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(n-2)-TIGHTNESS AND CURVATURE OF SUBMANIFOLDS WITH BOUNDARY

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ABSTRACT. The purpose of this note is to establish a connection between the notion of (n-2)-tightness in the sense of N.H. Kuiper and T.F. Banchoff and the total absolute curvature of compact submanifolds-with-boundary of even dimension in Euclidean space. The argument used is a certain geometric inequality similar to that of S.S. Chern and R.K. Lashof where equality characterizes (n-2)-tightness.

KEY WORDS AND PHRASES. *tight manifolds, total absolute curvature.*

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1. INTRODUCTION.

Let M be a compact n-dimensional smooth manifold with or without boundary - where the boundary is assumed to be smooth - and let

$$f : M \longrightarrow E^{n+k}$$

be a smooth immersion of M into the $(n+k)$-dimensional euclidean space. This leads to the notion of total absolute curvature

$$TA(f) = \frac{1}{c_{n+k-1}} \int_N |K| *1$$

where K denotes the Lipschitz-Killing curvature of f in each normal direction, N the unit normal bundle (with only the 'outer' normals at points of ∂M), and c_m denotes the volume of the unit sphere $S^m \subseteq E^{m+1}$. For detailed definitions, in particular in the case of manifolds with boundary, see [5] or [6].

Let us state the following equation ([6], 2.2)

$$TA(f) = TA(f|_{M\setminus\partial M}) + \tfrac{1}{2}TA(f|_{\partial M}) \qquad (1.1)$$

The famous result of S.S. Chern and R.F. Lashof gives a connection between total absolute curvature and the number of critical points of so-called height functions

$$zf : M \longrightarrow \mathbb{R}$$

defined by $(zf)(p) = \langle z, f(p) \rangle$, $z \in S^{n+k-1}$

Extending this result to the case of manifolds with boundary we can write

$$TA(f) = \frac{1}{c_{n+k-1}} \int_{z \in S^{n+k-1}} \sum_i (\mu_i(zf) + \overset{+}{\mu}_i(zf)) *1 \qquad (1.2)$$

where $\mu_i(zf)$ denotes the number of critical points of zf of index i in $M\setminus\partial M$, and $\overset{+}{\mu}_i(zf)$ denotes the number of (+)-critical points of zf of index i in ∂M. Here a point $p \in \partial M$ is called (+)-critical if p is critical for $zf|_{\partial M}$ and $\mathrm{grad}_p f$ is a nonvanishing inner vector on M (for details, see [2], [4] or [6]).

The i-th curvature τ_i introduced by N.H. Kuiper (cf. [7]) can be expressed by

$$\tau_i(f) = \frac{1}{c_{n+k-1}} \int_{z \in S^{n+k-1}} (\mu_i(zf) + \mu_i^+(zf)) *1$$

(cf. [6], lemma 4.2 or [9], lemma 3.1). So we get

$$TA(f) = \sum_i \tau_i(f).$$

The Morse-relations give the following connections between the curvatures and some topological invariants of M:

$$\tau_i(f) \geqq b_i(M) \tag{1.3}$$

$$TA(f) \geqq b(m) := \sum_i b_i(M)$$

$$\sum_i (-1)^i \tau_i(f) = \chi(M) = \sum_i (-1)^i b_i(M)$$

where $b_i(M)$ denotes the i-th Betti-number of homology with coefficients in a suitable field. (cf. [7]).

f is called <u>k-tight</u> if for all $k' \leqq k$ and for almost all $z \in S^{n+k-1}$ and all real numbers c the inclusion map

$$j: (zf)_c := \{p \in M / (zf)(p) \leqq c\} \longrightarrow M$$

induces a monomorphism in the k'-th homology :

$$H_{k'}(j) : H_{k'}((zf)_c) \longrightarrow H_{k'}(M)$$

As usual we write shortly 'tight' instead of 'n-tight'.

Then the results of N.H. Kuiper show

$$TA(f) = b(M) \quad \text{if and only if} \quad f \text{ is tight,}$$

$\tau_k(f) = b_k(M)$ if and only if $H_k(j)$ and $H_{k-1}(j)$ are monomorphisms for almost all z, all c (cf. [7]).

Results on tightness are collected in the survey article [10] by T.J. Willmore, for results on k-tightness we refer in addition to the notes [1] by T. Banchoff and [9] by L. Rodriguez, who has shown that in some sense (n-2)-tightness is closely related to convexity.

2. RESULTS

As mentioned above there is a relation between tightness on one hand and total absolute curvature and the sum of the Betti-numbers on the other hand. The following results give certain connections between (n-2)-tightness on one hand and usual curvature terms and the sum of the Betti-numbers on the other hand. Note that in case $\partial M = \phi$ by duality arguments tightness is equivalent to k-tightness for $k = \frac{n}{2} - 1$ if n is even and for $k = \frac{n-1}{2}$ if n is odd. But in case $\partial M \neq \phi$ there are examples of (n-2)-tight immersions which are not tight (for example: consider the round hemi-sphere).

THEOREM A . Let M^n be an even-dimensional manifold with non-void boundary and $f : M \to E^{n+k}$ be an immersion. Let N_0 be the unit normal bundle of $f_{M \setminus \partial M}$ and denote by $N_* \subsetneqq N_0$ the open set of unit normals where the second fundamental form of f is positive or negative definite.

Then there holds the following inequality

$$\tfrac{1}{2}TA(f|_{\partial M}) + \frac{1}{c_{n+k-1}} \int_{N_0 \setminus N_*} |K| \ {*1} \geqq b(m) \qquad (2.1)$$

where equality characterizes (n-2)-tightness of f .

In case of hypersurfaces (k = 1) (2.1) becomes

$$\tfrac{1}{2}TA(f|_{\partial M}) + TA(f|_{M \setminus M_* \setminus \partial M}) \geqq b(M) \qquad (2.2)$$

where M_* denotes the set of points in $M\setminus\partial M$ with positive or negative definite second fundamental form.
In case $n = 2$ M_* is just the set of points with positive Gaussian curvature, so we get

COROLLARY A 1. Assume $n = 2$ and $k = 1$. Then there holds the following inequality

$$\frac{1}{2\pi}\int_{K<0}|K|\ do + \frac{1}{2\pi}\int_{\partial M}|\kappa|\ ds \geq b(M) \geqq 2 - \chi(M) \qquad (2.3)$$

where equality characterizes 0-tightness of f. Here $|\kappa|$ denotes the usual curvature of $f|_{\partial M}$ considered as a space curve. For part of this result see [8], Prop. 9.

COROLLARY A 2. Assume $b(\partial M) = 2\,b(M)$. Then (n-2)-tightness of f implies that $f|_{\partial M}$ is tight and that the second fundamental form of f has either non-maximal rank of is positive or negative definite.

This is shown in [9], Prop. 5.2 under the assumption that M^n can be embedded in E^n. This condition implies $b(\partial M) = 2\,b(M)$ by Alexander duality.

Under the additional assumption that ∂M consists of a certain number of (n-1)-spheres L. Rodriguez has shown that (n-1)-tightness is equivalent to convexity (cf. [9], Theorem 2). This is not true in general, (See Corollary B 2 below).

THEOREM B. Let n be even and $f : M^n \to E^{n+1}$ be (n-2)-tight (if $\partial M \neq \phi$) or tight (if $\partial M = \phi$), and let $\tilde{M} \subsetneqq M\setminus\partial M$ be a compact submanifold of dimension n which is contained in some coordinate neighborhood in M. As above M_* denotes the set of points in $M\setminus\partial M$ with positive or negative definite second fundamental form. Then there holds the following inequality

$$\mathrm{TA}(f|_{\partial\tilde{M}}) \geq b(\partial\tilde{M}) + 2\ \mathrm{TA}(f|_{\tilde{M}\setminus M_*\setminus\partial\tilde{M}}) \qquad (2.4)$$

where equality characterizes (n-2)-tightness of $f|_{M\setminus(\tilde{M}\setminus\partial\tilde{M})}$.

REMARK. If $\tilde{M}$ contains only points of vanishing curvature or definite second fundamental form, then $\tilde{M}\setminus M_* = \phi$ and (2.4) reduces to the inequality of S.S Chern and R.K. Lashof for $\partial\tilde{M}$, otherwise (2.4) is sharper and reflects the additional condition that $\tilde{M}$ lies inside of some given M . For example in case $n = 2$ and $\tilde{M}$ being a disk we get

$$\int_{\partial\tilde{M}} |\chi| \, ds \geq 2\pi + \int_{\tilde{M}\cap\{K<0\}} |K| \, do \qquad (2.5)$$

COROLLARY B 1. Let f be as in Theorem B and assume moreover that there is an open region $U \subsetneqq M$ which is embedded by f in a hyperplane of E^{n+1} which implies $K|_U = 0$. Let $\tilde{M}^n$ be an embedded compact submanifold of E^n and assume by changing the scale $\tilde{M} \subsetneqq f(U)$.

Then $f|_{M\setminus f^{-1}(\tilde{M}\setminus\partial\tilde{M})}$ is (n-2)-tight if and only if $\partial\tilde{M}$ is tightly embedded in E^n .

Note that for $\tilde{M}^n \subsetneqq E^n$ tightness of $\tilde{M}$ and tightness of $\partial\tilde{M}$ are equivalent: this can be obtained easily using the equations $\mathrm{TA}(\tilde{M}) = \frac{1}{2}\mathrm{TA}(\partial\tilde{M})$ and $b(\tilde{M}) = \frac{1}{2}b(\partial\tilde{M})$.

Roughly spoken Corollary B 1 says: (n-2)-tight minus tight gives (n-2)-tight. In particular we get the following

COROLLARY B 2. In each even dimension there exist (n-2)-tight hypersurfaces which are not tight and not convex in the sense of [9] , in particular where $f(\partial M)$ is not contained in the boundary of the convex hull of $f(M)$.

3. PROOFS.

In all proofs the immersion f is fixed and so we may write $\mathrm{TA}(\partial M)$ instead of $TA(f|_{\partial M})$ and so on.

PROOF OF THEOREM A.

From

$$TA(M) = \sum_i \tau_i(M)$$

and

$$\chi(M) = \sum_i (-1)^i \tau_i(M)$$

we get

$$TA(M) + \chi(M) = 2\sum_i \tau_{2i}(M)$$

On the other hand by definition $\tau_n(M)$ is the average of the number of critical points of zf of index n which are precisely the strict local maxima in $M \setminus \partial M$. But a point is a strict local extremum of some height function zf if and only if the second fundamental form in the direction of z is positive or negative definite. Hence we get

$$2\,\tau_n(M) = \frac{1}{c_{n+k-1}} \int_{N_*} |K| \;*1$$

leading to

$$\begin{aligned} TA(M) - \frac{1}{c_{n+k-1}} \int_{N_*} |K| \;*1 &= 2\,(\tau_0(M) + \tau_2(M) + \ldots + \tau_{n-2}(M)) - \chi(M) \\ &\geq 2\,(b_0(M) + b_2(M) + \ldots + b_{n-2}(M)) - \chi(M) \\ &= b(M)\,, \end{aligned}$$

where we have used the assumption that n is even and $\partial M \neq \phi$ which implies $b_n(M) = 0$.

The case of equality is equivalent to the following equations:

$$\tau_o(M) = b_o(M) \ , \ \tau_2(M) = b_2(M) \ , \ \dots \ , \ \tau_{n-2}(M) = b_{n-2}(M) \tag{2.6}$$

But the equality $\tau_i(M) = b_i(M)$ is equivalent to injectivity of $H_i(j)$ and $H_{i-1}(j)$ for all inclusions $j : (zf)_c \to M$, so (2.6) is equivalent to (n-2)-tightness of f.

The assertion of the theorem then follows from the inequality above using the equation (1.1)

$$TA(M) = TA(M \setminus \partial M) + \tfrac{1}{2}TA(\partial M)$$

PROOF of Corollary A 2. By theorem A (n-2)-tightness of f implies

$$b(M) = \tfrac{1}{2}TA(\partial M) + \frac{1}{c_{n+k-1}} \int_{N_0 \setminus N_*} |K| \ *1$$

$$\geqq \tfrac{1}{2}TA(\partial M) \geqq \tfrac{1}{2}b(\partial M) = b(M)$$

which implies tightness of $f|_{\partial M}$ and moreover the vanishing of the integral of $|K|$ over $N_o \setminus N_*$, hence $K = 0$ on $N_o \setminus N_*$.

PROOF of Theorem B. By assumption and by theorem A we have

$$TA(M \setminus M_* \setminus \partial M) + \tfrac{1}{2}TA(\partial M) = b(M) \ , \text{ if } \partial M \neq \phi \ , \tag{2.7}$$

or $$TA(M) = b(M) \ , \text{ if } \partial M = \phi$$

which last equality is equivalent to

$$TA(M \setminus M_*) = b(M) - 2 \tag{2.8}$$

For $f|_{M \setminus (\tilde{M} \setminus \partial\tilde{M})}$ theorem A yields

$$TA(M \setminus \tilde{M} \setminus M_* \setminus \partial M \setminus \partial\tilde{M}) + \tfrac{1}{2}TA(\partial M) + \tfrac{1}{2}TA(\partial\tilde{M}) \geqq b(M \setminus \tilde{M}) \tag{2.9}$$

where equality characterizes (n-2)-tightness of $f|_{M \setminus (\tilde{M} \setminus \partial\tilde{M})}$.
Subtracting (2.9) from (2.7) or (2.8) respectively we get

$$TA(\tilde{M}\backslash M_{*}\backslash \partial\tilde{M}) - \tfrac{1}{2}TA(\partial\tilde{M}) \leqq b(M) - b(M\backslash\tilde{M}) \tag{2.10}$$

$$TA(\tilde{M}\backslash M_{*}\backslash \partial\tilde{M}) - \tfrac{1}{2}TA(\partial\tilde{M}) \leqq b(M) - b(M\backslash\tilde{M}) - 2 \tag{2.11}$$

respectively.

Now the assertion follows directly from the following lemma

LEMMA. Let M , $\tilde{M}$ be n-dimensional dompact connected manifolds with $\tilde{M} \subsetneqq M\backslash\partial M$ and assume that $\tilde{M}$ is contained in some coordinate neighborhood of M Then

$$b(M\backslash\tilde{M}) - b(M) = \tfrac{1}{2}b(\partial\tilde{M}) \quad \text{if} \quad \partial M \neq \phi \ ,$$

or

$$b(M\backslash\tilde{M}) - b(M) = \tfrac{1}{2}b(\partial\tilde{M}) - 2 \quad \text{if} \quad \partial M = \phi$$

PROOF. Let B be an open coordinate neighborhood in M such that $\overline{B}$ is topologically a closed n-ball . We can assume $\tilde{M} \subsetneqq B \subsetneqq \overline{B} \subsetneqq M\backslash\partial M$. To compute the Betti-numbers of $M\backslash\tilde{M}$ in terms of that of M and $\tilde{M}$ we apply the Mayer-Vietoris sequence to the following three decompostions

I. $$M = (M\backslash B) \cup \overline{B}$$
$$(M\backslash B) \cap \overline{B} = \partial\overline{B} \cong S^{n-1} \ ,$$

II. $$\overline{B} = (\overline{B}\backslash(\tilde{M}\backslash\partial\tilde{M})) \cup \tilde{M}$$
$$(\overline{B}\backslash(\tilde{M}\backslash\partial\tilde{M})) \cap \tilde{M} = \partial\tilde{M} \ ,$$

III. $$M\backslash(\tilde{M}\backslash\partial\tilde{M}) = (M\backslash B) \cup (\overline{B}\backslash(\tilde{M}\backslash\partial\tilde{M}))$$
$$(M\backslash B) \cap (\overline{B}\backslash(\tilde{M}\backslash\partial\tilde{M})) = \partial\overline{B} \cong S^{n-1} \ .$$

The first decomposition leads to

$$b(M) = b(M\backslash B) - 1 \quad \text{if } \partial M \neq \phi \tag{2.12}$$

$$b(M) = b(M\backslash B) + 1 \quad \text{if } \partial M = \phi \ , \tag{2.13}$$

the second one to

$$b(B\setminus\tilde{M}) + b(\tilde{M}) = b(\partial\tilde{M}) + 1 \tag{2.14}$$

the third one to

$$b(M\setminus\tilde{M}) = b(M\setminus B) + b(\overline{B}\setminus\tilde{M}) - 2 \tag{2.15}$$

At last we have the equation

$$b(\partial\tilde{M}) = 2\, b(\tilde{M}) \tag{2.16}$$

because by assumption $\tilde{M}$ can be embedded in $B \subsetneqq E^n$ (cf. [9] Prop. 5.1). Now the lemma follows directly from (2.12) - (2.16) .

PROOF of Corollary B 2 . Consider for example an embedding of $S^k \times S^{n-k}$ in E^{n+1} ($k \geqq 1$ arbitrary) as a tight hypersurface of rotation (like the standard-torus in E^3) and change this embedding a little bit such that there is an open region U contained in some hyperplane of E^{n+1}. Now define M by removing a small tight 'solid torus' of type $S^m \times B^{n-m}$ from U ($m \geqq 1$) . By Corollary B 1 M is (n-2)-tight but of course it is not tight. By suitable choice of the embedding of $S^k \times S^{n-k}$ we started from we can assume that U lies not in the boundary of the convex hull $\mathcal{H} M$. So we can obtain an example where ∂M lies not in the boundary of $\mathcal{H} M$.

<u>REMARK</u>. In the examples of corollary B 2 the boundary ∂M was always tightly embedded in E^{n+1}. The natural question whether there exist in higher dimensions (n-2)-tight immersions with non-tight boundary seems to be open. For $n = 2$ an example is due to L. Rodriguez.

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