLASSIFICATION CODE: 33C90.

1. INTRODUCTION.

The problem of evaluating the integral

$$\int_0^{\pi/2}\int_0^{\pi/2}\frac{(1-4\cos^2 s\cos^2 t)}{(1+8\cos^2 s\cos^2 t)^{3/2}}dsdt$$

has been proposed by A. Lundell. The computer algebra language Maple tells the user that it can not be evaluated explicitly but evaluates it numerically to seven decimal places in a couple of seconds. Mathematica, on the other hand, reduces it to the evaluation of a single integral by performing one of the single integrals.

The integral arose as a reduction of a surface integral on a torus which came in relating the cohomology of $\mathbf{R}^3-(C\cup L)$ and $\mathbf{R}^3-C$ where C is the circle $x^2+y^2=a^2$ in the xy-plane and L is the z-axis and where numerical calculations suggested the value $\pi/4$ [2, p.19]. The purpose of this note is to prove this conjecture.

We first consider the more general integral

$$I(a,b,c):=\int_0^{\pi/2}\int_0^{\pi/2}\frac{(1+b\cos^2 s\cos^2 t)}{(1+a\cos^2 s\cos^2 t)^c}dsdt. \tag{1.1}$$

We find that $I(a,b,c)$ can be expressed as a sum of two ${}_3F_2$'s with arguement $-a$. Although there are no explicit general formulas for the analytic continuation of ${}_3F_2$'s something remarkable happens when $c=3/2$. In this case each ${}_3F_2$ can be expressed as a product of ${}_2F_1$'s of arguement $-a$ which may now be analyticly continued throughout the complex a-plane cut along $(-\infty,-1]$. A further simplification occurs when $b=-4$ with $I(a,-4,3/2)$ being expressed as a single product of two ${}_2F_1$'s. A final remarkable simplification occurs with $a=8$

when each of these ${}_2F_1$'s can be explicitly summed in terms of gamma functions. As an end result we then obtain

THEOREM 1.

$$I(8,-4,3/2)=\frac{\pi}{4}. \tag{1.2}$$

In the next section we prove this result using the theory of hypergeometric functions where

$${}_{r+1}F_r\left(\begin{matrix}a_1,a_2,\cdots,a_{r+1}\\ b_1,b_2,\cdots,b_r\end{matrix};z\right):=\sum_{n=0}^{\infty}\frac{(a_1,a_2,\cdots,a_{r+1})_n}{(b_1,b_2,\cdots,b_r)_n}\frac{z^n}{n!}, \tag{1.3}$$

$$(a)_n=\Gamma(a+n)/\Gamma(a),\quad (a_1,a_2,\cdots,a_r)_n=\prod_{j=1}^{r}(a_j)_n.$$

The following formulas will be needed.

$${}_3F_2\left(\begin{matrix}2\alpha-1,2\beta,\alpha+\beta-1\\ 2\alpha+2\beta-2,\alpha+\beta-1/2\end{matrix};z\right)={}_2F_1\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta-1/2\end{matrix};z\right){}_2F_1\left(\begin{matrix}\alpha-1,\beta\\ \alpha+\beta-1/2\end{matrix};z\right), \tag{1.4}$$

$${}_3F_2\left(\begin{matrix}2\alpha,2\beta,\alpha+\beta\\ 2\alpha+2\beta-1,\alpha+\beta+1/2\end{matrix};z\right)={}_2F_1\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta-1/2\end{matrix};z\right){}_2F_1\left(\begin{matrix}\alpha,\beta\\ \alpha+\beta-1/2\end{matrix};z\right), \tag{1.5}$$

$${}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right)=(1-z)^{-a}{}_2F_1\left(\begin{matrix}a,c-b\\ c\end{matrix};\frac{z}{z-1}\right), \tag{1.6}$$

$${}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right)=(1-z)^{c-a-b}{}_2F_1\left(\begin{matrix}c-a,c-b\\ c\end{matrix};z\right), \tag{1.7}$$

$$\begin{aligned}c(c-1)(z-1)\,{}_2F_1\left(\begin{matrix}a,b\\ c-1\end{matrix};z\right)+c[c-1-(2c-a-b-1)z]\,{}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right)\\ +z(c-a)(c-b)\,{}_2F_1\left(\begin{matrix}a,b\\ c+1\end{matrix};z\right)=0,\end{aligned} \tag{1.8}$$

$${}_2F_1\left(\begin{matrix}a,b\\ a+b+1/2\end{matrix};z\right)={}_2F_1\left(\begin{matrix}2a,2b\\ a+b+1/2\end{matrix};\frac{1}{2}-\frac{1}{2}(1-z)^{1/2}\right), \tag{1.9}$$

$${}_2F_1\left(\begin{matrix}a,b\\ a+b-1/2\end{matrix};z\right)=(1-z)^{-1/2}{}_2F_1\left(\begin{matrix}2a-1,2b-1\\ a+b-1/2\end{matrix};\frac{1}{2}-\frac{1}{2}(1-z)^{1/2}\right), \tag{1.10}$$

$${}_2F_1\left(\begin{matrix}a,b\\ 1+a-b\end{matrix};-1\right)=2^{-a}\frac{\Gamma(1+a-b)\Gamma(1/2)}{\Gamma(1-b+a/2)\Gamma(1/2+a/2)}. \tag{1.11}$$

These formulas are in [1], (9) and (8) p. 186, (3) and (2) p. 105, (30) p. 103, (10) and (13) p. 111, and (47) p. 104 respectively.

2. THE PROOF.

To prove Theorem 1 we first establish four lemmas.

LEMMA 2.1. Let

$$u_n:=\int_0^{\pi/2}\cos^{2n}t\,dt,\ n=0,1,\cdots. \tag{2.1}$$

Then

$$u_n=\frac{\pi}{2}\frac{(1/2)_n}{n!}. \tag{2.2}$$

PROOF. This result is well known. An integration by parts yields $u_n = \frac{2n-1}{2n} u_{n-1}, n \geq 1$. Clearly $u_0 = \pi/2$. Iterating we get (2.2).

LEMMA 2.2. If $|a| < 1$ then

$$I(a,b,c) = \frac{\pi^2}{4}\left[{}_3F_2\left({c, \frac{1}{2}, \frac{1}{2} \atop 1, 1} ; -a \right) + \frac{b}{4}\, {}_3F_2\left({c, \frac{3}{2}, \frac{3}{2} \atop 2, 2} ; -a \right)\right]. \tag{2.3}$$

PROOF. In (1.1) we expand $(1 + a\cos^2 s \cos^2 t)^{-c}$ using the binomial theorem and do the integration. Using Lemma 2.1 we then obtain (2.3).

We now specialize to the value $c = 3/2$.

LEMMA 2.3. If $|a| < 1$ or $a = 1$ then

$$\begin{aligned} I(a,b,3/2) = & \frac{\pi^2}{4}\left[{}_2F_1\left({\frac{5}{4}, \frac{1}{4} \atop 1} ; -a \right) {}_2F_1\left({\frac{1}{4}, \frac{1}{4} \atop 1} ; -a \right) \right. \\ & \left. + \frac{b}{4}\, {}_2F_1\left({\frac{3}{4}, \frac{3}{4} \atop 1} ; -a \right) {}_2F_1\left({\frac{3}{4}, \frac{3}{4} \atop 2} ; -a \right)\right]. \end{aligned} \tag{2.4}$$

PROOF. We use (1.4) for the first ${}_3F_2$ on the right of (2.3) and (1.5) for the second ${}_3F_2$ on the right of (2.3).

Having established (2.4) for $|a| < 1$ one may use the properties of ${}_2F_1$'s to obtain an analytic continuation of (2.4) throughout the complex a-plane cut along $(-\infty, -1]$.

We now specialize to the values $b = -4, c = 3/2$.

LEMMA 2.4.

$$I(a,-4,3/2) = \frac{15\pi^2 a}{128}\, {}_2F_1\left({\frac{1}{4}, \frac{1}{4} \atop 1} ; -a \right) {}_2F_1\left({\frac{5}{4}, \frac{9}{4} \atop 3} ; -a \right). \tag{2.5}$$

PROOF. In (2.4) we put $b = -4$ and apply (1.7) to the first and third ${}_2F_1$ on the right of (2.4). The result is

$$I(a,-4,3/2) = \frac{\pi^2}{4(1+a)^{1/2}}\, {}_2F_1\left({\frac{1}{4}, \frac{1}{4} \atop 1} ; -a \right) \left[{}_2F_1\left({\frac{3}{4}, -\frac{1}{4} \atop 1} ; -a \right) - {}_2F_1\left({\frac{3}{4}, \frac{3}{4} \atop 2} ; -a \right)\right]. \tag{2.6}$$

We now apply (1.6) to the ${}_2F_1$'s in the brackets above and then use (1.8). This gives

$$I(a,-4,3/2) = \frac{15\pi^2 a}{128(1+a)^{5/4}}\, {}_2F_1\left({\frac{1}{4}, \frac{1}{4} \atop 1} ; -a \right) {}_2F_1\left({\frac{3}{4}, \frac{5}{4} \atop 3} ; \frac{a}{1+a} \right). \tag{2.7}$$

After another application of (1.6) to the second ${}_2F_1$ above we obtain (2.5).

PROOF OF THEOREM 1. We now specialize to the case $a = 8, b = -4, c = 3/2$. In (2.5) we put $a = 8$. We use (1.9) and (1.11) to get

$${}_2F_1\left({\frac{1}{4}, \frac{1}{4} \atop 1} ; -8 \right) = {}_2F_1\left({1/2, 1/2 \atop 1} ; -1 \right) = \frac{\Gamma(1)\Gamma(1/2)}{2^{1/2}\Gamma^2(3/4)}. \tag{2.8}$$

Using (1.10) and (1.11) we also get

$${}_2F_1\left({\frac{5}{4}, \frac{9}{4} \atop 3} ; -8 \right) = \frac{1}{3}\, {}_2F_1\left({\frac{3}{2}, \frac{7}{2} \atop 3} ; -1 \right) = \frac{\Gamma(3)\Gamma(1/2)}{3\Gamma(5/4)\Gamma(9/4)2^{7/2}}. \tag{2.9}$$

Thus

$$I(8, -4, 3/2) = \frac{\pi^2\Gamma^2(1/2)}{32\Gamma^2(3/4)\Gamma^2(5/4)} \tag{2.10}$$

where we have used the above ${}_2F_1$ evaluations together with $\Gamma(1) = 1, \Gamma(3) = 2$ and $\Gamma(9/4) = 5\Gamma(5/4)/4$. A final use of the duplication formula [1,(15), p. 5] yields $\Gamma^2(1/2) = \pi, \Gamma^2(3/4)\Gamma^2(5/4) = \pi^2/8$ and the theorem is established.

AKNOWLEDGEMENT. This research was partially supported by NSERC (Canada).

REFERENCES

1. A. Erdélyi, Ed., Higher Transcendental Functions, Vol. 1, McGraw-Hill, New York, 1953.
2. A. Lundell, Potential Theory and Cohomology, Lecture Notes, Department of Mathematics, University of Colorado, Boulder, 1993.