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RESEARCH NOTES

ON VON NEUMANN'S INEQUALITY

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Von Neumann's inequality states that for a contraction T acting on a Hilbert space H

(v) $\quad ||p(T)|| \leq \sup \{|p(z)|:|z| < 1\}$

holds for all polynomials p. The analog for a set of commuting contractions $\{T_1,\ldots,T_n\}$,

(v_n) $\quad ||p(T_1,\ldots,T_n)|| \leq \sup \{|p(z_1,\ldots,z_n)|:|z_i| < 1\}$

is known to be false for n > 2. In fact, for any c > o, there exist $\{T_1,\ldots,T_n\}$, where n is sufficiently large, and a polynomial p such that

$$||(p(T_1,\ldots,T_n)|| > c \sup\{|p(z_1,\ldots,z_n)|:|z_i| < 1\}, \qquad (2)$$

In this note we establish the following weakened version of (v_n):

<u>PROPOSITION 1</u>. Let $\{T_1,\ldots,T_n\}$ be commuting contractions on a Hilbert space H. Then for any polynomial p,

$$||p(T_1,\ldots,T_n)|| \leq \sup \{|p(z_1,\ldots,z_n)|:|z_i| < n^{\frac{1}{2}}\},$$

i.e., $D_n = \{(z_1,\ldots,z_n): |z_i| < n^{\frac{1}{2}}\}$ is a spectral set for $(T_1,\ldots,T_n)$.

Our proof is an easy consequence of the following proposition.

PROPOSITION 2 (3, I.9.2). Let $\{S_1,\ldots,S_n\}$ be commuting contractions with $\sum_{i=1}^{n} ||S_i||^2 \leq 1$. Then $\{S_1,\ldots,S_n\}$ has a commuting unitary dilation (in fact a regular one) and it therefore follwos immediately that $\{S_1,\ldots,S_n\}$ satisfies (v_n).

PROOF OF PROPOSITION 1. Given $\{T_1,\ldots,T_n\}$, let $S_i = n^{-\frac{1}{2}} T_i$, $i = 1,\ldots,n$. Then $\sum_{i=1}^{n} ||S_i||^2 = n^{-1} \sum_{i=1}^{n} ||T_i||^2 \leq 1$ so (v_n) holds for $\{S_1,\ldots,S_n\}$.

Given any polynomial $p(z_1,\ldots,z_n)$, let $q(z_1,\ldots,z_n) = p(n^{\frac{1}{2}}z_1,\ldots,n^{\frac{1}{2}}z_n)$. Then

$$\begin{aligned}
||p(T_1,\ldots,T_n)|| &= ||p(n^{\frac{1}{2}}S_1,\ldots,n^{\frac{1}{2}}S_n)|| \\
&= ||q(S_1,\ldots,S_n)|| \\
&\leq \sup \{|q(w_1,\ldots,w_n)|:|w_i| < 1\} \\
&= \sup \{|p(n^{\frac{1}{2}}w_1,\ldots,n^{\frac{1}{2}}w_n)|:|w_i| < 1\} \\
&= \sup \{|p(z_1,\ldots,z_n)|:|z_i| < n^{\frac{1}{2}}\}
\end{aligned}$$

COROLLARY 3. (see (1) p. 279). Any set $\{T_1,\ldots,T_n\}$ of commuting contractions on H has the polydisc $D_n = \{(z_1,\ldots,z_n):|z_i| < n^{\frac{1}{2}}\}$ as a complete spectral set.

PROOF. By proposition 2, there exist commuting unitary operators $U_1,\ldots,U_n$ on a Hilbert space K containing H such that $q(S_1,\ldots,S_n) = P\, q(U_1,\ldots,U_n)$ for all polynomials q, where $S_i = n^{-\frac{1}{2}} T_i$ and P projects K onto H. Setting $N_i = n^{\frac{1}{2}} U_i$, we have that $\{N_1,\ldots,N_n\}$ is a normal dilation of $\{T_1,\ldots,T_n\}$ with joint spectrum sp(N) contained in the boundary of D_n and the corollary follows as in (1).

Similarly, it follows that $D_a = \{(z_1,\ldots,z_n):|z_i| < a_i\}$ is a complete spectral set for all commuting contractions $\{T_1,\ldots,T_n\}$ if $\Sigma\, a_i^{-2} < 1$.

Since the common intersection of such D_a is the unit polydisc D, which is not in general a complete spectral set since (v_n) can fail if $n \geq 3$, we have

COROLLARY 4. If $\{T_1,\ldots,T_n\}$ is a set of commuting contractions such that the intersection of any two complete spectral sets is also a complete spectral set, then the unit polydisc D is also a complete spectral set.

We note that von Neumann's original paper (4) showed that for a single contraction the intersection of two spectral sets need not be a spectral set.

Since (v_n) holds for n = 2, we see that proposition 1 is not the best possible result. This prompts the following

PROBLEM. Find

$$V(n) = \inf\{r: \|p(T_1,\ldots,T_n)\| \leq \sup\{|p(z_1,\ldots,z_n)|:|z_i| < r\}\}$$

We note that Theorem 1.2(b) of (2) yields information concerning the growth of V(n) as n increases. Since $\sup\{|p(z_1,\ldots,z_n)|:|z_i| < r\} = r^s \sup\{|p(z)|:|z_i| < 1\}$ for homogeneous polynomials of degree s, we have for any $\varepsilon > o$, $V(n) \geq n^{(\frac{1}{4}-\varepsilon)}$ for n sufficiently large.

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