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FATOU MAPS IN $\mathbb{P}^n$ DYNAMICS

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We study the dynamics of a holomorphic self-map f of complex projective space of degree $d > 1$ by utilizing the notion of a Fatou map, introduced originally by Ueda (1997) and independently by the author (2000). A Fatou map is intuitively like an analytic subvariety on which the dynamics of f are a normal family (such as a local stable manifold of a hyperbolic periodic point). We show that global stable manifolds of hyperbolic fixed points are given by Fatou maps. We further show that they are necessarily Kobayashi hyperbolic and are always ramified by f (and therefore any hyperbolic periodic point attracts a point of the critical set of f). We also show that Fatou components are hyperbolically embedded in $\mathbb{P}^n$ and that a Fatou component which is attracted to a taut subset of itself is necessarily taut.

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1. Introduction. All complex spaces used in this note are assumed to be reduced and to have a countable basis of open sets.

Given a holomorphic self-map $f : \mathbb{P}^n \to \mathbb{P}^n$ of degree $d > 1$ we present the following definition.

Definition 1.1. A *Fatou map* $g : Z \to \mathbb{P}^n$ for f from a complex space Z is a holomorphic map such that the collection of iterates $\{f^{\circ n} \circ g\}_{n \geq 0}$ is a normal family of maps from Z to $\mathbb{P}^n$. For an arbitrary complex space Z, let $\text{Fatou}_Z(f)$ denote the set of all Fatou maps from Z for the self-map $f : \mathbb{P}^n \to \mathbb{P}^n$.

Fatou maps were originally defined in [8] and independently in [6]. Note that the definition of a Fatou map depends both upon the map $f : \mathbb{P}^n \to \mathbb{P}^n$ used and upon the complex space Z. A Fatou map generalizes the notion of the Fatou set of f. If an open subset $U \subset \mathbb{P}^n$ lies in the Fatou set of f, then the inclusion $i : U \to \mathbb{P}^n$ is clearly a Fatou map, and conversely.

One might wonder whether there is any advantage of considering Fatou maps rather than considering varieties already lying in $\mathbb{P}^n$, on which the iterates of f are a normal family. The advantage lies in the fact that for f, Z fixed, the set of Fatou maps from Z to $\mathbb{P}^n$ has been shown to be compact. Using this, we will be able to prove that Fatou components are hyperbolically embedded and that a Fatou component which is attracted to a taut subset of itself is in fact taut. It will follow, using a theorem of Fornaess and Sibony, that all recurrent Fatou components in $\mathbb{P}^2$, which are not Siegel domains, are taut.

(Brendan Wieckert has an unpublished proof that basins of attracting periodic points in $\mathbb{P}^n$ are taut.)

We will show that if $g : Z \to \mathbb{P}^n$ is an injective Fatou map, then Z must be Kobayashi hyperbolic.

Given a hyperbolic fixed point p, we will use the term “global stable variety” to refer to all points whose forward iterates converge to p. We use the term global stable variety instead of global stable manifold because, in our setting, this set could hypothetically have singularities. In fact, this set could plausibly even fail to be a subvariety of $\mathbb{P}^n$, due to bad behaviour at the boundary. However, we will show that the global stable variety of a hyperbolic fixed point is always the image of a holomorphic map from some complex space and that this map is in fact a Fatou map. We further show that this global stable variety is necessarily ramified by f, and thus must intersect the critical set of f. It follows that every hyperbolic periodic point attracts a point of the critical set.

2. Fatou maps. We recall that given a holomorphic self-map $f : \mathbb{P}^n \to \mathbb{P}^n$ of degree $d \ge 2$, there is a lift of f to a polynomial map $F : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ whose coordinate functions are homogeneous of degree d such that $F^{-1}(0) = 0$ and such that the diagram

$$\begin{array}{ccc} \mathbb{C}^{n+1} \setminus \{0\} & \xrightarrow{F} & \mathbb{C}^{n+1} \setminus \{0\} \\ \downarrow \rho & & \downarrow \rho \\ \mathbb{P}^n & \xrightarrow{f} & \mathbb{P}^n \end{array} \tag{2.1}$$

commutes. Such a lift always exists and is unique up to constant multiple. The Green’s function $G : \mathbb{C}^{n+1} \to \mathbb{R} \cup \{-\infty\}$ of f is then defined as

$$G(z) = \lim_{n\to\infty} \frac{\log ||F^{\circ n}(z)||}{d^n}. \tag{2.2}$$

Then, $G : \mathbb{C}^{n+1} \to \mathbb{R} \cup \{-\infty\}$ is continuous and the only point mapped to $-\infty$ by G is the origin. It is easy to verify that $G(\lambda z) = G(z) + \log|\lambda|$ for $\lambda \in \mathbb{C}^*$.

A basic property of the Green’s function is that $G(F(z)) = d \cdot G(z)$. The zero set of the Green’s function is therefore completely invariant under F. We let $\mathfrak{Z} = \{z \in \mathbb{C}^{n+1} \mid G(z) = 0\}$ be the zero set of the Green’s function. Then the set $\mathfrak{Z}$ is a compact subset of $\mathbb{C}^{n+1} \setminus \{0\}$, invariant under multiplication by elements of the unit circle in $\mathbb{C}$ and $\mathfrak{Z}$ intersects every complex line through the origin in $\mathbb{C}^{n+1}$ in a circle.

The following theorem was originally proven by Ueda in [8] and independently by the author in [6]. It generalizes the work of Hubbard and Papadopol [4], Fornaess and Sibony [2], and Ueda [7] from statements about Fatou components to statements about Fatou maps.

Theorem 2.1. *For a holomorphic map $g : Z \to \mathbb{P}^n$, the following properties are equivalent:*

(1) *g is a Fatou map for f;*
(2) *the sequence $\{f^{\circ k} \circ g\}_{k \geq 0}$ contains a convergent subsequence;*
(3) *if U is any open subset of Z and $\hat{g}_U : U \to \mathbb{C}^{n+1} \setminus \{0\}$ is a holomorphic map such that $\rho \circ \hat{g}_U = g|_U$, then $G \circ \hat{g}_U$ is pluriharmonic on U. (Where a function will be said to be pluriharmonic on a complex space if and only if it is locally the real part of a holomorphic function);*
(4) *there is a complex cover $p : \hat{Z} \to Z$ and a holomorphic map $\hat{g} : \hat{Z} \to \mathbb{C}^{n+1} \setminus \{0\}$ such that $\hat{g}$ lands in the zero set of G and $\rho \circ \hat{g} = g \circ p$.*

It is also worth noting that being a Fatou map is a local property, since being a normal family of maps is a local property.

The following was also proven originally in [8] and independently in [6].

Theorem 2.2. *The set of maps* $\text{Fatou}_Z(f)$ *is compact for any complex space Z.*

Ueda showed in [7] that Fatou components in $\mathbb{P}^n$ are necessarily Kobayashi hyperbolic. Here we prove that the same will be true more generally (the idea is still to lift the zero set of the Green's function).

Lemma 2.3. *If $g : Z \to \mathbb{P}^n$ is an injective Fatou map, then Z is Kobayashi hyperbolic.*

Proof. Let $\hat{g} : \hat{Z} \to \mathbb{C}^{n+1} \setminus \{0\}$ be a lift of $g : Z \to \mathbb{P}^n$ which lands in the zero set of the Green's function as given by Theorem 2.1. Choose an open ball B in $\mathbb{C}^{n+1}$ which is large enough that the zero set of the Green's function lies inside B. Given z_1 and z_2 arbitrary distinct points of Z, then $g(z_1)$ and $g(z_2)$ are distinct points of $\mathbb{P}^n$ and thus $\ell_1 = \rho^{-1}(g(z_1))$ and $\ell_2 = \rho^{-1}(g(z_2))$ are distinct complex lines in $\mathbb{C}^{n+1} \setminus \{0\}$. Let $C_1 = \ell_1 \cap \mathfrak{Z}$ and $C_2 = \ell_2 \cap \mathfrak{Z}$ be the intersections of the lines ℓ_1 and ℓ_2, respectively, with the zero set of the Green's function. Then C_1 and C_2 are disjoint circles which lie in B.

Now choose any $y_1, y_2 \in \hat{Z}$ such that $p(y_1) = z_1$ and $p(y_2) = z_2$. Then, $\hat{g}(y_1) \in C_1$ and $\hat{g}(y_2) \in C_2$ (since $\rho(\hat{g}(y_1)) = g(z_1)$, $\rho(\hat{g}(y_2)) = g(z_2)$, and $\hat{g}$ lands in $\mathfrak{Z}$). Thus, for any such y_1 and y_2, we see $d_{\hat{Z}}(y_1, y_2) \geq d_B(C_1, C_2) > 0$ (where $d_{\hat{Z}}$ and d_B represent the Kobayashi pseudometric on the spaces $\hat{Z}$ and B, resp.).

Since $d_Z(z_1, z_2) = \inf_{y_1, y_2} d_{\hat{Z}}(y_1, y_2) \geq d_B(C_1, C_2) > 0$ (see Kobayashi [5, page 61]) and since z_1 and z_2 are arbitrary distinct points, it follows that Z is Kobayashi hyperbolic. □

3. The stable variety of a hyperbolic fixed point. If p is a hyperbolic fixed point of f, then we will show that the global stable variety of p (meaning the set of all points in $\mathbb{P}^n$ which eventually converge to p) is given by a complex space X with a holomorphic inclusion map $i : X \to \mathbb{P}^n$ (the set $i(X)$ could potentially fail to be an analytic subset of $\mathbb{P}^n$ because of bad behavior at the boundary; similarly, the topology of X could potentially be finer than the subspace topology on $i(X)$ for the same reason).

If p is a hyperbolic fixed point of f, then, by replacing f with an iterate if necessary, we can assume that there is some open neighborhood U_0 of p such that the local stable manifold X_0 at p is a closed complex submanifold of U_0 and $f(X_0) \Subset X_0$. (We assume here the basic folklore of local stable and unstable manifolds. Particularly that the local stable manifold of a holomorphic map about a hyperbolic fixed point is a complex analytic manifold.)

Lemma 3.1. *The inclusion $i : X_0 \to \mathbb{P}^n$ is a Fatou map.*

Proof. This is immediate since the iterates of f converge uniformly to p on X_0. □

We will now make some definitions in order to prove that there is a natural Fatou map which corresponds to the global stable manifold of p.

For each positive integer j, we let $X_j = f^{\circ -j}(X_0)$ and $U_j = f^{\circ -j}(U_0)$. Then, since X_j is the preimage of the analytic subset X_0 of U_0 under the holomorphic map $f^{\circ j} : U_j \to U_0$, then X_j is a closed analytic subset of U_j. Since $f(X_0) \Subset X_0$ and since f is proper, then applying f^{-1} we see that $X_0 \subset f^{-1}(f(X_0)) \Subset f^{-1}(X_0)$ and thus, inductively, $X_0 \Subset f^{-1}(X_0) \Subset f^{\circ -2}(X_0) \Subset \cdots$ or rather $X_0 \Subset X_1 \Subset X_2 \Subset \cdots$. Now each X_i is an open subset of X_j (in the subspace topology of X_j) for all $i \le j$.

We now let

$$X = \bigcup_{i=0}^{\infty} X_i. \tag{3.1}$$

Because each X_i is an open subset of X_j for $j \ge i$, then the inclusion $X_i \subset X_j$ is a biholomorphism onto its image. We note that X has the natural structure of a complex space by considering the collection $\{X_i\}$ as an atlas of open sets. Since given $0 \le i \le j$, the inclusion $X_i \subset X_j$ is a biholomorphism onto its image. The topology defined by the atlas $\{X_i\}$ is not necessarily the topology X inherits as a subset of $\mathbb{P}^n$. By the definition of the topology induced by an atlas of open sets, an arbitrary subset N of X is open in X if and only if $N \cap X_i$ is an open subset of X_i for each i (where each set X_i has the subspace topology it inherits from either U_i or $\mathbb{P}^n$, these being equivalent since U_i is open in $\mathbb{P}^n$). We note that since X_i is an open subset of X_j whenever $j \ge i$, then the subspace topology each X_i inherits from X is the same as the topology we have already defined on it. Thus, the topology we have defined on X is at least as fine as the subspace topology X inherits from $\mathbb{P}^n$ since if U is open in $\mathbb{P}^n$, then $U \cap X_i$ is also open in X_i for each $i \ge 1$; hence $U \cap X$ is open in X. Thus, the inclusion $i : X \to \mathbb{P}^n$ with the topology we have given X is continuous.

Theorem 3.2. *The inclusion $i : X \to \mathbb{P}^n$ is a Fatou map (and hence X is Kobayashi hyperbolic).*

Proof. We first need to show that it is a holomorphic map. To see this, we note that each point x of X lies in one of the open sets X_i, and the inclusion map is holomorphic on X_i. Hence, the inclusion map is locally holomorphic and therefore holomorphic.

We next need to show that the collection of maps $\{f^{\circ i}\}$ is a normal family of maps on X. To see this, let K be any compact subset of X. There is some $n \geq 0$ such that $K \subset X_n$. Then $f^{\circ n}(X_n) \subset X_0$ by the definition of X_n and so $f^{\circ(n+i)}(X_n) \subset f^{\circ i}(X_0)$ from which it follows that the iterates of f converge uniformly to p on K. □

Because i is a local biholomorphism, it is clear that f induces a self-map of X (whose induced map we will still refer to as f). Let S_p be the irreducible component of the global stable variety X containing p. Then f maps S_p into itself. We can think of p as still being a point of S_p in which case $f : S_p \to S_p$ has p as an attracting fixed point.

We will prove the following theorem.

Theorem 3.3. *Given a hyperbolic fixed point p, the stable variety S_p of p is ramified by f. Thus, S_p intersects the critical set of f and hence p attracts a point of the critical set of f.*

Proof. Assume that S_p is not ramified by f. Then $f : S_p \to S_p$ must be a local biholomorphism. Clearly, $f : S_p \to S_p$ is a proper map, so $f : S_p \to S_p$ is actually a covering map. Let $s : \tilde{S}_p \to S_p$ be the (complex) universal cover of S_p and choose a point $q \in \tilde{S}_p$ such that $s(q) = p$. Now $f \circ s : \tilde{S}_p \to S_p$ is a composition of covering projections and is therefore a covering projection (since complex spaces are locally contractible [1]). Since $\tilde{S}_p$ is simply connected, then $f \circ s : \tilde{S}_p \to S_p$ is a (complex) universal cover, just as s is. Thus there is an automorphism $\tilde{f} : \tilde{S}_p \to \tilde{S}_p$ such that the diagram

$$\begin{array}{ccc} \tilde{S}_p & \xrightarrow{\tilde{f}} & \tilde{S}_p \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ S_p & \xrightarrow{f} & S_p \end{array} \tag{3.2}$$

commutes. This automorphism can be chosen so that $\tilde{f}(q) = q$. This automorphism is also holomorphic since it is locally defined by mapping consecutively by s and f and then lifting by a local section of s. It follows that $D_q(\tilde{f})$ must be conjugate to $D_p(f|_{S_p})$. However, the eigenvalues at a fixed point of an automorphism of a hyperbolic complex space must all have norm one (Kobayashi [5, page 268] applied to the automorphism and its inverse), contrary to the fact that the eigenvalues of $f|_{S_p}$ all have norm smaller than one. □

4. Applications to the geometry of the Fatou set. We now study the geometry of the Fatou set. We will show that all Fatou components are hyperbolically embedded in $\mathbb{P}^n$, and we will derive a criterion for a fixed Fatou component to be taut. Combining our criterion with a theorem of Fornaess and Sibony, we will be able to show that any recurrent Fatou component of $\mathbb{P}^2$ which is not a Siegel domain is taut.

Corollary 4.1. *Every component of the Fatou set is hyperbolically embedded in $\mathbb{P}^n$.*

Proof. The proof is immediate since if U is a Fatou component, then the set of maps $\mathrm{Hol}(D,U)$, where D is the unit disk, lies inside $\mathrm{Fatou}_D(f)$ which is compact in $\mathrm{Hol}(D,\mathbb{P}^n)$. Thus, $\overline{\mathrm{Hol}(D,U)}$ is compact in $\mathrm{Hol}(D,\mathbb{P}^n)$, so U is tautly embedded which is equivalent to being hyperbolically embedded (see Kobayashi [5, pages 244, 246]). □

Theorem 4.2. *If U is any fixed Fatou component which is attracted to a set S contained in U (meaning that if $x_0 \in U$, then any convergent subsequence $\{f^{\circ n_i}(x_0)\}$ converges to a point of S) and if S is taut, then U is taut.*

Proof. Let $\mathcal{L}$ be the set of all maps $g : U \to \mathbb{P}^n$ such that some subsequence $f^{\circ n_i}$ converges to g. Thus $\mathcal{L}$ is the set of forward limit maps of $f^{\circ n_i}$ on U. It follows that $\mathcal{L}$ is a closed and hence compact subset of $\mathrm{Fatou}_U(f)$. If K is any compact subset of U, then $\mathcal{L}(K) = \{g(z) \mid g \in \mathcal{L},\ z \in K\}$ is a compact subset of $\mathbb{P}^n$ and from the hypothesis, we see that $\mathcal{L}(K) \subset S$.

If U is not taut, there is a sequence of maps h_i of the unit disk D into U, which is not compactly divergent and does not have a subsequence which converges to a map into U. Since it is not compactly divergent, then, by definition, there are compact subsets $K \subset U$ and $L \subset D$ such that $h_i(L) \cap K \neq \emptyset$ for arbitrarily large values i. We replace our sequence of maps h_i with a subsequence if necessary so that $h_i(L) \cap K \neq \emptyset$ for all i. Now the image of each map h_i lies in U and hence $h_i \in \mathrm{Fatou}_D(f)$ for each i. Since $\mathrm{Fatou}_D(f)$ is compact, we can replace the sequence of maps h_i with a subsequence if necessary so that the sequence of maps h_i converges to a map $h \in \mathrm{Fatou}_D(f)$. As $h_i(L) \cap K \neq \emptyset$ for each i, then we see that $h(L) \cap K \neq \emptyset$ as convergence is uniform on L (for otherwise $h(L)$ and K would be disjoint compact sets, hence they would be separated by some finite distance ϵ, and for all sufficiently large i, then $h_i(L)$ would have to be at least a distance $\epsilon/2$ away from $h(L)$ and thus be disjoint from K). Now we also know that $h(D) \not\subset U$ by hypothesis on the sequence h_i. Thus, $h(D)$ must meet ∂U. Specifically $h^{-1}(\partial U)$ is not empty. Our plan is to now push down the maps h_i and h to maps into S, and from this obtain a contradiction.

Choose any member g of $\mathcal{L}$ and assume that f_{m_i} is a subsequence which converges to g. Since each h_i lands in U, then we see that, for each i, the sequence $\{f^{\circ m_j} \circ h_i\}$ converges to $g \circ h_i \in \mathrm{Fatou}_D(f)$. By our hypothesis on U, we see that $g \circ h_i$ lands in $S \subset U$ for each i. Similarly, on the open subset $W = h^{-1}(U)$ of D, the limit of $\{f^{\circ m_j} \circ h\}$ is $g \circ h \in \text{Fatou}_W(f)$. We now need to show that some subsequence of $\{g \circ h_i\}$ converges to a map which is $g \circ h$ on W.

As $f^{\circ m_j} \circ h \in \text{Fatou}_D(f)$ for each j, there is a convergent subsequence to some map $h' \in \text{Fatou}_D(f)$ and, of course, this must agree with $g \circ h$ on W. Since ∂U is closed and forward invariant under f, we see that $h'(y) \in \partial U$ for each point $y \in h^{-1}(\partial U)$.

Now, we only need to note that since g is uniformly continuous on each compact set, then $g \circ h_i$ converges to $g \circ h$ on any open subset V of W such that $\overline{h(V)} \subset U$. Since $g \circ h_i$ is in $\text{Fatou}_D(f)$ for each i, then the sequence $g \circ h_i$ has a convergent subsequence. We replace $g \circ h_i$ with this subsequence. This subsequence must then converges to $g \circ h$ on W, and hence to h' on D (due to unique continuation of analytic maps). Moreover, $h'(L)$ must meet $\mathcal{L}(K)$ since $h(L) \cap K \neq \emptyset$, so $g \circ h(L) \cap g(K) \neq \emptyset$ since it contains $g(h(L) \cap K)$ (note that $h(L) \cap K$ lies inside U, so g is defined on it). As we noted before, the set $\mathcal{L}(K)$ must be compact in S, so $h'(L)$ intersects a compact subset of S. Thus, the sequence $g \circ h_i$ is neither compactly divergent nor does it have a convergent subsequence to a map into S and hence S is not taut. Thus, if U is not taut, then S is not taut. Hence, the theorem is proved. □

This theorem has specific applications in the $\mathbb{P}^2$ case in the case of recurrent Fatou components. We recall the definition of a recurrent Fatou component.

Definition 4.3. A Fatou component U is *recurrent* if there is a point $p_0 \in U$ and a subsequence $\{f^{\circ n_i}\}_{i \geq 0}$ such that $\{f^{\circ n_i}(p_0)\}_{i \geq 0}$ is relatively compact in U.

Fornaess and Sibony [3] classified the recurrent Fatou components of a holomorphic self-map $f : \mathbb{P}^2 \to \mathbb{P}^2$ as follows.

Theorem 4.4 (Fornaess and Sibony). *Suppose that f is a holomorphic self-map of $\mathbb{P}^2$ of degree $d \geq 2$. Suppose that U is a fixed, recurrent Fatou component. Then, one of the following holds:*

(1) *U is an attracting basin of some fixed point in U;*
(2) *there exists a one-dimensional closed complex submanifold R of U and $\{f^{\circ n}\}$ converges uniformly to R on any compact subset K of U. The Riemann surface R is biholomorphic to a disk, a punctured disk, or an annulus and the restriction of f to R is conjugate to an irrational rotation;*
(3) *U is a Siegel Domain.*

Applying Theorem 4.2, we immediately have the following corollary.

Corollary 4.5. *Every recurrent Fatou component in $\mathbb{P}^2$ which is not a Siegel Domain is taut.*

Proof. This follows from the above theorem since a point, a disk, and an annulus are all taut sets. □

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References

[1] D. Burghelea and A. Verona, *Local homological properties of analytic sets*, Manuscripta Math. **7** (1972), 55–66.

[2] J. E. Fornaess, *Dynamics in Several Complex Variables*, CBMS Regional Conference Series in Mathematics, vol. 87, American Mathematical Society, Rhode Island, 1996.

[3] J. E. Fornaess and N. Sibony, *Classification of recurrent domains for some holomorphic maps*, Math. Ann. **301** (1995), no. 4, 813–820.

[4] J. H. Hubbard and P. Papadopol, *Superattractive fixed points in* $\mathbf{C}^n$, Indiana Univ. Math. J. **43** (1994), no. 1, 321–365.

[5] S. Kobayashi, *Hyperbolic Complex Spaces*, Grundlehren der Mathematischen Wissenschaften, vol. 318, Springer-Verlag, Berlin, 1998.

[6] J. W. Robertson, *Complex dynamics in higher dimensions*, Ph.D. thesis, University of Michigan, Michigan, 2000.

[7] T. Ueda, *Fatou sets in complex dynamics on projective spaces*, J. Math. Soc. Japan **46** (1994), no. 3, 545–555.

[8] ———, *Complex dynamics on* $\mathbf{P}^n$ *and Kobayashi metric*, Sūrikaisekikenkyūsho Kōkyūroku (1997), no. 988, 188–191.

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