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GENERALIZED WHITTAKER'S EQUATIONS FOR HOLONOMIC MECHANICAL SYSTEMS

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ABSTRACT. In this paper the classical theorem "a conservative holonomic dynamic system is invariantly connected with a certain differential form" is generalized to group variables. This general theorem is then used to reduce the order of a Hamiltonian system by the use of the integral of energy. Equations of motion of the reduced system so obtained are derived which are the so-called generalized Whittaker's equations. Finally an example is given as an application of the theory.

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1. INTRODUCTION

It is known [4] that canonical equations for a conservative holonomic system whose Hamiltonian is H are obtained by forming the first Pfaff's system of differential equations of the differential form,

$$p_i dq_i - H\,dt, \qquad (i = 1,2,---,n),$$

where $p_1, p_2, ---, p_n$ are the generalized momenta corresponding to the generalized coordinates $q_1, q_2, ---, q_n$ of the system and summation over a repeated suffix is implied. This result leads to the theorem "A dynamical system is invariantly connected with the differential form $p_i dq_i - Hdt$". Further this theorem has been used to reduce the order of the system by means of the integral of energy. Canonical equations of the reduced system so obtained are known as Whittaker's equations. In what follows in this section we state a few basic results from the theory of group variables in order to generalize the above mentioned results.

Consider a conservative holonomic system having n degrees of freedom and whose position is specified by group variables $x_1, x_2, ---, x_n$. Let $\eta_1, \eta_2, ---, \eta_n$ be the parameters of real displacement and $X_1, X_2, ---, X_n$ be the corresponding displacement operators expressed by the relations

$$X_i = \overset{j}{\xi_i} \frac{\partial}{\partial x_j}, \quad (i,j = 1,2, ---,n) \tag{1}$$

where $\overset{j}{\xi_i}$ are functions of $x_1, x_2, --, x_n$, then for an arbitrary function $f(x_1, x_2, ---, x_n, t)$ the infinitesimal change df is expressed by

$$df = [X_o(f) + \eta_i X_i(f)]dt \tag{2}$$

where $X_o = \frac{\partial}{\partial t}$. The X's satisfy the relations

$$(X_o, X_i) = 0, \ (X_i, X_j) = C_{ijk} X_k \qquad (k=1,2,---,n). \tag{3}$$

Putting $f = x_j$ in (2), we get

$$\frac{dx_j}{dt} = \dot{x}_j = \eta_i \overset{j}{\xi_i} \tag{4}$$

Since the operators X_i are independent therefore the matrix $||\overset{j}{\xi_i}||$ is non-singular and consequently (4) yields

$$\eta_i = A_{ij} \dot{x}_j \tag{5}$$

Let L be the Lagrangian of the system then the canonical equations of the system as obtained in [1,2] are

$$\eta_i = \frac{\partial H}{\partial y_i}, \frac{dy_i}{dt} = C_{jik}\, \eta_j\, y_k - X_i(H), \tag{6}$$

where $(i,j,k = 1,2,---,n)$,

$$y_i = \frac{\partial L}{\partial \eta_i}$$

and

$$H(x_1, x_2, ----, x_n; y_1, ---, y_n) = \eta_i y_i - L \tag{7}$$

is the Hamiltonian of the system and is equal to the total energy of the system.

2. DERIVATION OF CANONICAL EQUATIONS FROM A CERTAIN DIFFERENTIAL FORM.

In order to establish the invariant relation between the system (6) and a certain differential form we prove the following theorem:

THEOREM. The system of equations (6) is equivalent to the first Pfaff's system of differential equations of the differential form $(\eta_i y_2 - H)dt$.

PROOF. We put

$$\theta_d = (\eta_i y_i - H)dt$$

or using (5), we obtain

$$\theta_d = y_i A_{ij}\, d\, x_j - H\, dt \tag{8}$$

therefore

$$\theta_\delta = y_i\, A_{ij}\, \delta x_j - H\, \delta t \tag{9}$$

where d and δ denote two independent variations in each of the variables x_1, x_2, ---, x_n, y_1, ---, y_n, t. The bilinear coreariant of (8) is given by

$$\delta\theta_d - d\theta_\delta = \delta y_i [A_{ij} dx_j - \frac{\partial H}{\partial y_i} dt] + \delta x_k [y_i \frac{\partial A_{ij}}{\partial x_k} dx_j - \frac{\partial H}{\partial x_k} dt - dy_i\, A_{ik} - y_i \frac{\partial A_{ik}}{\partial x_j} dx_j]$$

$$+ \delta t [dH - \frac{\partial H}{\partial t} dt] \tag{10}$$

where we have used the relations

$$d\delta x_i = \delta d\, x_i \quad (i = 1,2,---,n)$$

$$d\delta t = \delta dt.$$

Equating to zero the coefficients of δx_1, δx_2, ---, δx_n, δy_1,---, δy_n, δt in (10), we get the first Pfaff's system of equation in the form

$$A_{ij} dx_j - \frac{\partial H}{\partial y_i} dt = 0, \quad (i = 1,2,---,n) \tag{11}$$

$$y_i \frac{\partial A_{ij}}{\partial x_k} dx_j - \frac{\partial H}{\partial x_k} dt - dy_i A_{ik} - y_i \frac{\partial A_{ik}}{\partial x_j} dx_j = 0 \tag{12}$$

$$dH = \frac{\partial H}{\partial t} dt, \quad (i = 1,2,---,n). \tag{13}$$

By virtue of (5), the equations (11) assume the form

$$\eta_i = \frac{\partial H}{\partial y_i}, \quad (i = 1,2,---,n). \tag{14}$$

With the help of (4), the equations (12) become

$$\frac{dy_i}{dt} = y_\ell \eta_m \frac{\partial A_{\ell j}}{\partial x_k} \overset{j}{\xi_m} \overset{k}{\xi_i} - y_\ell \eta_m \frac{\partial A_{\ell k}}{\partial x_j} \overset{j}{\xi_m} \overset{k}{\xi_i} - \overset{k}{\xi_i} \frac{\partial H}{\partial x_k}$$

which, by means of the relations (1) and (3), finally takes the form

$$\frac{dy_i}{dt} = \eta_j\, y_k\, C_{jik} - X_i(H). \tag{15}$$

The relation (13) is a consequence of (14) and (15) and skew symmetric property of C_{jik} with respect to the first two indices. Since the equations (14) and (15) are identical with (6) the theorem is thus proved.

3. GENERALIZED WHITTAKER'S EQUATIONS:

Assume that H does not involve the time explicitly and

$$H + h = 0, \tag{16}$$

is the integral of energy of the system. Let the equation (16) be solved for the variable y_1 so that it is algebraically equivalent to

$$K(x_1,---, x_n, y_2,---, y_n, t, h) + y_1 = 0. \tag{17}$$

The differential form associated with the system is

$$(\eta_1 y_1 + \eta_2 y_2 + --- + \eta_n y_n + h)dt,$$

where the variables $x_1, ---, x_n, y_1, ---, y_n, h$ are connected by (17); the differential form can therefore be written as

$$(\eta_2 y_2 + \eta_3 y_3 + --- + \eta_n y_n + h)dt - \eta_1 K\, dt \tag{18}$$

where we can regard $(x_1, ---, x_n, y_2, ---, y_n, h, t)$ as the 2n+1 variables in the phase space. If we express (18) in the form

$$\eta_1 dt\, [\eta_2' y_2 + --- + \eta_n' y_n + \frac{1}{\eta_1} h - K] \tag{19}$$

and put

$$\eta_1 dt = d\tau$$

then we take τ as the new time variable and $\frac{1}{\eta_1}$, $\eta_1' = 1$, $\eta_2' = \frac{\eta_2}{\eta_1}$, ---, $\eta_n' = \frac{\eta_n}{\eta_1}$

as the parameters of real displacement, the corresponding displacement operators and new momenta are respectively $X_o, X_1, ---, X_n$ and $h, y_1, ---, y_n$. Using the result of section (2), the differential equation corresponding to the form (19) are

$$\eta_p' = \frac{\partial K}{\partial y_p}, \quad \frac{dy_p}{d\tau} = \eta_j' y_k C_{jpk} - X_p(K), \quad (p = 2,3,---,n) \tag{20}$$

$$\frac{dt}{d\tau} = \frac{\partial K}{\partial h}, \quad \frac{dh}{d\tau} = 0.$$

The last pair of equations can be separated from the rest of the system since the first (2n-2) equations do not involve t and h is a constant. The equations (20) can be further simplified to take the form

$$\eta_p' = \frac{\partial K}{\partial y_p}, \quad \frac{dy_p}{d\tau} = - K[C_{1p1} + \eta_r' C_{rp1}] + y_r C_{1pr} + \eta_r' y_q C_{rpq} - X_p(K) \tag{21}$$

$$(p,q,r = 2,3,---,n).$$

The original differential equations can therefore be replaced by the reduced system (21) which has only n-1 degrees of freedom. The equations (21) are the desired Whittaker's equations.

4. AN EXAMPLE

Consider a rigid body which is moving about one of its fixed points O under the action of gravity. We introduce a fixed frame of reference Oxyz such that Oz is vertically upwards and a moving frame Ox'y'z' which coincides with the principal axes of inertia of the body at O. Let us choose the Eulerian angles θ, ϕ, ψ (θ is the angle of nutation, ϕ the angle of precession and ψ the angle of proper rotation) as the group variables which specify the position of the body at time t. Obviously the dynamical system under consideration is a conservative one and it has three degrees of freedom. Choosing the parameters of real displacement as the components of angular velocity along the moving axes, we have the relations

$$\eta_1 = \dot{\theta} \operatorname{Cos} \psi + \dot{\phi} \operatorname{Sin} \theta \operatorname{Sin} \psi,$$

$$\eta_2 = -\dot{\theta} \operatorname{Sin} \psi + \dot{\phi} \operatorname{Sin} \theta \operatorname{Cos} \psi,$$

$$\eta_3 = \dot{\psi} + \dot{\phi} \operatorname{Cos} \theta$$

Consequently the displacement operators X_1, X_2, X_3 are given by

$$\left.\begin{aligned} X_1 &= \operatorname{Cos} \psi \frac{\partial}{\partial \theta} + \operatorname{Cosec} \theta \operatorname{Sin} \psi \frac{\partial}{\partial \phi} - \operatorname{Cot} \theta \operatorname{Sin} \psi \frac{\partial}{\partial \psi} \\ X_2 &= -\operatorname{Sin} \psi \frac{\partial}{\partial \theta} + \operatorname{Cosec} \theta \operatorname{Cos} \psi \frac{\partial}{\partial \phi} - \operatorname{Cot} \theta \operatorname{Cos} \psi \frac{\partial}{\partial \psi} \\ X_3 &= \frac{\partial}{\partial \psi} \end{aligned}\right] \quad (22)$$

which satisfy the commutation relations

$$\left.\begin{aligned} (X_1, X_2) &= X_1X_2 - X_2X_1 = X_3 \\ (X_2, X_3) &= X_2X_3 - X_3X_2 = X_1 \\ (X_3, X_1) &= X_3X_1 - X_1X_3 = X_2 \end{aligned}\right] \quad (23)$$

The non-vanishing C's are therefore expressed by the relations

$$\left.\begin{aligned} C_{123} &= -C_{213} = 1, \\ C_{231} &= -C_{321} = 1, \\ C_{312} &= -C_{132} = 1. \end{aligned}\right] \tag{24}$$

Let T and U denote the kinetic and potential energies of the system respectively, then

$$\left.\begin{aligned} T &= \frac{1}{2}(A\,\eta_1^2 + B\,\eta_2^2 + C\eta_3^2), \\ U &= -Mg(\bar{x}\,\mathrm{Sin}\,\theta\,\mathrm{Sin}\,\psi + \bar{y}\,\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\psi + \bar{z}\,\mathrm{Cos}\,\theta \end{aligned}\right] \tag{25}$$

where A, B, C are the principal moments of inertia at O; $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coordinates of the centre of gravity of the body with respect to the moving axis and M is the mass of the body. Using (25), we have the Lagrangian L and momenta y_1, y_2, y_3 expressed by the relations:

$$L = T - U = \frac{1}{2}(A\eta_1^2 + B\eta_2^2 + C\eta_3^2) + Mg\,(\bar{x}\,\mathrm{Sin}\,\theta\,\mathrm{Sin}\,\psi + \bar{y}\,\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\psi + \bar{z}\,\mathrm{Cos}\,\theta),$$

$$y_1 = A\,\eta_1,\ y_2 = B\,\eta_2,\ y_3 = C\,\eta_3. \tag{26}$$

In view of (26) the Hamiltonian H is given by

$$H = \frac{1}{2}\left(\frac{y_1^2}{A} + \frac{y_2^2}{B} + \frac{y_3^2}{C}\right) - Mg(\bar{x}\,\mathrm{Sin}\,\theta\,\mathrm{Sin}\,\psi + \bar{y}\,\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\psi + \bar{z}\,\mathrm{Cos}\,\theta) \tag{27}$$

Using (6), (22), (24), (26), and (27), canonical equations of the system are

$$\left.\begin{aligned} &\eta_1 = \frac{y_1}{A},\ \eta_2 = \frac{y_2}{B},\ \eta_3 = \frac{y_3}{C} \\ &\frac{dy_1}{dt} = \frac{B-C}{BC}\,y_2y_3 + Mg\,(\bar{y}\,\mathrm{Cos}\,\theta - \bar{z}\,\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\psi), \\ &\frac{dy_2}{dt} = \frac{C-A}{CA}\,y_3y_1 + Mg\,(-\bar{x}\,\mathrm{Cos}\,\theta + \bar{z}\,\mathrm{Sin}\,\theta\,\mathrm{Sin}\,\psi) \\ &\frac{dy_3}{dt} = \frac{A-B}{AB}\,y_1y_2 + Mg\,\mathrm{Sin}\,\theta(\bar{x}\,\mathrm{Cos}\,\psi - \bar{y}\,\mathrm{Sin}\,\psi). \end{aligned}\right] \tag{28}$$

Now the relation (16) gives

$$\frac{y_1^2}{A} + \frac{y_2^2}{B} + \frac{y_3^2}{C} - 2Mg(\bar{x}\ \text{Sin}\ \theta\ \text{Sin}\ \psi + \bar{y}\ \text{Sin}\ \theta\ \text{Cos}\ \psi + \bar{z}\ \text{Cos}\ \theta) + 2h = 0,$$

and consequently

$$y_1 = A[2Mg(\bar{x}\ \text{Sin}\ \theta\ \text{Sin}\ \psi + \bar{y}\ \text{Sin}\ \theta\ \text{Cos}\ \psi + \bar{z}\ \text{Cos}\ \theta) - \frac{y_2^2}{B} - \frac{y_3^2}{C} - 2h]^{\frac{1}{2}}.$$

Comparing this relation with (17), we get

$$K = -A[2Mg(\bar{x}\ \text{Sin}\ \theta\ \text{Sin}\ \psi + \bar{y}\ \text{Sin}\ \theta\ \text{Cos}\ \psi + \bar{z}\ \text{Cos}\ \theta) - \frac{y_2^2}{B} - \frac{y_3^2}{C} - 2h]^{\frac{1}{2}}.$$

Therefore by the application of (21) the canonical equations of the system reduce to

$$\left.\begin{aligned} \eta_2' &= \frac{\partial K}{\partial y_2}, \quad \eta_3' = \frac{\partial K}{\partial y_3}, \\ \frac{dy_2}{d\tau} &= y_3 - \eta_3' y_1 - X_2(K), \\ \frac{dy_3}{d\tau} &= -y_2 + \eta_2' y_1 - X_3(K). \end{aligned}\right] \qquad (29)$$

Now

$$\frac{\partial K}{\partial h} = -\frac{A}{K}, \quad \frac{\partial K}{\partial y_2} = -\frac{A}{BK} y_2, \quad \frac{\partial K}{\partial y_3} = -\frac{A}{CK} y_3,$$

$$\frac{dy_2}{d\tau} = \frac{dy_2}{dt}\frac{\partial K}{\partial h} = -\frac{A}{K}\frac{dy_2}{dt}$$

$$\frac{dy_3}{dt} = \frac{dy_3}{dt}\frac{\partial K}{\partial h} = -\frac{A}{K}\frac{dy_3}{dt}$$

$$X_2(K) = \frac{A}{K} \cdot Mg(-\bar{x}\ \text{Cos}\ \theta + \bar{z}\ \text{Sin}\ \theta\ \text{Sin}\ \psi),$$

$$X_3(K) = \frac{A}{K} \cdot Mg(\bar{x}\ \text{Cos}\ \psi - \bar{y}\ \text{Sin}\ \psi)\ \text{Sin}\ \theta.$$

Therefore equation (29) assume the form

$$\eta_2' = -\frac{A}{BK} y_2, \quad \eta_3' = -\frac{A}{CK} y_3$$

$$\frac{dy_2}{dt} = \frac{K(A-C)}{CA} y_3 + Mg(-\bar{x} \text{ Cos } \theta + \bar{z} \text{ Sin } \theta \text{ Sin } \psi),$$

$$\frac{dy_3}{dt} = \frac{K(B-A)}{AB} y_2 + Mg \text{ Sin } \theta(\bar{x} \text{ Cos } \psi - \bar{y} \text{ Sin } \psi).$$

These are the Whittaker's equations for the system under consideration.

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