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ON THE DEFINITION OF A CLOSE-TO-CONVEX FUNCTION

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Dedicated to Professor S.M. Shah on the occasion of his 70th birthday and in recognition of his outstanding mathematical contributions.

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ABSTRACT. The standard definition of a close-to-convex function involves a complex numerical factor $e^{i\beta}$ which is on occasion erroneously replaced by 1. While it is known to experts in the field that this replacement cannot be made without essentially changing the class, explicit reasons for this fact seem to be lacking in the literature. Our purpose is to fill this gap, and in so doing we are lead to a new coefficient problem which is solved for $n = 2$, but is open for $n > 2$.

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1. THE DEFINITION OF A CLOSE-TO-CONVEX FUNCTION.

The most common form for this definition is

DEFINITION 1. The function

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

regular in $E : |z| < 1$, is said to be close-to-convex in E, if there is a $\phi(z)$,

$$\phi(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \quad b_1 = e^{i\beta}, \tag{1.2}$$

that is convex in E and such that in E

$$\operatorname{Re} \frac{f'(z)}{\phi'(z)} > 0. \tag{1.3}$$

We denote the set of all such functions by CL.

One can begin with more general expressions but any additive constants disappear on differentiation and hence may be dropped at the beginning. Further there is no loss of generality in assuming that $f'(0) = 1$ and $\phi'(0) = e^{i\beta}$. Here it seems natural to set $\phi'(0) = 1$, but there are several places in the literature (for example [3, p. 51]) where this second normalization is expressly forbidden. As far as we are aware, the reason for maintaining $b_1 = e^{i\beta}$ is never explicitly given.

In this note we prove that for each β in the open interval $(-\pi/2, \pi/2)$ there is a corresponding function F(z) that should be regarded as close-to-convex, but would not be in CL if that particular β is forbidden in Definition 1.

2. THE EXAMPLE FUNCTION.

It is well known and easy to prove that the function

$$F(z) = \frac{z - e^{2i\alpha}\cos\alpha z^2}{(1 - e^{i\alpha}z)^2}, \quad 0 < \alpha < \pi, \tag{2.1}$$

maps E onto the complex plane minus a vertical slit (see [1] where some interesting properties of this function are obtained). Hence the boundary curve of F(E) has no "hairpin" bend that exceeds 180°. Consequently, on geometric grounds (see Kaplan [2]) F(z) is close-to-convex. But we do not need these geometric facts because we can prove that F(z) satisfies Definition 1. Indeed

$$F'(z) = \frac{1-e^{3i\alpha}z}{(1-e^{i\alpha}z)^3} . \tag{2.2}$$

If we select for our convex function

$$\phi(z) = -ie^{i\alpha}\frac{z}{1-e^{i\alpha}z} , \tag{2.3}$$

then

$$\mathrm{Re}\,\frac{F'(z)}{\phi'(z)} = \mathrm{Re}\,\frac{1-e^{3i\alpha}z}{(1-e^{i\alpha}z)^3}\, ie^{-i\alpha}(1-e^{i\alpha}z)^2$$

$$\mathrm{Re}\,\frac{F'(z)}{\phi'(z)} = \mathrm{Re}\; i\,\frac{e^{-i\alpha}-e^{2i\alpha}z}{1-e^{i\alpha}z} > 0 \tag{2.4}$$

in E, because this last function carries E onto Re $w > 0$. Thus by Definition 1 with $e^{i\beta} = -ie^{i\alpha}$, F(z) is close-to-convex. Here

$$\beta = \arg(-ie^{i\alpha}) = \alpha - \pi/2, \tag{2.5}$$

and since $0 < \alpha < \pi$, we have $-\pi/2 < \beta < \pi/2$.

We now prove that if $\phi(z)$ is any convex function different from the one given by equation (2.6) then the condition

$$\mathrm{Re}\,\frac{F'(z)}{\phi'(z)} = \mathrm{Re}\,\frac{1-e^{3i\alpha}z}{(1-e^{i\alpha}z)^3}\,\frac{1}{\phi'(z)} > 0 \tag{2.6}$$

fails to be satisfied. Indeed, if (2.6) holds in E then we can write

$$\phi'(z) = P(z)\,\frac{1-e^{3i\alpha}z}{(1-e^{i\alpha}z)^3}, \tag{2.7}$$

where P(z) has positive real part in E. When P(0) = 1 it is well-known that

$$|P(z)| \geqq \frac{1-r}{1+r}\,, \quad \text{for } z = re^{i\theta},\ 0 \leqq r < 1,$$

and hence for arbitrary P(0) we have $|P(z)| \geqq c(1-r)$, for $z = re^{i\theta}$, $0 \leqq r < 1$, where c is a positive constant that depends only on P(0). This last inequality, together with (2.7), and the condition $\alpha \neq 0, \pi$, imply that

$$|\phi'(re^{-i\alpha})| \geqq \frac{\lambda}{(1-r)^2}\,, \text{ as } r \to 1^-, \tag{2.8}$$

where $\lambda > 0$ is some constant.

Now suppose that $\phi(E)$ is not a half-plane. Since $\phi(E)$ is a convex region, $\phi(E)$ must have two distinct lines of support and hence $\phi(E)$ is contained in a sector with vertex angle $< \pi$. Consequently, by subordination (see Rogosinski [8]) it follows that for some τ, $0 < \tau < 1$,

$$|\phi(re^{i\theta})| = O\left(\frac{1}{(1-r)^{\tau}}\right). \tag{2.9}$$

But then, using Cauchy estimates it is easy to see that (2.9) implies that

$$|\phi'(re^{i\theta})| = O\left(\frac{1}{(1-r)^{\tau+1}}\right),$$

and this contradicts (2.8) since $\tau + 1 < 2$. Hence $\phi(E)$ is a half-plane, and $\phi(z)$ has the form $\eta z/(1-e^{i\alpha}z)$, $|\eta| = 1$. But then η must be the factor selected in (2.3), otherwise the inequality (2.4) is false. Consequently, if the associated value of β is not permitted in the definition of a close-to-convex function, then the function (2.1) would not be classified as close-to-convex. This completes the justification of the factor $e^{i\beta}$ in Definition 1, if $-\pi/2 < \beta < \pi/2$.

The remaining cases, $\pi/2 \leqq \beta \leqq 3\pi/2$, are trivial. If β lies in this interval, then $\mathrm{Re}\{f'(0)/\phi'(0)\} \leqq 0$ and the condition (3) is not satisfied.

We have proved that Definition 1 is proper for the class we wish to describe, if we add the condition $-\pi/2 < \beta < \pi/2$. Further no single point of this interval can be dropped without losing at least one function from the class CL.

3. A REMARK ON THE COEFFICIENTS.

The class CL is naturally divided into subclasses CL(β) in accordance with the value of β that may be used in

$$\mathrm{Re}\,\frac{f'(z)}{e^{i\beta}\phi'(z)} > 0, \qquad z \text{ in } E, \tag{3.1}$$

where now $\phi(z) = z + \cdots$. The subclasses CL(β) are exhaustive, but they are certainly not mutually exclusive. Thus if $f(z)$ is itself convex then we may take $\phi(z) \equiv f(z)$ in (3.1). Hence a convex function is in CL(β) for every β in $(-\pi/2, \pi/2)$. In fact, it is easy to show using a normal families argument that the intersection

$$\bigcap_{-\pi/2<\beta<\pi/2} \mathrm{CL}(\beta)$$

is precisely the collection of all normalized convex functions.

We can ask for extreme properties of functions in the subclasses CL(β). Here we pause only to discuss the magnitude of the coefficients.

THEOREM 1. If $f(z)$ given by (1.1) is in CL(β), then for $n=2,3,\ldots$

$$|a_n| \leqq 1 + (n-1)\cos\beta. \tag{3.2}$$

If $n=2$, the result $|a_2| \leqq 1 + \cos\beta$ is sharp.

PROOF. If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ is a normalized function with positive real part and $\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is the associated convex function, then (3.1) yields

$$\frac{1}{\cos\beta}\left[\frac{f'(z)}{e^{i\beta}\phi'(z)} + i\sin\beta\right] = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{3.3}$$

or

$$1 + \sum_{n=2}^{\infty} na_n z^{n-1} = \left(1 + \sum_{n=2}^{\infty} nb_n z^{n-1}\right)\left(1 + e^{i\beta}\cos\beta \sum_{n=1}^{\infty} p_n z^n\right).$$

Hence

$$na_n = nb_n + e^{i\beta}\cos\beta\left(p_{n-1} + \sum_{k=2}^{n-1} kb_k p_{n-k}\right). \tag{3.4}$$

The known bounds, $|b_k| \leqq 1$ (Loewner), and $|p_k| \leqq 2$ (Caratheodory) yield the inequality (3.2).

If we define G(z) to be the solution of

$$G'(z) = \frac{1+e^{i\beta}z}{1-e^{-i\beta}z}\,\frac{1}{(1-z)^2} = 1 + \sum_{n=2}^{\infty} n\,A_n z^{n-1}, \tag{3.5}$$

then for $n \geqq 1$

$$A_n = 1 + \frac{2\cos\beta}{n}\sum_{k=1}^{n-1} ke^{-i(n-k-1)\beta}. \tag{3.6}$$

If $G(0) = 0$ and $\cos\beta \neq 1$, then

$$G(z) = \frac{e^{i\beta}}{1-\cos\beta}\left\{\cos\beta\, \ell n\left(\frac{1-e^{-i\beta}z}{1-z}\right) - i\sin\beta\,\frac{z}{1-z}\right\}. \tag{3.7}$$

If $\cos\beta = 1$, then $G(z) = z/(1-z)^2$. In either case, for $n = 2$, equation (3.6) gives $A_2 = 1 + \cos\beta$, the sharp upper bound. To see that G(z) is in CL(β) we put (3.5) in the form

$$\frac{G'(z)(1-z)^2}{e^{i\beta}} = \frac{1}{e^{i\beta}}\,\frac{1+e^{i\beta}z}{1-e^{-i\beta}z} \tag{3.8}$$

$$= -i\sin\beta + \cos\beta\,\frac{1+e^{-i\beta}z}{1-e^{-i\beta}z}.$$

Since the convex function $\phi(z) = z/(1-z)$ yields $1/\phi'(z) = (1-z)^2$, the function

G(z) satisfies the condition (3.1).

The problem of finding the maximum for $|a_n|$ in the class CL(β) seems to be difficult for $n \geqq 3$. Although G(z), given by (3.7) furnishes the maximum when $n = 2$, there is no reason to believe that it continues to play this role when $n > 2$. Indeed if we use equation (3.7), we can obtain an alternate form for the coefficient A_n (the sum indicated in (3.6)):

$$A_n = \frac{e^{i\beta}}{1-\cos\beta}\left\{\frac{(1-e^{-in\beta})\cos\beta}{n} - i\sin n\beta\right\}. \tag{3.9}$$

Thus with β fixed, $A_n \to$ a constant as $n \to \infty$.

In contrast, if F(z), given by equation (2.1), has the expansion $\sum_1^\infty B_n z^n$, then

$$B_n = 1 + (n-1)\cos^2\beta + i(n-1)\sin\beta\cos\beta \tag{3.10}$$

where $\beta = \alpha - \pi/2$, and B_n/n approaches a nonzero constant as $n \to \infty$.

We observe that for $\beta \neq 0$, the extremal function (for $|a_2|$) G(z) maps E onto a slit half-plane. Both the boundary and the slit make an angle β with the real axis. Thus the complement of G(E) contains a half-plane. It is very unusual for the extremal solution of a coefficient problem to omit an open set when there are competing functions (such as F(z)) in the same class that do not omit any open set. As $\beta \to 0$, the function $G(z) \to z/(1-z)^2$, the Koebe function.

We return to the bound $|a_n| \leqq 1 + (n-1)\cos\beta$ given in Theorem 1. Since every convex function belongs to CL(β) for every β in $(-\pi/2, \pi/2)$ this inequality includes the Loewner bound $|a_n| \leqq 1$ for convex functions as a special case. It also includes the inequality $|a_n| \leqq n$ for all close-to-convex functions; a result that was obtained much earlier by M. Reade [5].

Work on the coefficients of subclasses of CL has been done by Renyi [7], Pommerenke [4], and Reade [6], but their subclasses are different from the ones considered here. Reade [6] proves that if the condition (3) is replaced by

$$\left|\arg \frac{f'(z)}{\phi'(z)}\right| < \frac{\pi}{2}\alpha, \quad 0 < \alpha \leqq 1, \tag{3.11}$$

then

$$|a_n| \leqq 1 + (n-1)\alpha \tag{3.12}$$

and the result is sharp when n = 2.

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