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ON NILPOTENT FILIFORM LIE ALGEBRAS OF DIMENSION EIGHT

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The aim of this paper is to determine both the Zariski constructible set of characteristically nilpotent filiform Lie algebras g of dimension 8 and that of the set of nilpotent filiform Lie algebras whose group of automorphisms consists of unipotent automorphisms, in the variety of filiform Lie algebras of dimension 8 over C.

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1. Introduction. Characteristically nilpotent Lie algebras were defined by Dixmier and Lister in [4] and filiform Lie algebras by Vergne in [8]. A complete classification of nilpotent filiform Lie algebras of dimension 8 is available since 1988 in [1] due to Ancochéa-Bermúdez and Goze. Then, Echarte-Reula et al. [6], considering that a filiform Lie algebra g is characteristically nilpotent if and only if g is not a derived algebra, obtained a list of characteristically nilpotent filiform Lie algebras of dimension 8. Recently, Castro-Jiménez and Núñez-Valdés studied extensively in [2, 3] the cases of dimension 9 and 10 and gave the sets of the corresponding characteristically nilpotent Lie algebras as a finite union of Zariski locally closed subsets. In 1970, Dyer in [5] gave an example of a characteristically nilpotent Lie algebra of dimension 9 and showed that each automorphism of this Lie algebra is unipotent. Some years later, Favre in [7] reached the same result working on an example of a characteristically nilpotent Lie algebra of dimension 7.

In this paper, we study the Lie algebras of dimension 8. We first express the set of characteristically nilpotent filiform Lie algebras g as a finite union of locally closed subsets, then we prove that the set of nilpotent filiform Lie algebras g, whose group of automorphisms consists of unipotent automorphisms, is a Zariski constructible set in the variety of nilpotent filiform Lie algebras, and we express it as a finite union of locally closed subsets. Furthermore, we prove that the group of automorphisms $\mathrm{Aut}(g)$ of each one of the above characteristically nilpotent filiform Lie algebras consists of unipotent automorphisms except two, in each of which the set of their unipotent automorphisms forms a proper subgroup of the group $\mathrm{Aut}(g)$.

2. Preliminaries. Let g be a Lie algebra of dimension n over C of characteristic zero. If we consider the descending central sequence $C^1 g = g$,

$C^2g = [g,g],\dots$, and $C^qg = [g,C^{q-1}g],\dots$, of the above Lie algebra, then the Lie algebra g is called filiform if $\dim_C C^qg = n-q$ for $2 \le q \le n$ [8].

Let g be a filiform Lie algebra of dimension n. Then, there exists a basis $E = \{e_1,e_2,\dots,e_n\}$ of g such that $e_1 \in g \setminus C^2g$, the matrix of $\mathrm{ad}(e_1)$ with respect to E has a Jordan block of order $n-1$, and C^ig is the vector space generated by $\{e_2,e_3,\dots,e_{n-i+1}\}$ with $2 \le i \le n-1$. Such a basis is called an adapted basis.

Let W be a vector space over a field C of dimension n. A subset A of W is an algebraic one if there exists a set B of polynomial functions on W such that $A = \{x \in W / p(x) = 0$, for all $p \in B\}$. We consider the set $C[x]$ of all polynomials in n variables $x = \{x_1,\dots,x_n\}$ over C and I an ideal of $C[x]$. We denote by $\mathcal{V}(I)$ the set $\mathcal{V}(I) = \{a \in C^n / p(a) = 0$, for all $p \in I\}$. As a consequence of the above definitions, $\mathcal{V}(I)$ is an algebraic subset of the vector space C^n and so the Zariski topology on the space C^n is the one whose closed sets are $\mathcal{V}(I)$. Finally, we denote by $D(I)$ the complement of $\mathcal{V}(I)$ in C^n.

3. The equations. It has been proved in [1] that there exists a basis $\{e_1,e_2,\dots,e_8\}$ such that every nilpotent filiform Lie algebra g over a field C of characteristic zero of dimension 8 is isomorphic to one of the Lie algebras belonging to the nine-parameter family given in [1].

We now consider a change of the previous base of the nilpotent filiform Lie algebra g such that $Y_i = e_i$, $i = 1,2,\dots,7$, and $Y_8 = e_8 + a_1e_1$. So, the set of nilpotent filiform Lie algebras over C can be parametrized by the points $(a_2,a_3,\dots,a_9)$ of the algebraic set $V' \in C^8$, and the above-mentioned equations of the nine-parameter family, with respect to the new base $A = \{e_1,e_2,\dots,e_8\}$, takes the form

$$\begin{aligned}
&[e_1,e_i] = e_{i-1}, \quad i \ge 3,\\
&[e_4,e_7] = a_2e_2,\\
&[e_4,e_8] = a_2e_3 + a_3e_2,\\
&[e_5,e_6] = a_4e_2,\\
&[e_5,e_7] = (a_2+a_4)e_3 + a_5e_2,\\
&[e_5,e_8] = (2a_2+a_4)e_4 + (a_3+a_5)e_3 + a_6e_2,\\
&[e_6,e_7] = (a_2+a_4)e_4 + a_5e_3 + a_7e_2,\\
&[e_6,e_8] = (3a_2+2a_4)e_5 + (a_3+2a_5)e_4 + (a_6+a_7)e_3 + a_8e_2,\\
&[e_7,e_8] = (3a_2+2a_4)e_6 + (a_3+2a_5)e_5 + (a_6+a_7)e_4 + a_8e_3 + a_9e_2,
\end{aligned} \tag{3.1}$$

with $a_j \in C$, $j = 2,3,\dots,9$, verifying the equations

$$\begin{aligned}
a_2 + a_4 &= 0,\\
a_2(5a_5 + 2a_3) &= 0.
\end{aligned} \tag{3.2}$$

Those two equations are consequences of the Jacobi's identities.

4. Characteristically nilpotent filiform Lie algebras. Let g be a nilpotent Lie algebra of dimension n over C of characteristic zero. A Lie algebra g is said to be characteristically nilpotent if the Lie algebra of its derivations D is nilpotent. By $\mathrm{D}: g \to g$ verifying $\mathrm{D}[x,y] = [\mathrm{D}x,y]+[x,\mathrm{D}y]$ for all $(x,y) \in g$, we mean a derivation of g.

Let $\mathrm{D} = (d_{ij}) \in \mathrm{Mat}(8\times 8, C)$ be the set of matrices representing the derivations D of the filiform Lie algebras over C of dimension 8 with respect to the new base $A = \{e_1, e_2, \dots, e_8\}$.

Suppose that

$$\begin{gathered} \mathrm{D}e_k = \sum d_{k\lambda} e_\lambda, \quad 1 \le k,\ \lambda \le 8,\ d_{k\lambda} \in C, \\ \mathrm{D}[e_i, e_j] - \mathrm{D}e_k = 0, \quad 1 \le i < j \le 8,\ 1 \le k \le 8. \end{gathered} \tag{4.1}$$

From

$$\mathrm{D}[e_1, e_2] = 0, \qquad \mathrm{D}[e_1, e_i] = \mathrm{D}e_{i-1}, \quad i \ge 3, \tag{4.2}$$

we deduce that

$$d_{ij} = 0, \quad 2 \le i \le 7,\ i < j,\ 3 \le j \le 8, \qquad d_{i1} = 0, \quad 2 \le i \le 8. \tag{4.3}$$

For each (i,j,k), $1 \le i < j \le 8$, $1 \le k \le 8$, we denote by $b(i,j,k)$ the coefficient of e_k in the expression $\mathrm{D}[e_i,e_j] - [\mathrm{D}e_i, e_j] - [e_i, \mathrm{D}e_j]$ with respect to the base A. From above, we obtain a homogeneous linear system defined by

$$S = \{b(i,j,k) = 0,\ 1 \le i < j \le 8,\ 1 \le k \le 8\}. \tag{4.4}$$

The solutions satisfying system (4.4) are elements of the set of matrices $\mathrm{D} = (d_{ij}) \in \mathrm{Mat}(8 \times 8, C)$.

In case that D are nilpotent matrices, according to the previous definition, the filiform Lie algebra g is characteristically nilpotent.

4.1. The system of equations. Let $t = (a_2, a_3, a_5, a_6, a_7, a_8, a_9)$ be a point of $V \in C^7$, g_t the corresponding filiform Lie algebra of dimension 8, and S_t the homogeneous linear system corresponding to (4.4). We consider the linear system S_t as a system with coefficients in the quotient ring R/I where $R = C[a_2, a_3, a_5, a_6, a_7, a_8, a_9]$ and I is the ideal generated by (3.2). In that case, system S in (4.4) is reduced to the following equivalent system S_t:

$$\begin{aligned}
&d_{11}-d_{22}+d_{33}=0\\
&d_{11}-d_{77}+d_{88}=0\\
&a_2d_{17}-d_{76}+d_{87}=0\\
&a_2d_{22}-a_2d_{44}-a_2d_{77}=0\\
&a_2d_{22}-a_2d_{55}-a_2d_{66}=0\\
&a_2d_{33}-a_2d_{44}-a_2d_{88}=0\\
&a_5d_{33}-a_5d_{66}-a_5d_{77}=0\\
&a_2d_{44}-a_2d_{55}-a_2d_{88}=0\\
&a_2d_{55}-a_2d_{66}-a_2d_{88}=0\\
&a_2d_{66}-a_2d_{77}-a_2d_{88}=0\\
&d_{11}-a_2d_{18}-d_{33}+d_{44}=0\\
&d_{11}-a_2d_{18}-d_{44}+d_{55}=0\\
&d_{11}-a_2d_{18}-d_{55}+d_{66}=0\\
&d_{11}-a_2d_{18}-d_{66}+d_{77}=0\\
&(a_3+a_5)d_{18}+d_{43}-d_{54}=0\\
&(a_6+a_7)d_{18}+d_{64}-d_{75}=0\\
&(a_3+2a_5)d_{18}+d_{54}-d_{65}=0\\
&(a_3+2a_5)d_{18}+d_{65}-d_{76}=0\\
&a_5d_{16}-a_8d_{18}-d_{63}+d_{74}=0\\
&a_2d_{17}+a_3d_{18}+d_{32}-d_{43}=0\\
&a_5d_{17}+(a_6+a_7)d_{18}+d_{53}-d_{64}=0\\
&a_2d_{15}+a_7d_{17}+a_8d_{18}+d_{52}-d_{63}=0\\
&a_2d_{16}-a_5d_{17}-a_6d_{18}-d_{42}+d_{53}=0\\
&a_2d_{16}+(a_3+2a_5)d_{17}-d_{75}+d_{86}=0\\
&a_3d_{22}+a_2d_{32}-a_3d_{44}-a_2d_{87}-a_3d_{88}=0\\
&a_5d_{22}-a_2d_{54}+a_5d_{55}+a_2d_{76}+a_5d_{77}=0\\
&a_2d_{14}+a_5d_{15}+a_7d_{16}-a_9d_{18}-d_{62}+d_{73}=0\\
&a_3d_{14}+a_6d_{15}+a_8d_{16}+a_9d_{17}-d_{72}+d_{83}=0\\
&(a_3+a_5)d_{15}+(a_6+a_7)d_{16}+a_8d_{17}-d_{73}+d_{84}=0\\
&a_2d_{15}+(a_3+2a_5)d_{16}+(a_6+a_7)d_{17}-d_{74}+d_{85}=0
\end{aligned}$$

$$
\begin{aligned}
&a_7d_{22}+a_5d_{32}-a_2d_{64}-a_5d_{65}-a_7d_{66}-a_2d_{75}-a_7d_{77}=0\\
&(a_3+a_5)d_{33}+a_2d_{43}-a_2d_{54}-(a_3+a_5)d_{55}-(a_3+a_5)d_{88}=0\\
&(a_3+2a_5)d_{44}+a_2d_{54}-a_2d_{65}-(a_3+2a_5)d_{66}-(a_3+2a_5)d_{88}=0\\
&(a_3+2a_5)d_{55}+a_2d_{65}-a_2d_{76}-(a_3+2a_5)d_{77}-(a_3+2a_5)d_{88}=0\\
&a_6d_{22}+(a_3+a_5)d_{32}+a_2d_{42}-a_3d_{54}-a_6d_{55}+a_2d_{86}-a_5d_{87}-a_6d_{88}=0\\
&(a_6+a_7)d_{33}+(a_3+2a_5)d_{43}+a_2d_{53}-a_2d_{64}-(a_3+a_5)d_{65}\\
&\qquad-(a_6+a_7)d_{66}-a_5d_{87}-(a_6+a_7)d_{88}=0\\
&(a_6+a_7)d_{44}+(a_3+2a_5)d_{54}+a_2d_{64}-a_2d_{75}-(a_3+2a_5)d_{76}\\
&\qquad-(a_6+a_7)d_{77}-(a_6+a_7)d_{88}=0\\
&a_8d_{22}+(a_6+a_7)d_{32}+(a_3+2a_5)d_{42}+a_2d_{52}-a_3d_{64}-a_6d_{65}\\
&\qquad-a_8d_{66}-a_2d_{85}-a_7d_{87}-a_8d_{88}=0\\
&a_8d_{33}+(a_6+a_7)d_{43}+(a_3+2a_5)d_{53}+a_2d_{63}-a_2d_{74}-(a_3+a_5)d_{75}\\
&\qquad-(a_6+a_7)d_{76}-a_8d_{77}+a_5d_{86}-a_8d_{88}=0\\
&a_9d_{22}+a_8d_{32}+(a_6+a_7)d_{42}+(a_3+2a_5)d_{52}+a_2d_{62}-a_3d_{74}-a_6d_{75}\\
&\qquad-a_8d_{76}-a_9d_{77}+a_2d_{84}+a_5d_{85}+a_7d_{86}-a_9d_{88}=0.
\end{aligned}
\tag{4.5}
$$

The solutions satisfying S_t are derivations of the nilpotent filiform Lie algebra g_t. If all the derivations of g_t are nilpotent, then g_t is characteristically nilpotent.

We will prove that the set of points $t\in V\subset C^7$, such that there exists a solution of S_t satisfying the conditions of g_t being a characteristically nilpotent filiform Lie algebra, is a Zariski constructible set, and we will express it as a finite union of Zariski locally closed subsets. To realize the above idea, we study S_t in suitable subsets of V.

4.2. Main results. We consider two cases: first, $a_2\neq 0$ and then, $a_2=0$.

4.2.1. $a_2\neq 0$**.** Let the open set $V\cap D(a_2)$. Because of the equation $a_2(5a_5+2a_3)=0$, we can distinguish the following two subcases.

(1) ($a_3\neq 0$). First, we consider the set $T^{(1)}=V\cap D(a_2\cdot a_3)$. From $5a_5+2a_3=0$, we obtain $a_5=-(2/5)a_3$. By doing the necessary calculations in system S_t, we prove that, in the set of points $T^{(1)}\cap D(Q_1)$ with $Q_1=2a_3^2-25a_2a_6-25a_2a_7$, the corresponding Lie algebra is characteristically nilpotent.

(2) ($a_3=0$). Now, we consider the set $T^{(2)}=V\cap D(a_2)\cap\mathcal{V}(a_3)$. From $5a_5+2a_3=0$, we obtain $a_5=0$. In case that $a_6+a_7\neq 0$ and $a_8\neq 0$, that means in $T^{(2)}\cap D((a_6+a_7)\cdot a_8)$, only one Lie algebra is characteristically nilpotent.

From the above, we can state the following theorem.

Theorem 4.1. *Consider the set of complex filiform Lie algebras. Consider C^8 with $(a_2,a_3,\dots,a_9)$ as coordinates given by (3.1) and let V be the hypersurface defined in C^7 by (3.2). In the Zariski open set $V\cap D(a_2)$, the Zariski constructible subset of characteristically nilpotent Lie algebras is defined as the union of the following subsets:*

$$\begin{gathered} D(a_3\cdot(2a_3^2-25a_2a_6-25a_2a_7)),\\ \mathcal{V}(a_3)\cap D((a_6+a_7)\cdot a_8). \end{gathered} \tag{4.6}$$

4.2.2. $a_2=0$. We consider the set $T^{(3)}=V\cap\mathcal{V}(a_2)$. Because of the equation $a_2(5a_5+2a_3)=0$, we can distinguish the following subcases.

(1) ($a_5\neq 0$). So, we obtain the set $T^{(3)}\cap D(a_5)$ and we distinguish the following:

(1A) ($a_3+2a_5\neq 0$). In the subset $T^{(3)}\cap D(a_5\cdot(a_3+2a_5)\cdot Q_2)$ with $Q_2=2a_3^2a_7-3a_3a_5a_6+5a_3a_5a_7-3a_5^2a_6+5a_5^2a_7$, the corresponding Lie algebra is characteristically nilpotent,

(1B) ($a_3+2a_5=0$). The corresponding Lie algebra in the set of points $T^{(3)}\cap D(a_5\cdot(a_6+a_7))\cap\mathcal{V}(a_3+2a_5)$ is characteristically nilpotent.

(2) ($a_5=0$). First, we distinguish two subcases $a_3\neq 0$ and $a_3=0$.

(2A) ($a_3\neq 0$). Then, we consider the set $T^{(3)}\cap\mathcal{V}(a_5)\cap D(a_3)$. By doing some calculations, we distinguish two more subcases.

(i) ($a_7\neq 0$). In this case, the Lie algebra corresponding to the set of points $T^{(3)}\cap\mathcal{V}(a_5)\cap D(a_3\cdot a_7)$ is characteristically nilpotent.

(ii) ($a_7=0$). Now, we study S_t in the set of points $Z=T^{(3)}\cap\mathcal{V}(a_5,a_7)\cap D(a_3)$. The Lie algebras corresponding to the set of points $Z\cap(D(Q_{31})\cup D(Q_{32}))$, with $Q_{31}=4a_3a_8-5a_6^2$ and $Q_{32}=2a_3^2a_9-2a_3a_6a_8-a_6^3$, are characteristically nilpotent.

(2B) ($a_3=0$). We operate in $T^{(3)}\cap\mathcal{V}(a_3,a_5)$ and we distinguish the cases $a_7\neq 0$ and $a_7=0$.

(i) ($a_7\neq 0$). The Lie algebra corresponding to the set of points $T^{(3)}\cap\mathcal{V}(a_3,a_5)\cap D(a_7\cdot a_8)$ is characteristically nilpotent.

(ii) ($a_7=0$). We now consider the subset $T^{(3)}\cap\mathcal{V}(a_3,a_5,a_7)$. We distinguish another two subcases, $a_6\neq 0$ and $a_6=0$.

(iiA) ($a_6\neq 0$). The Lie algebra corresponding to $T^{(3)}\cap\mathcal{V}(a_3,a_5,a_7)\cap D(a_6\cdot a_8)$ is characteristically nilpotent.

(iiB) ($a_6=0$). The Lie algebra in the set $T^{(3)}\cap\mathcal{V}(a_3,a_5,a_6,a_7)\cap D(a_8\cdot a_9)$ is characteristically nilpotent.

So, we have proved the following theorem.

Theorem 4.2. *Consider the set of complex filiform Lie algebras. Consider C^8 with $(a_2,a_3,\dots,a_9)$ as coordinates given by (3.1), and let V be the hypersurface defined in C^7 by (3.2). The Zariski constructible subset of characteristically nilpotent Lie algebras in the Zariski closed set $V\cap\mathcal{V}(a_2)$ is defined as the union* *of the following subsets:*

$$\begin{cases} D(a_5)\cap\begin{cases} D((a_3+2a_5)\cdot(2a_3^2a_7-3a_3a_5a_6+5a_3a_5a_7-3a_5^2a_6+5a_5^2a_7)), \\ \mathcal{V}(a_3+2a_5)\cap D(a_6+a_7), \end{cases} \\ \mathcal{V}(a_5)\cap\begin{cases} D(a_3)\cap\begin{cases} D(a_7), \\ \mathcal{V}(a_7)\cap(D(4a_3a_8-5a_6^2)\cup D(2a_3^2a_9-2a_3a_6a_8-a_6^3)), \end{cases} \\ \mathcal{V}(a_3)\cap\begin{cases} D(a_7\cdot a_8), \\ \mathcal{V}(a_7)\cap\begin{cases} D(a_6\cdot a_8), \\ \mathcal{V}(a_6)\cap D(a_8\cdot a_9). \end{cases} \end{cases} \end{cases} \end{cases} \tag{4.7}$$

By ($\{$), we mean the union of the corresponding sets.

5. Unipotent automorphisms of nilpotent filiform Lie algebras. Let g be a nilpotent Lie algebra of dimension n over C of characteristic zero. The automorphism θ of a Lie algebra g over C is defined by the mapping $[x,y]\to\theta([x,y])=[\theta(x),\theta(y)]$ for all $(x,y)\in g$. An automorphism θ is called unipotent if its representation, with respect to the base $\{e_1,e_2,\dots,e_n\}$, has the form

$$B=(b_{ij})\in\operatorname{Mat}(n\times n,C),\quad b_{ij}=0,\ j<i,\ b_{ii}=1,\ 1\le i,j\le n. \tag{5.1}$$

Let $B=(b_{ij})\in\operatorname{Mat}(8\times 8,C)$ be the set of matrices representing the automorphisms θ of the filiform Lie algebras over C of dimension 8 with respect to the new base $A=\{e_1,e_2,\dots,e_8\}$.

Suppose that

$$\begin{gathered}\theta(e_k)=\sum b_{k\lambda}e_\lambda,\quad 1\le k,\ \lambda\le 8,\ b_{k\lambda}\in C,\\ \theta([e_i,e_j])-\theta(e_k)=0,\quad 1\le i<j\le 8,\ 1\le k\le 8.\end{gathered} \tag{5.2}$$

From

$$\theta([e_1,e_2])=0,\qquad \theta([e_1,e_i])=\theta(e_{i-1}),\quad i\ge 3,\qquad \theta([e_3,e_8])=0, \tag{5.3}$$

we deduce that

$$b_{ij}=0,\quad 3\le i\le 8,\ j<i,\ 2\le j\le 7,\qquad b_{1j}=0,\quad 2\le j\le 8. \tag{5.4}$$

For each (i,j,k), $1\le i<j\le 8$, $1\le k\le 8$, we denote by $c(i,j,k)$ the coefficient of e_k in the expression $\theta([e_i,e_j])-[\theta(e_i),\theta(e_j)]$ with respect to the base A.

From the above, we obtain a homogeneous system defined by

$$S'=\{c(i,j,k)=0,\ 1\le i<j\le 8,\ 1\le k\le 8\}. \tag{5.5}$$

The solutions satisfying system (5.5) are elements of the set of matrices $B = (b_{ij}) \in \mathrm{Mat}(8\times 8, C)$.

In case that $B = (b_{ij}) \in \mathrm{Mat}(8\times 8, C)$ are matrices of the form (5.1), according to the above definition, the group of automorphisms $\mathrm{Aut}(g)$ of the corresponding filiform Lie algebra g consists of unipotent automorphisms.

5.1. The system of equations. Let $t = (a_2, a_3, a_5, a_6, a_7, a_8, a_9)$ be a point of $V \in C^7$, g_t the corresponding filiform Lie algebra of dimension 8, and S'_t the homogeneous system corresponding to (5.5). We will consider the linear system S'_t as a system with coefficients in the quotient ring R/I where $R = C[a_2, a_3, a_5, a_6, a_7, a_8, a_9]$ and I is the ideal generated by (3.2). In that case, system S' in (5.5) is reduced to the following equivalent system S'_t:

$$b_{22} - b_{11}b_{33} = 0$$
$$b_{77} - b_{11}b_{88} = 0$$
$$a_2b_{22} - a_2b_{44}b_{77} = 0$$
$$a_2b_{22} - a_2b_{55}b_{66} = 0$$
$$a_2b_{33} - a_2b_{44}b_{88} = 0$$
$$a_2b_{44} - a_2b_{55}b_{88} = 0$$
$$a_2b_{55} - a_2b_{66}b_{88} = 0$$
$$a_2b_{66} - a_2b_{77}b_{88} = 0$$
$$a_5b_{33} - a_5b_{66}b_{77} = 0$$
$$b_{33} - b_{11}b_{44} + a_2b_{44}b_{81} = 0$$
$$b_{44} - b_{11}b_{55} + a_2b_{55}b_{81} = 0$$
$$b_{55} - b_{11}b_{66} + a_2b_{66}b_{81} = 0$$
$$b_{66} - b_{11}b_{77} + a_2b_{77}b_{81} = 0$$
$$b_{23} - b_{11}b_{34} + a_2b_{44}b_{71} + a_3b_{44}b_{81} = 0$$
$$b_{67} - b_{11}b_{78} - a_2b_{71}b_{88} + a_2b_{78}b_{81} = 0$$
$$a_3b_{22} + a_2b_{23} - a_2b_{44}b_{78} - a_3b_{44}b_{88} = 0$$
$$a_5b_{22} - a_2b_{45}b_{77} + a_2b_{55}b_{67} - a_5b_{55}b_{77} = 0$$
$$b_{34} - b_{11}b_{45} + a_2b_{45}b_{81} + (a_3 + a_5)b_{55}b_{81} = 0$$
$$b_{45} - b_{11}b_{56} + a_2b_{56}b_{81} + (a_3 + 2a_5)b_{66}b_{81} = 0$$
$$b_{56} - b_{11}b_{67} + a_2b_{67}b_{81} + (a_3 + 2a_5)b_{77}b_{81} = 0$$
$$(a_3 + a_5)b_{33} + a_2b_{34} - a_2b_{45}b_{88} - (a_3 + a_5)b_{55}b_{88} = 0$$
$$(a_3 + 2a_5)b_{44} + a_2b_{45} - a_2b_{56}b_{88} - (a_3 + 2a_5)b_{66}b_{88} = 0$$
$$(a_3 + 2a_5)b_{55} + a_2b_{56} - a_2b_{67}b_{88} - (a_3 + 2a_5)b_{77}b_{88} = 0$$
$$b_{46} - b_{11}b_{57} + a_2b_{57}b_{81} + (a_3 + 2a_5)b_{67}b_{81} + (a_6 + a_7)b_{77}b_{81} = 0$$
$$b_{24} - b_{11}b_{35} + a_2b_{45}b_{71} + a_3b_{45}b_{81} - a_2b_{55}b_{61} + a_5b_{55}b_{71} + a_6b_{55}b_{81} = 0$$

$$
\begin{aligned}
&a_7b_{22}+a_5b_{23}-a_2b_{46}b_{77}+a_2b_{56}b_{67}-a_5b_{56}b_{77}-a_2b_{57}b_{66}-a_7b_{66}b_{77}=0\\
&b_{35}-b_{11}b_{46}+a_2b_{46}b_{81}+(a_3+a_5)b_{56}b_{81}+a_5b_{66}b_{71}+(a_6+a_7)b_{66}b_{81}=0\\
&b_{57}-b_{11}b_{68}-a_2b_{61}b_{88}+a_2b_{68}b_{81}-(a_3+2a_5)b_{71}b_{88}+(a_3+2a_5)b_{78}b_{81}=0\\
&(a_6+a_7)b_{44}+(a_3+2a_5)b_{45}+a_2b_{46}-a_2b_{57}b_{88}-(a_3+2a_5)b_{67}b_{88}\\
&\qquad-(a_6+a_7)b_{77}b_{88}=0\\
&b_{36}-b_{11}b_{47}+a_2b_{47}b_{81}+(a_3+a_5)b_{57}b_{81}-a_5b_{61}b_{77}+a_5b_{67}b_{71}\\
&\qquad+(a_6+a_7)b_{67}b_{81}+a_8b_{77}b_{81}=0\\
&a_6b_{22}+(a_3+2a_5)b_{23}+a_2b_{24}-a_2b_{45}b_{78}-a_3b_{45}b_{88}+a_2b_{55}b_{68}\\
&\qquad-a_5b_{55}b_{78}-a_6b_{55}b_{88}=0\\
&(a_6+a_7)b_{33}+(a_3+2a_5)b_{34}+a_2b_{35}-a_2b_{46}b_{88}-(a_3+a_5)b_{56}b_{88}\\
&\qquad-a_5b_{66}b_{78}-(a_6+a_7)b_{66}b_{88}=0\\
&b_{25}-b_{11}b_{36}+a_2b_{46}b_{71}+a_3b_{46}b_{81}+a_2b_{51}b_{66}-a_2b_{56}b_{61}+a_5b_{56}b_{71}\\
&\qquad+a_6b_{56}b_{81}+a_7b_{66}b_{71}+a_8b_{66}b_{81}=0\\
&b_{47}-b_{11}b_{58}-a_2b_{51}b_{88}+a_2b_{58}b_{81}-(a_3+2a_5)b_{61}b_{88}+(a_3+2a_5)b_{68}b_{81}\\
&\qquad-(a_6+a_7)b_{71}b_{88}+(a_6+a_7)b_{78}b_{81}=0\\
&a_8b_{33}+(a_6+a_7)b_{34}+(a_3+2a_5)b_{35}+a_2b_{36}-a_2b_{47}b_{88}-(a_3+a_5)b_{57}b_{88}\\
&\qquad-a_5b_{67}b_{78}-(a_6+a_7)b_{67}b_{88}+a_5b_{68}b_{77}-a_8b_{77}b_{88}=0\\
&a_8b_{22}+(a_6+a_7)b_{23}+(a_3+2a_5)b_{24}+a_2b_{25}-a_2b_{46}b_{78}-a_3b_{46}b_{88}\\
&\qquad+a_2b_{56}b_{68}-a_5b_{56}b_{78}-a_6b_{56}b_{88}-a_2b_{58}b_{66}\\
&\qquad-a_7b_{66}b_{78}-a_8b_{66}b_{88}=0\\
&b_{26}-b_{11}b_{37}-a_2b_{41}b_{77}+a_2b_{47}b_{71}+a_3b_{47}b_{81}+a_2b_{51}b_{67}-a_5b_{51}b_{77}\\
&\qquad-a_2b_{57}b_{61}+a_5b_{57}b_{71}+a_6b_{57}b_{81}-a_7b_{61}b_{77}+a_7b_{67}b_{71}\\
&\qquad+a_8b_{67}b_{81}+a_9b_{77}b_{81}=0\\
&b_{37}-b_{11}b_{48}-a_2b_{41}b_{88}+a_2b_{48}b_{81}-(a_3+a_5)b_{51}b_{88}+(a_3+a_5)b_{58}b_{81}\\
&\qquad-a_5b_{61}b_{78}-(a_6+a_7)b_{61}b_{88}+a_5b_{68}b_{71}+(a_6+a_7)b_{68}b_{81}\\
&\qquad-a_8b_{71}b_{88}+a_8b_{78}b_{81}=0\\
&a_9b_{22}+a_8b_{23}+(a_6+a_7)b_{24}+(a_3+2a_5)b_{25}+a_2b_{26}-a_2b_{47}b_{78}-a_3b_{47}b_{88}\\
&\qquad+a_2b_{48}b_{77}+a_2b_{57}b_{68}-a_5b_{57}b_{78}-a_6b_{57}b_{88}-a_2b_{58}b_{67}+a_5b_{58}b_{77}\\
&\qquad-a_7b_{67}b_{78}-a_8b_{67}b_{88}+a_7b_{68}b_{77}-a_9b_{77}b_{88}=0\\
&b_{27}-b_{11}b_{38}-a_2b_{41}b_{78}-a_3b_{41}b_{88}+a_2b_{48}b_{71}+a_3b_{48}b_{81}+a_2b_{51}b_{68}\\
&\qquad-a_5b_{51}b_{78}-a_6b_{51}b_{88}-a_2b_{58}b_{61}+a_5b_{58}b_{71}+a_6b_{58}b_{81}\\
&\qquad-a_7b_{61}b_{78}-a_8b_{61}b_{88}+a_7b_{68}b_{71}+a_8b_{68}b_{81}\\
&\qquad-a_9b_{71}b_{88}+a_9b_{78}b_{81}=0.
\end{aligned}
\tag{5.6}
$$

The solutions satisfying S_t' (5.6) are elements of the set of matrices $B = (b_{ij}) \in \mathrm{Mat}(8\times 8, C)$. In case that B are matrices of the form

$$B = (b_{ij}) \in \mathrm{Mat}(8\times 8, C), \quad b_{ij} = 0,\ j < i,\ b_{ii} = 1, \tag{5.7}$$

the group of automorphisms $\mathrm{Aut}(g_t)$ of g_t consists of unipotent automorphisms.

We will prove that the set of points $t \in V \subset C^7$, such that there exists a solution $B = (b_{ij}) \in \mathrm{Mat}(8\times 8, C)$ of S_t' satisfying the conditions (5.7), is a Zariski constructible set and we will express it as a finite union of Zariski locally closed subsets.

To realize the above idea, we study S_t' in suitable subsets of V.

5.2. Main results. We consider two cases: first, $a_2 \neq 0$ and then, $a_2 = 0$.

5.2.1. $a_2 \neq 0$**.** Let the open set $V \cap D(a_2)$. Because of the equation $a_2(5a_5 + 2a_3) = 0$, we can distinguish the following two subcases.

(1) ($a_3 \neq 0$). First, we consider the set $T^{(1)} = V \cap D(a_2 \cdot a_3)$. From $5a_5 + 2a_3 = 0$, we obtain $a_5 = -(2/5)a_3$. By doing the necessary calculations in S_t', we can deduce

$$b_{11} = b_{88}^2, \qquad b_{22} = b_{88}^9, \qquad b_{ii} = b_{88}^{10-i}, \quad i = 3,\dots,7, \qquad Q_4(b_{88} - 1) = 0 \tag{5.8}$$

with $Q_4 = 2a_3^2 - 25a_2a_6 - 25a_2a_7$.

So, the set of points in $T^{(1)}$ in which the group $\mathrm{Aut}(g_t)$ of the corresponding Lie algebra consists of unipotent automorphisms is $T^{(1)} \cap D(Q_4)$.

(2) ($a_3 = 0$). Now, we consider the set $T^{(2)} = V \cap D(a_2) \cap \mathcal{V}(a_3)$. From $5a_5 + 2a_3 = 0$, we obtain $a_5 = 0$. By doing some calculations as above, in case that $a_6 + a_7 \neq 0$, we deduce

$$b_{11} = b_{88}^3, \qquad b_{22} = b_{88}^{11}, \qquad b_{ii} = b_{88}^{11-i}, \quad i = 3,\dots,7, \qquad a_8(b_{88} - 1) = 0. \tag{5.9}$$

So, in the set $T^{(2)} \cap D((a_6 + a_7) \cdot a_8)$, the group $\mathrm{Aut}(g_t)$ of only one Lie algebra consists of unipotent automorphisms. On the other hand, the group $\mathrm{Aut}(g_t)$ of each of the corresponding Lie algebras in the set $T^{(2)} \cap D(a_6 + a_7) \cap \mathcal{V}(a_8)$ and $T^{(2)} \cap \mathcal{V}(a_6 + a_7)$ do not contain unipotent automorphisms.

From the above, we can state the following theorem.

Theorem 5.1. *Consider the set of complex filiform Lie algebras. Consider C^8 with $(a_2, a_3, \dots, a_9)$ as coordinates given by (3.1) and let V be the hypersurface defined in C^7 by (3.2). In the Zariski open set $V \cap D(a_2)$, the Zariski constructible* *subset of filiform Lie algebras whose group of automorphisms consists of unipotent automorphisms is defined as the union of the following subsets:*

$$\begin{gathered} D(a_3\cdot(2a_3^2-25a_2a_6-25a_2a_7)),\\ \mathcal{V}(a_3)\cap D((a_6+a_7)\cdot a_8). \end{gathered} \tag{5.10}$$

5.2.2. $a_2 = 0$**.** We consider the set $T^{(3)} = V\cap\mathcal{V}(a_2)$. Because of the equation $a_2(5a_5+2a_3) = 0$, we can distinguish the following subcases.

(1) ($a_5 \neq 0$). So, we obtain the set $T^{(3)}\cap D(a_5)$ and we distinguish the following:

(1A) ($a_3+2a_5 \neq 0$). In the subset $T^{(3)}\cap D(a_5\cdot(a_3+2a_5))$, after the necessary calculations in system (5.6), we deduce

$$b_{ii} = b_{11}^{10-i},\quad i = 2,\dots,8,\qquad Q_5(b_{11}-1) = 0 \tag{5.11}$$

with $Q_5 = 2a_3^2a_7-3a_3a_5a_6+5a_3a_5a_7-3a_5^2a_6+5a_5^2a_7$.

So, in the set of points $T^{(3)}\cap D(a_5\cdot(a_3+2a_5)\cdot Q_5)$, the group $\mathrm{Aut}(g_t)$ of the corresponding Lie algebra consists of unipotent automorphisms, whereas, in the set $T^{(3)}\cap D(a_5\cdot(a_3+2a_5))\cap\mathcal{V}(Q_5)$, the group $\mathrm{Aut}(g_t)$ of the corresponding Lie algebra does not contain unipotent automorphisms.

(1B) ($a_3+2a_5 = 0$). We study S_t' in $T^{(3)}\cap D(a_5)\cap\mathcal{V}(a_3+2a_5)$ and we obtain

$$b_{ii} = b_{11}^{10-i},\quad i = 2,\dots,8,\qquad (a_6+a_7)(b_{11}-1) = 0. \tag{5.12}$$

So, the group $\mathrm{Aut}(g_t)$ of the corresponding Lie algebra in the set of points $T^{(3)}\cap D(a_5\cdot(a_6+a_7))\cap\mathcal{V}(a_3+2a_5)$ consists of unipotent automorphisms, but the group $\mathrm{Aut}(g_t)$ of the Lie algebra corresponding to the set $T^{(3)}\cap D(a_5)\cap\mathcal{V}(a_3+2a_5,a_6+a_7)$ does not contain unipotent automorphisms.

(2) ($a_5 = 0$). First, we distinguish two subcases $a_3 \neq 0$ and $a_3 = 0$.

(2A) ($a_3 \neq 0$). Then, we consider the set $T^{(3)}\cap\mathcal{V}(a_5)\cap D(a_3)$. By doing some calculations in S_t', we obtain

$$b_{ii} = b_{11}^{10-i},\quad i = 2,\dots,8,\qquad a_7(b_{11}-1) = 0. \tag{5.13}$$

In this case, the group $\mathrm{Aut}(g_t)$ of the Lie algebra corresponding to the set of points $T^{(3)}\cap\mathcal{V}(a_5)\cap D(a_3\cdot a_7)$ consists of unipotent automorphisms.

(2B) ($a_3 = 0$). In $T^{(3)}\cap\mathcal{V}(a_3,a_5)$, we distinguish the cases $a_7 \neq 0$ and $a_7 = 0$.

(i) ($a_7 \neq 0$). In the subset $T^{(3)}\cap\mathcal{V}(a_3,a_5)\cap D(a_7)$, after the necessary calculations in system S_t', we deduce

$$b_{ii} = b_{11}^{11-i},\quad i = 2,\dots,8,\qquad a_8(b_{11}-1) = 0. \tag{5.14}$$

So, the group $\mathrm{Aut}(g_t)$ of the Lie algebra corresponding to the set of points $T^{(3)}\cap\mathcal{V}(a_3,a_5)\cap D(a_7\cdot a_8)$ consists of unipotent automorphisms.

(ii) ($a_7 = 0$). We now consider the subset $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_7)$. We distinguish another two subcases $a_6 \neq 0$ and $a_6 = 0$.

(iiA) ($a_6 \neq 0$). So, we obtain the set $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_7) \cap D(a_6)$. By doing some calculations in S'_t, we deduce

$$b_{ii} = b_{11}^{11-i}, \quad i = 2, \dots, 8, \qquad a_8(b_{11} - 1) = 0. \tag{5.15}$$

From the above, we conclude that the group $\mathrm{Aut}(g_t)$ of the Lie algebra corresponding to $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_7) \cap D(a_6 \cdot a_8)$ consists of unipotent automorphisms, but the groups $\mathrm{Aut}(g_t)$ of those that correspond to $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_7, a_8) \cap D(a_6)$ do not contain unipotent automorphisms.

(iiB) ($a_6 = 0$). The set of points in which we are acting now is $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_6, a_7)$. In case that $a_8 \neq 0$ and by using similar techniques as we did previously in system S'_t, we deduce

$$b_{ii} = b_{11}^{12-i}, \quad i = 2, \dots, 8, \qquad a_9(b_{11} - 1) = 0. \tag{5.16}$$

Hence, the group $\mathrm{Aut}(g_t)$ of the Lie algebra in the set $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_6, a_7) \cap D(a_8 \cdot a_9)$ consists of unipotent automorphisms. On the other hand, the groups $\mathrm{Aut}(g_t)$ of each of the algebras in the subsets $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_6, a_7, a_9) \cap D(a_8)$, $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_6, a_7, a_8) \cap D(a_9)$, and $T^{(3)} \cap \mathcal{V}(a_3, a_5, a_6, a_7, a_8, a_9)$ do not contain unipotent automorphisms.

So, we have proved the following theorem.

Theorem 5.2. *Consider the set of complex filiform Lie algebras. Consider C^8 with $(a_2, a_3, \dots, a_9)$ as coordinates given by (3.1) and let V be the hypersurface defined in C^7 by (3.2). The Zariski constructible subset of filiform Lie algebras whose group of automorphisms consists of unipotent automorphisms in the Zariski closed set $V \cap \mathcal{V}(a_2)$ is defined as the union of the following subsets:*

$$\begin{cases} D(a_5) \cap \begin{cases} D((a_3 + 2a_5) \cdot (2a_3^2 a_7 - 3a_3 a_5 a_6 + 5a_3 a_5 a_7 - 3a_5^2 a_6 + 5a_5^2 a_7)), \\ \mathcal{V}(a_3 + 2a_5) \cap D(a_6 + a_7), \end{cases} \\ \mathcal{V}(a_5) \cap \begin{cases} D(a_3 \cdot a_7), \\ \mathcal{V}(a_3) \cap \begin{cases} D(a_7 \cdot a_8), \\ \mathcal{V}(a_7) \cap \begin{cases} D(a_6 \cdot a_8), \\ \mathcal{V}(a_6) \cap D(a_8 \cdot a_9). \end{cases} \end{cases} \end{cases} \end{cases} \tag{5.17}$$

By ($\{$), we mean the union of the corresponding sets.

Let $g_1 = \mu_8^{10,a}$ and $g_2 = \mu_8^{11}$ be the following Lie algebras belonging to the family (3.1) as they were defined in [1]:

$$\begin{aligned} &[e_1,e_i] = e_{i-1}, \quad i \ge 3, && [e_4,e_8] = e_2, \\ &[e_5,e_8] = e_3, && [e_6,e_8] = e_2+e_4, \\ &[e_7,e_8] = ae_2+e_3+e_5, \quad a \in C, && \end{aligned} \tag{5.18}$$

$$\begin{aligned} &[e_1,e_i] = e_{i-1}, \quad i \ge 3, \\ &[e_i,e_8] = e_{i-2}, \quad 4 \le i \le 6, \qquad [e_7,e_8] = e_2+e_5. \end{aligned} \tag{5.19}$$

Now, we can state the following theorems.

Theorem 5.3. *Consider the set of complex filiform Lie algebras. If $C^8 = \{(a_2,\dots,a_9)/a_i \in C$ and a_i satisfy (3.1)$\}$, we define the hypersurface V in C^7 by (3.2). The group $\mathrm{Aut}(g_1)$ of g_1 corresponding to the set $V \cap \mathcal{V}(a_2,a_5,a_7) \cap D(a_3 \cdot (4a_3a_8 - 5a_6^2))$ consists of automorphisms of the type*

$$\begin{aligned} &L = (l_{ij}), \quad l_{ij} = 0,\ j < i,\ l_{ii} = 1,\ 1 \le i,j \le 8. \\ &L = (l_{ij}), \quad l_{ij} = 0,\ j < i,\ l_{ii} = \begin{cases} 1, & i \text{ is even}, \\ -1, & i \text{ is odd}, \end{cases} \end{aligned} \tag{5.20}$$

where $1 \le i,j \le 8$.

So, the set of unipotent automorphisms form a proper subgroup of the group $\mathrm{Aut}(g_1)$.

Proof. If we consider the set of points $V \cap \mathcal{V}(a_2,a_5,a_7) \cap D(a_3)$ after the necessary calculations in system S_t', we deduce

$$b_{ii} = b_{11}^{10-i}, \quad i = 2,\dots,8, \qquad (4a_3a_8 - 5a_6^2)(b_{11}^2 - 1) = 0. \tag{5.21}$$

Obviously, the group $\mathrm{Aut}(g_1)$ of the Lie algebras corresponding to the set of points $V \cap \mathcal{V}(a_2,a_5,a_7) \cap D(a_3 \cdot (4a_3a_8 - 5a_6^2))$ does not contain only unipotent automorphisms. So, the group $\mathrm{Aut}(g_1)$, except the unipotent, contains automorphisms of the following type:

$$L = (l_{ij}), \quad l_{ij} = 0,\ j < i,\ l_{ii} = \begin{cases} 1, & i \text{ is even}, \\ -1, & i \text{ is odd}, \end{cases} \tag{5.22}$$

□

where $1 \le i,j \le 8$.

In case that $4a_3a_8-5a_6^2=0$, by acting as we previously did in system S_t', we obtain

$$b_{ii}=b_{11}^{10-i},\quad i=2,\dots,8,\qquad (4a_3^2a_9-7a_6^3)(b_{11}^3-1)=0. \tag{5.23}$$

Thus, the group $\mathrm{Aut}(g_2)$ of the Lie algebra corresponding to the set of points $V\cap\mathcal{V}(a_2,a_5,a_7,4a_3a_8-5a_6^2)\cap D(a_3\cdot(4a_3^2a_9-7a_6^3))$ does not contain only unipotent automorphisms but also automorphisms of the type

$$K=(k_{ij}),\quad k_{ij}=0,\ j<i,\ k_{ii}=\begin{cases} z, & i=1,3,6,\\ \bar{z}, & i=2,5,8,\\ 1, & i=4,7,\end{cases} \tag{5.24}$$

where $j=1,\dots,8$ and z is a cubic root of 1.

So, we have proved the following theorem.

Theorem 5.4. *Consider the set of complex filiform Lie algebras. If $C^8=\{(a_2,\dots,a_9)/a_i\in C$ and a_i satisfy (3.1)$\}$, we define the hypersurface V in C^7 by (3.2). The group $\mathrm{Aut}(g_2)$ of the filiform Lie algebra g_2 corresponding to the set $V\cap\mathcal{V}(a_2,a_5,a_7,4a_3a_8-5a_6^2)\cap D(a_3\cdot(4a_3^2a_9-7a_6^3))$ consists of automorphisms of the type*

$$K=(k_{ij}),\quad k_{ij}=0,\ j<i,\ k_{ii}=\begin{cases} z, & i=1,3,6,\\ \bar{z}, & i=2,5,8,\\ 1, & i=4,7,\end{cases} \tag{5.25}$$

where $j=1,\dots,8$ and z is a cubic root of 1.

Remark 5.5. The group $\mathrm{Aut}(g)$ of the Lie algebra corresponding to the set of points $V\cap\mathcal{V}(a_2,a_5,a_7,4a_3a_8-5a_6^2,4a_3^2a_9-7a_6^3)\cap D(a_3)$ does not contain unipotent automorphisms.

6. General conclusions. From Theorems 4.1, 4.2, 5.1, 5.2, 5.3, and 5.4, we conclude.

Theorem 6.1. *The group of automorphisms $\mathrm{Aut}(g)$ of each one of the characteristically nilpotent filiform Lie algebras of dimension 8 over C given by (4.6) and (4.7) consists of unipotent automorphisms, except that of the Lie algebras $g_1=\mu_8^{10,a}$, $a\in C$, and $g_2=\mu_8^{11}$ given by (5.18), and (5.19).*

Theorem 6.2. *The unipotent automorphisms of each one of the characteristically nilpotent filiform Lie algebras of dimension 8 over C, $g_1=\mu_8^{10,a}$, $a\in C$,* *and* $g_2 = \mu_8^{11}$*, given by (5.18) and (5.19), form a proper subgroup of the group* $\mathrm{Aut}(g_1)$ *and* $\mathrm{Aut}(g_2)$*, respectively.*

Theorem 6.3. *The group of automorphisms* $\mathrm{Aut}(g_1)$ *of the characteristically nilpotent filiform Lie algebras of dimension* 8 *over* C*,* $g_1 = \mu_8^{10,a}$*,* $a \in C$*, given by (5.18), consists of automorphisms of the type*

$$L = (l_{ij}), \quad l_{ij} = 0,\ j < i,\ l_{ii} = 1,\ 1 \le i,j \le 8,$$
$$L = (l_{ij}), \quad l_{ij} = 0,\ j < i,\ l_{ii} = \begin{cases} 1, & i \text{ is even}, \\ -1, & i \text{ is odd}, \end{cases} \tag{6.1}$$

where $1 \le i,j \le 8$*.*

Theorem 6.4. *The group of automorphisms* $\mathrm{Aut}(g_2)$ *of the characteristically nilpotent filiform Lie algebra of dimension* 8 *over* C*,* $g_2 = \mu_8^{11}$*, given by (5.19), consist of automorphisms of the type*

$$K = (k_{ij}), \quad k_{ij} = 0,\ j < i,\ k_{ii} = \begin{cases} z, & i = 1,3,6, \\ \overline{z}, & i = 2,5,8, \\ 1, & i = 4,7, \end{cases} \tag{6.2}$$

where $j = 1,\dots,8$ *and* z *is a cubic root of* 1*.*

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